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T-ORTHOGONALITY AND NONLINEAR FUNCTIONALS ON TOPOLOGICAL VECTOR SPACES

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In recent years the problem of concretely representing a class of nonlinear functionals on Banach spaces has received considerable attention. Suppose B is a Banach space equipped with an orthogonality relation $\perp \subset B \times B$. Denoting $(x, y) \in \perp$ by $x \perp y$, a real valued function F on B is said to be orthogonally additive if

$$x \perp y \text{ implies } F(x + y) = F(x) + F(y).$$

For example when B is a vector lattice, a natural orthogonality relation is the lattice theoretic one: $x \perp_1 y$ if $|x| \wedge |y| = 0$. The problem of representing orthogonally additive functions on normed vector lattices of measurable functions has been dealt in Drewnowskii and Orlica [**1**], Mizel and Sundaresan [**2**], Friedman and Katz [**4**], Koshi [**5**], and several others. If B is the Hilbert space $L_2[0, 1]$ with the usual concept of orthogonality, i.e., $x \perp_2 y$ if the inner product $(x, y) = 0$, the problem of representing orthogonally additive functionals has been considered by Pinsker [**3**]. If B is an arbitrary Banach space there are several orthogonality relations which are generalisations of the usual concept of orthogonality when B is a Hilbert space. One such concept of considerable geometric and analytic interest is the following. Let $(B, || \ ||)$ be a Banach space. If $x, y \in B$, $x \perp_3 y$ if $||x + \lambda y|| \geqq ||x||$ for all real values of λ. The problem of representing orthogonally additive functionals on B with respect to the relation $\perp_3$ has been dealt with in Sundaresan [**7**].

None of the preceding concepts of orthogonality extend to arbitrary topological vector spaces. We introduce here a useful orthogonality concept in an arbitrary topological vector space. Let E be a Hausdorff topological vector space and let $T : E \rightarrow E^*$, where E^* is the dual of E, be a linear mapping. If $x, y \in E$, then x is T-orthogonal to y if $Tx(y)$, denoted by (Tx, y) equals zero. In the present paper we deal with the problem of characterizing T-orthogonally additive functionals on a topological vector space.

In the next section we recall briefly the basic terminology and establish a few results useful in the subsequent discussion. In section 3 we discuss T-orthogonally additive functionals when T-orthogonality is not symmetric. In section 4 we consider the same problem when T-orthogonality is symmetric.

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2. Throughout the paper E is a Hausdorff topological vector space on the real field R. E^* is the vector space of continuous linear functionals on E. To avoid trivialities we always assume that $\dim E \geqq 2$. If $T : E \to E^*$ is a linear mapping and $x, y \in E$, then x is T-orthogonal to y or briefly $x \perp y$, when T is understood, if $(Tx, y) = 0$. T-orthogonality is said to be symmetric, if $(Tx, y) = 0$ implies $(Ty, x) = 0$. A vector x is said to be T-isotropic or simply isotropic if $(Tx, x) = 0$. The operator T is said to be symmetric if $(Tx, y) = (Ty, x)$ for all $x, y \in E$. If $x, y, z, \ldots$ are vectors in E, the span of $x, y, z, \ldots$ is denoted by $[x, y, z, \ldots]$.

We conclude this section with a few useful lemmas.

LEMMA 1. *If $T : E \to E^*$ is a linear mapping such that T-orthogonality is symmetric and if there is a nonisotropic vector, then T is symmetric.*

Proof. Let $y, z \in E$. Suppose $(Ty, z) \neq (Tz, y)$. Since the relation $\perp$ is symmetric $(Ty, z) \neq 0 \neq (Tz, y)$. If $y \not\perp y$ it is verified that there is a real number $\alpha \neq 0$, such that $y \perp y + \alpha z$. Hence $y + \alpha z \perp y$. Thus $\alpha(Tz, y) = -(Ty, y) = \alpha(Ty, z)$. Hence $(Ty, z) = (Tz, y)$. If $z \not\perp z$ it is verified similarly that $(Ty, z) = (Tz, y)$. Let now $y \perp y$ and $z \perp z$. Let x be a vector such that $x \not\perp x$. The preceding observation shows that $(Tx, p) = (Tp, x)$ for all $p \in E$. Further since $x \not\perp x$ either $x + y$ or $x - y$ is not isotropic. Hence $(T(x + y), z) = (Tz, (x + y))$ or $(T(x - y), z) = (Tz, (x - y))$. Thus $(Ty, z) = (Tz, y)$ and T is a symmetric mapping.

