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A NOTE ON A THEOREM ON A NONLINEAR COMPLEMENTARITY PROBLEM

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In the paper "A nonlinear complementarity problem for monotone functions", *Bull. Austral. Math. Soc.* 20 (1979), 227-231, Nanda and Patel proved that for a monotone function that fixes the origin, the complementarity problem admits a solution. In this note we give a short proof of the same result under weaker assumptions.

Introduction and statement of the theorem

Let C^n denote the n-dimensional complex space with hermitian norm and the usual inner product and let S be a closed convex cone in C^n. The polar of S, denoted by S^*, is the cone defined by

$$S^* = \{y \in C^n : \text{re}(x, y) \geq 0 \text{ for all } x \in S\} .$$

For each $r \geq 0$ we write

$$D_r = \{x \in S : \|x\| \leq r\} .$$

A mapping $g : C^n \to C^n$ is said to be monotone on S if $\text{re}\big(g(x)-g(y), x-y\big) \geq 0$ for each $(x, y) \in S \times S$ and strictly monotone if strict inequality holds whenever $x \neq y$.

Given a continuous function $g : C^n \to C^n$ the nonlinear complementarity problem in C^n consists of finding a z such that

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(1) $$z \in S \ , \quad g(z) \in S^* \ \text{ and } \ \operatorname{re}\big(g(z), z\big) = 0 \ .$$

The purpose of this note is to prove the following.

THEOREM. *Let* $g : S \to C^n$ *be a continuous monotone function on a closed convex cone* S *satisfying* $g(0) \in S^*$ *. Then there is a* z *which satisfies* (1)*. If further* g *is strictly monotone, then zero is the unique solution to* (1).

This work has been motivated by the work of Nanda and Patel [3] who have proved the same result under the assumption that $g(0) = 0$. We obtain the result under the assumption that $g(0) \in S^*$ (which is weaker) and our proof is much shorter and direct.

Proof of the theorem

The following result, which will be needed in the sequel, is a modified version of a lemma of Hartman and Stampacchia [1]. See [2] for a short proof.

LEMMA. *Let* $g : C^n \to C^n$ *be a continuous map on a nonempty, compact, convex set* $K \subset C^n$ *. Then there is a* $z_0 \in K$ *such that*

$$\operatorname{re}\big(g(z_0), z-z_0\big) \geq 0$$

for all $z \in K$.

Proof of the theorem. Since D_r is a nonempty, compact, convex set, it follows from the lemma that for each $r \geq 0$ there is a $z_r \in D_r$ such that

$$\operatorname{re}\big(g(z_r), z-z_r\big) \geq 0$$

for all $z \in D_r$. Since $0 \in D_r$ it follows that

$$\operatorname{re}\big(g(z_r), z_r\big) \leq 0 \ .$$

Since g is monotone we have

$$\operatorname{re}\big(g(z_r)-g(0), z_r\big) \geq 0 \ .$$

Now if $g(0) \in S^*$ $\big($or $g(0) = 0\big)$ we obtain

$$\mathrm{re}\big(g(z_r),\ z_r\big) \geq 0 .$$

It now follows that $\mathrm{re}\big(g(z_r),\ z_r\big) = 0$ for all $r \in (0, \infty)$. Thus for each $r \in (0, \infty)$, z_r is a solution to (1). If further g is strictly monotone, (1) can have at most one solution, say y. Then $y = x_r \in D_r$ for each r and

$$\|y\| = \|x_r\| \leq r$$

for each r. Therefore $y = 0$ and this completes the proof.

REMARK. Note that the assumption that $g(0) = 0$ or $g(0) \in S^*$ may fail to hold. For example, take $n = 1$. Let

$$S = \{z = (x,\ y) \in C : x \geq 0,\ y = 0\} .$$

Define $g : C \to C$ by $g(z) = -1/(1+z)$. Then $\mathrm{re}\big(g(z),\ z\big) = 0$ implies $z = 0$. However, $g(0) = -1$ and since $S = S^*$, $g(0) \notin S^*$.

References

[1] Philip Hartman and Guido Stampacchia, "On some nonlinear elliptic differential-functional equations", *Acta Math.* **115** (1966), 271-310.

[2] Sribatsa Nanda and Sudarsan Nanda, "A complex nonlinear complementarity problem", *Bull. Austral. Math. Soc.* **19** (1978), 437-444.

[3] Sribatsa Nanda and Ujagar Patel, "A nonlinear complementarity problem for monotone functions", *Bull. Austral. Math. Soc.* **20** (1979), 227-231.

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