LEMMA 2. *If $T : E \to E^*$ is a linear mapping and if the rank of T is an odd integer, then there is at least one non-isotropic vector.*

Proof. Suppose every vector is isotropic. The hypothesis of the lemma implies there exists a $(2K + 1)$-dimensional subspace E^{2K+1} of E, for some positive integer K, such that $T(E^{2K+1})$ is also $(2K + 1)$-dimensional. Thus if T_1 is the restriction of T to E^{2K+1}, T_1 might be considered as a linear isomorphism on E^{2K+1} to E^{2K+1} such that the inner-product $(T_1x, x) = 0$ for all $x \in E^{2K+1}$. Thus there exists continuous nonvanishing tangential vector field on the sphere in E^{2K+1}, contradicting Poincare-Brouwer theorem [**6**].

LEMMA 3. *If $T : E \to E^*$ is a 1-dimensional linear mapping then the following two statements are equivalent.*

(1) *T-orthogonality is symmetric.*

(2) *There is a nonisotropic vector x such that $x \perp y$ implies $Ty = 0$.*

Proof. Let $x \not\perp x$. Let $y \in Tx^{-1}(0)$. Then (1) implies $y \perp x$. Since T is 1-dimensional and $Tx \neq 0$, $Ty \in [Tx]$. Let $Ty = \lambda Tx$. Then since $y \perp x$ it is verified that either, $\lambda = 0$ or $(Tx, x) = 0$. Since $x \not\perp x$, $\lambda = 0$. Hence $Ty = 0$. Thus (1) implies (2). Conversely suppose (2) holds and $x \in E$ such that $x \perp y$ implies $Ty = 0$. Since Tx is a non-zero member of E^*, $Tx^{-1}(0)$ is a subspace of E of codimension 1. Thus each $\xi \in E$ determines uniquely a real number λ and a vector h, $x \perp h$, such that $\xi = \lambda x + h$. Thus if $\xi_i = \lambda_i x + h_i$, $i = 1, 2$, then $\xi_1 \perp \xi_2$ if and only if $\lambda_1\lambda_2 = 0$ since $Th_i = 0$. Hence $\perp$ is symmetric.

Remark 1. From the proof of the preceding lemma it is clear that (2) could as well be replaced by "for every nonisotropic vector x, $x \perp y$ implies $Ty = 0$."

3. Let $T : E \to E^*$ be a linear mapping such that $\perp$ is not symmetric. Let the rank of $T = 1$. Then from Lemma 2 it is inferred that there is a nonisotropic vector. Let x be one such vector. Let $M = Tx^{-1}(0)$. If $y, z \in M$ then since $Tx \neq 0$ and rank $T = 1$, Ty, $Tz \in [Tx]$. Since $(Tx, z) = 0$ it is verified that $y \perp z$. In particular for all $y \in M$, $y \perp y$. Now if F is a continuous T-orthogonally additive functional on E then the preceding observation implies that F is homogeneous and additive on M. Thus $F|M$ is a continuous linear functional on M. Since $\perp$ is not symmetric it follows from Lemma 3 that there is a vector $y \in M$ such that $Ty \neq 0$. Since M is a subspace and $Ty \in [Tx]$ we can as well assume that $Ty = Tx$. Thus $x - y \perp x$. Hence if $F(\lambda x) = \varphi(\lambda)$ then since $\lambda(x - y) \perp \mu x$ for all pairs of real numbers λ, μ it is verified from the orthogonal additivity of F and linearity of F on M that $\varphi(\lambda + \mu) = \varphi(\lambda) + \varphi(\mu)$. Since F is a continuous function, $\varphi : R \to R$ is a continuous additive function. Thus φ is linear. Now if $\xi \in E$ and $\xi = \lambda x + y, y \in M$, then $F(\lambda x + y) = \varphi(\lambda) + F(y)$. Since φ is linear on R it follows that $F \in E^*$. Since every linear functional on E is orthogonally additive it is proved that under the above hypothesis on T that a continuous function $F : E \to R$ is T-orthogonally additive if and only if $F \in E^*$.

Next we proceed to the case when rank $T > 1$. First we deal with the case of dim $E = 2$ or 3.

PROPOSITION 1. *If* dim $E = 2$ *or* 3 *and if* $T : E \to E^*$ *is a linear mapping such that* rank $T > 1$ *and T-orthogonality is not symmetric, then every continuous orthogonally additive functional on E is linear.*

Proof. Let dim $E = 2$. Suppose that $e_1, e_2 \in E$ such that $e_1 \perp e_2$ but $e_2 \not\perp e_1$. Thus e_1, e_2 are linearly independent. Since the rank $T = 2$, $Te_1 \neq 0$. Hence $(Te_1, e_2) = 0$ implies that $(Te_1, e_1) \neq 0$. Thus, there is a real number $a \neq 0$ such that $ae_1 + e_2 \perp e_1$. Hence if λ, μ are two real numbers then $\lambda(ae_1 + e_2) \perp \mu e_1$. Hence $F(\lambda ae_1 + \lambda e_2 + \mu e_1) = F(\lambda(ae_1 + e_2)) + F(\mu e_1)$. Since $e_1 \perp e_2$, $F((\lambda a + \mu)e_1 + \lambda e_2) = F((\lambda a + \mu)e_1) + F(\lambda e_2)$. Thus

$$\begin{aligned} F((\lambda a + \mu)e_1) + F(\lambda e_2) &= F(\lambda(ae_1 + e_2)) + F(\mu e_1) \\ &= F(\lambda ae_1) + F(\lambda e_2) + F(\mu e_1). \end{aligned}$$

Hence F is additive on $[e_1]$. Since F is also continuous, F is homogeneous on $[e_1]$. Further noting that $ae_1 + e_2 \perp e_1$, $e_1 \not\perp ae_1 + e_2$, arguing as in the preceding sentences with e_1, e_2 replaced respectively by $ae_1 + e_2$, and e_1, it follows that F is additive and homogeneous on $[ae_1 + e_2]$. Since $ae_1 + e_2 \perp e_1$, the T-orthogonal additivity of F at once implies that F is a linear functional on E.

Next we proceed to the case when dim $E = 3$. Let the rank $T = 2$ and $e_1, e_2 \in E$ such that $e_1 \perp e_2$ and $e_2 \not\perp e_1$. If $(Te_1, e_1) \neq 0$ or $(Te_2, e_2) \neq 0$, then as in the preceding case it is verified that F is linear on $[e_1, e_2]$. If $(Te_1, e_1) = 0$, and $(Te_2, e_2) = 0$, then F is homogeneous on $[e_1]$, and $[e_2]$. Since $e_1 \perp e_2$, is linear on the subspace $[e_1, e_2]$. Thus in either case F is linear on $[e_1, e_2]$. Now if Te_1, Te_2 are linearly independent then since the rank $T = 2$ there exists a vector $e_3 \notin [e_1, e_2]$ such that $Te_3 = 0$. Since $e_3 \perp e_3$, F is homogeneous on $[e_3]$. Further since $e_3 \perp [e_1, e_2]$ and F is linear on $[e_1, e_2]$ it is verified that F is a linear functional. If Te_1, Te_2 are linearly dependent then either $Te_1 = 0$ or $Te_1 = \lambda Te_2$, $\lambda \neq 0$. If $(Te_2, e_3) = 0$ then $e_2 \perp e_3$. If $(Te_2, e_3) \neq 0$, then there are real numbers $a \neq 0 \neq b$ such that $(Te_2, ae_1 + be_3) = 0$, since $(Te_2, e_1) \neq 0$. Thus there is $x \notin [e_1, e_2]$, such that $e_2 \perp x$. Thus if $Te_1 = 0$, then $e_1 \perp x$. If $x \not\perp e_2$ or $x \not\perp e_1$ then as in the case of $\dim E = 2$, it is verified that F is homogeneous on $[x]$. Since $[e_1, e_2] \perp x$, F is a linear functional. If $x \perp e_2$ and $x \perp e_1$, then, since $e_2 \perp e_1$, $x + e_2 \perp e_1$. However since $e_1 \perp e_2$, $e_1 \perp x + e_2$. Once again F is verified to be homogeneous on $[x + e_2]$. Since $x \perp e_2$ and F is homogeneous on $[e_2]$ it is verified that F is homogeneous on $[x]$. Thus F is linear. Next suppose $Te_1 \neq 0$. Then since $Te_1 = \lambda Te_2$ for some $\lambda \neq 0$, and $e_2 \not\perp e_1$ there is a vector $x \notin [e_1, e_2]$ such that $[e_1, e_2] \perp x$. If $x \not\perp [e_1, e_2]$ then once again F is homogeneous on $[x]$ and F is a linear functional. If $x \perp [e_1, e_2]$, since the rank $T = 2$, $(Tx, x) \neq 0$. Further since $Te_1 \neq 0$, and $e_1 \perp [x, e_2]$ it follows that $(Te_1, e_1) \neq 0$. Since $e_1 \perp x$, $x \perp e_1$ and $(Te_1, e_1) \neq 0 \neq (Tx, x)$, it follows that

(*) there is a real number $a \neq 0$, such that $x + ae_1 \perp x + ae_1$ or $x + ae_1 \perp x - ae_1$.

In the case of the first alternative, F is homogeneous on $[x + ae_1]$. Then since $x \perp e_1$ and F is homogeneous on $[e_1]$, it is verified that F is homogeneous in $[x]$. Thus F is linear. If $x + ae_1 \perp x - ae_1$ then if λ, μ are two real numbers $F((\lambda + \mu)x + \lambda ae_1 - \mu ae_1) = F(\lambda(x + ae_1)) + F(\mu(x - ae_1)) = F(\lambda x) + F(\mu x) + F(\lambda ae_1) - F(\mu ae_1)$, since $\lambda(x + ae_1) \perp \mu(x - ae_1)$. Since $x \perp e_1$, $F((\lambda + \mu)x + \lambda ae_1 - \mu ae_1) = F((\lambda + \mu)x) + F(\lambda ae_1 - \mu ae_1)$. From the preceding equations it is verified that

$$F((\lambda + \mu)x) = F(\lambda x) + F(\mu x)$$

after noting that F is homogeneous on $[e_1]$. Since F is continuous, F is homogeneous on $[x]$. Hence F is a linear functional completing the proof in the case rank $T = 2$.

Next suppose $\dim E = 3$, and rank $T = 3$. Since T-orthogonality is not symmetric there exist linearly independent vectors e_1, e_2 such that $e_1 \perp e_2$ and $e_2 \not\perp e_1$. Thus as in the case of $\dim E = 2$ it is verified that F is linear on $[e_1, e_2]$. Suppose there is a vector $e_3 \notin [e_1, e_2]$ such that $e_3 \perp [e_1, e_2]$. If $e_1 \not\perp e_3$ or $e_2 \not\perp e_3$ then F is homogeneous on $[e_3]$ and F is a linear functional. Next let $e_1 \perp e_3$ and $e_2 \perp e_3$ or equivalently $[e_1, e_2] \perp e_3$. Since $e_3 \perp [e_1, e_2]$, $e_3 \notin [e_1, e_2]$ and rank $T = 3$, $(Te_3, e_3) \neq 0$. Similarly since $e_1 \perp e_2$, $e_1 \perp e_3$ it is verified that $(Te_1, e_1) \neq 0$. Thus since $e_1 \perp e_3$, $e_3 \perp e_1$ there is a nonzero real number a such that

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either $ae_1 + e_3 \perp ae_1 + e_3$ or $e_3 + ae_1 \perp e_3 - ae_1$. Thus as in the case of (*) in the preceding paragraph it follows that F is homogeneous on $[e_3]$. Hence F is a linear functional. Next suppose there is no vector $e_3 \notin [e_1, e_2]$ such that $e_3 \perp [e_1, e_2]$. Since rank $T = 3$, there is a vector $x \neq 0$ such that $x \perp [e_1, e_2]$ and $x \notin [e_1]$. Since such a vector $x \in [e_1, e_2]$ there are real numbers $a, b, b \neq 0$ such that $ae_1 + be_2 \perp e_2$, and $ae_1 + be_2 \perp e_1$. Thus since $e_1 \perp e_2$ and $e_2 \not\perp e_1$ it is verified that $(Te_2, e_2) = 0 = (Te_1, e_1)$. Hence we are in the case $e_1 \perp e_1$, $e_2 \perp e_2$, $e_1 \perp e_2$ and $e_2 \not\perp e_1$. Since $e_1 \perp [e_1, e_2]$, and Te_1, Te_2 are linearly independent there is a vector $e_3 \notin [e_1, e_2]$ such that $e_2 \perp e_3$. Identifying linear functionals f on E with points in E by the mapping

$$f \leftrightarrow \sum_{i=1}^{3} f(e_i)e_i$$

it is verified that there are real numbers a_3, b_1, c_1, c_2, and c_3 such that $Te_1 = a_3c_3$, $Te_2 = b_1e_1$, $Te_3 = \sum_{i=1}^{3} c_ie_i$. Since the rank $T = 3$, a_3, b_1, c_3, c_2 are nonzero real numbers. Thus $e_3 \not\perp e_2$ while $e_2 \perp e_3$. Hence F is homogeneous on $[e_3]$. Further it is verified that $e_3 \perp c_2e_3 - c_3e_2$ and $c_2e_3 - c_3e_2 \not\perp e_3$. Hence F is linear on $[e_3, c_2e_3 - c_3e_2]$. Now since $e_2 \perp [e_3, c_2e_3 - c_3e_2]$ and F is homogeneous on $[e_2]$ it follows that F is linear on E.

Next we proceed to the main theorem of this section.

THEOREM 1. *Let E be a real Hausdorff topological vector space and $T: E \to E^*$ be a linear mapping such that the T-orthogonality is not symmetric. Then every continuous orthogonally additive functional on E is linear.*

Proof. In view of the introductory comments in this section we may assume that rank $T \geqq 2$. Since the range of T is of dimension at least 2, and orthogonality is not symmetric we claim that there exist two vectors $e_1, e_2 \in E$ such that $e_1 \perp e_2$, $e_2 \not\perp e_1$ and Te_1, Te_2 are linearly independent. For let x, y be two vectors such that $x \perp y$, and $y \not\perp x$. If Tx, Ty are linearly dependent let $p \in E$ be such that Tp, Ty are linearly independent. If $y \not\perp p$ then since $y \not\perp x$ there exists a real number a such that $y \perp p + ax$. If $p + ax \perp y$ then since $x \perp y, p \perp y$. Thus $y \not\perp p$ and $p \perp y$ and Tp, Ty are linearly independent. Next, if $p + ax \not\perp y$, then $p + ax, y$ are vectors of the required type. If $y \perp p$, then if $p \not\perp y$, p, y have the desired properties. If $p \perp y$ then $p + x \perp y$ and $p + x, y$ have the desired properties. Thus there exist vectors e_1, e as claimed. Let now $x \in E \sim [e_1, e_2]$. Consider the linear mapping $T|[x, e_1, e_2] = T_1$. Then applying Proposition 1 to T_1 and the function F it follows $F|[x, e_1, e_2]$ is linear. This also implies in particular that F is linear on $[x]$ for all $x \in E$. Next, let x, y be two linearly independent vectors, $x, y \notin [e_1, e_2]$. If $x \perp y (y \perp x)$ F is verified to be linear on $[x, y]$ from the preceding observation. Next if $x \not\perp y$ and $y \not\perp x$, then if $(Tx, x) \neq 0$ or $(Ty, y) \neq 0$ it is possible to find a real number a such that $x \perp x + ay$ or $y \perp y + ax$. Then in either case as before F is

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linear on the span of $[x, y]$. If $(Tx, x) = 0 = (Ty, y)$, then since $(Tx, y) \neq 0 \neq (Ty, x)$ it is verified that there is a real number a such that $x = ay \perp y + x$, once again verifying F is linear on $[x, y]$. Thus in any case F is linear on $[x, y]$. Hence F is a linear functional.

4. We discuss here the case when the T-orthogonality is symmetric. We note that if $F:E \to R$ is orthogonally additive then the even and odd parts F_1, F_2 of F are also orthogonally additive. This is verified from the equations $F_1(x) = \frac{1}{2}[F(x) + F(-x)]$ and $F_2(x) = \frac{1}{2}[F(x) - F(-x)]$.

As in the preceding sections we assume that $\dim E \geqq 2$. Further we note that if $\dim T = 1$ then as observed in Lemma 2 there is a $x \in E$ such that $(Tx, x) \neq 0$. Now as in the case when T-orthogonality is not symmetric, $\dim T = 1$ (see first paragraph in section 3) it is verified that if F is a orthogonally additive functional on E and $M = Tx^{-1}(0)$ then $F|M$ is linear. Since $E = M \oplus [x]$ it is verified that F determines a unique continuous function $\varphi:R \to R$, $\varphi(0) = 0$ such that $F(\lambda x + y) = \varphi(\lambda) + l(y)$ if $y \in M$ and $F|M = l$. Conversely if $l \in E^*$ and $\varphi:R \to R$ is a continuous function, $\varphi(0) = 0$, then the function $F:E \to R$ defined by $F(\xi) = \varphi(\lambda) + l(y)$, if $\xi = \lambda x + y$, $y \in M$, determines a continuous orthogonally additive function. The preceding fact is verified by noting that for $y, z \in M$, $\lambda x + y \perp \mu x + z$ if and only if $\lambda\mu = 0$ since orthogonality is symmetric and $y \perp z$.

We proceed to discuss the case when rank $T \geqq 2$.

PROPOSITION 2. *Let* $\dim E = 2$. *If* $T:E \to E^*$ *is a linear mapping*, rank $T = 2$, *and if T-orthogonality is symmetric, then a continuous function $F:E \to R$ is even and orthogonally additive if and only if $F(x) = c(Tx, x)$ for some real number c*.

Proof. If $(Tx, x) = 0$ for all $x \in E$, then since F is even orthogonally additive functional it follows that $F(x) = F(-x)$, and $F(x) + F(-x) = F(0) = 0$. Thus $F(x) = 0$ for all $x \in E$.

Next if for some x $(Tx, x) \neq 0$, then from Lemma 1 it follows that T is a symmetric mapping. Let e_1 be a vector such that $e_1 \not\perp e_1$. Then there is a vector e_2, $e_2 \notin [e_1]$ such that $e_1 \perp e_2$. Since T is of rank 2, $Te_2 \neq 0$. Thus $e_2 \perp e_1$ implies $e_2 \not\perp e_2$. Hence we can assume that there are real numbers $a \neq 0 \neq b$, such that $Te_1 = ae_1$ and $Te_2 = be_2$. We can assume without loss of generality that $a > 0$. It is verified that $x_1e_1 + x_2e_2 \perp y_1e_1 + y_2e_2$ if and only if $ax_1y_1 + bx_2y_2 = 0$. Now if $b > 0$ then there are vectors $x, y, x \in [e_1], y \in [e_2]$ such that $(Tx, x) = 1 = (Ty, y)$. If $b < 0$ then there are vectors x, y, as above such that $(Tx, x) = 1 = -(Ty, y)$. For such a pair x, y, for all real numbers K, $K(x + y) \perp K(x - y)$ or $K(x + y) \perp K(x + y)$ according as $b > 0$ or $b < 0$. Since F is even and $Kx \perp Ky$, it is verified from the orthogonal additivity of F that $F(Kx) = F(Ky)$ or $F(Kx) = -F(Ky)$. Now it follows that there is a real number c such that for all K, $F(Kz) = c(TKz, Kz)$ where $z = x$ or $z = y$, noting that $F(Kx) = F(Ky)$ and $F(Kx) = -F(Ky)$ according as $(Tx, x) = (Ty, y)$ or $(Tx, x) = -(Ty, y)$. Now let ξ be an arbitrary vector in E. Let $\xi =$

$\lambda x + \mu y$. Then from the orthogonal additivity of F it follows that

$$F(\lambda x + \mu y) = F(\lambda x) + F(\mu y) = c(T\lambda x, \mu x) + c(T\mu y, \mu y)$$
$$= c(T(\lambda x + \mu y), \lambda x + \mu y).$$

Hence $F(\xi) = c(T\xi, \xi)$.

THEOREM 2. *Let* dim $E \geqq 2$ *and* $T:E \to E^*$ *be a linear mapping such that* rank $T \geqq 2$. *If* T*-orthogonality is symmetric, then a continuous real valued function* F *on* E *is even and orthogonally additive only if there is a real number* c *such that for all* $\xi \in E$,

$$F(\xi) = c(T\xi, \xi).$$

Proof. If $(Tx, x) = 0$ for all $x \in E$, then since $x \perp x$ for all x, F is linear on $[x]$. Since F is also even $F(x) = 0$ for all $x \in E$ and it follows that $F(x) = c(Tx, x)$ for all x, where c is an arbitrary real number.

Next let x be a vector such that $(Tx, x) \neq 0$. Let F be a continuous orthogonally additive function, and let $M = Tx^{-1}(0)$. There exists a $y \in M$ such that $(Ty, y) \neq 0$. For, let every vector in M be isotropic. Since the rank $T \geqq 2$ there is a vector $p \in M$ such that $Tp \notin [Tx]$. Thus there exists a $z \in M$ such that $p \not\perp z$. Now $p + z \in M$. Since $p + z \perp p + z$, $(Tp, z) + (Tz, p) = 0$, since every vector in M is isotropic. Since the mapping T is symmetric the preceding equation implies $p \perp z$ contradicting the choice of z. Thus there is a vector $y \in M$ with $(Ty, y) \neq 0$. Let $T_1 = T|[x, y]$. Since $(Ty, y) \neq 0$ and $(Ty, y) = 0$, T_1y, T_1x are linearly independent and the rank $T_1 = 2$. Noting that T-orthogonality coincides with T_1-orthogonality on the plane $[x, y]$ it follows from the preceding proposition that $F(\xi) = c(T\xi, \xi)$ for all $\xi \in [x, y]$ where c is independent of ξ. In particular $F(Kx) = K^2F(x)$ for all $K \neq 0$. Now let $z \in E$, and write $z = \lambda x + \eta$ where $x \perp \eta$ and λ is a real number. Then

$$F(z) = F(\lambda x + \eta) = F(\lambda x) + F(\eta) = \lambda^2 F(x) + F(\eta).$$

If $(T\eta, \eta) = 0$ then $F(\eta) = 0$. If $(T\eta, \eta) \neq 0$, from the preceding it follows that $F(\eta) = c(T\eta, \eta)$ where c is such that $F(x) = c(Tx, x)$. Thus

$$F(z) = \lambda^2 F(x) + c(T\eta, \eta) = c(T(\lambda x + \eta), \lambda x + \eta).$$

This completes the proof of the theorem.

Next we proceed to the case when T-orthogonality is symmetric and F is an odd functional. In this case if $x \perp x$, then F is linear on $[x]$. Thus, if $(Tx, x) = 0$ for all x, we expect F to be a linear functional. However we provide an example to show that this need not be the case when every vector x in E is isotropic and rank $T = 2$.

THEOREM 3. *Let* $T:E \to E^*$ *be a linear mapping such that* T*-orthogonality is symmetric and* rank $T \geqq 2$. *Then every odd continuous* T*-orthogonally additive real valued function on* E *is linear if there is at least one nonisotropic vector*.

Proof. Since there is a nonisotropic vector and T-orthogonality is symmetric, the linear mapping T is symmetric. Further we note that since F is an odd orthogonally additive function, F is linear on $[x]$ if x is isotropic. We proceed to verify that F is linear on $[x]$ even if x is nonisotropic. As already noted in the second paragraph of the proof of the preceding theorem there is a vector $y \perp x$ such that $(Ty, y) \neq 0$. We may even assume that $(Ty, y) = \pm(Tx, x)$. If $(Ty, y) = (Tx, x)$ then since $x \perp y$, $K(x + y) \perp K(x - y)$ for all real numbers K. Thus noting that F is an odd function it is verified that $F(2Kx) = 2F(Kx)$ and $F(2Ky) = 2F(Ky)$. Further since for any real number m, $m(x + y) \perp (x - y)$ it is verified that

$$F((m + 1)x) + F((m - 1)y) = F(mx) + F(x) + F(my) - F(y).$$

Now by straightforward induction it is verified that for integers m, $F(mx) = mF(x)$ and $F(my) = mF(y)$. Since x, y could be replaced by rx, ry, r a real number, $F(mrx) = mF(rx)$ for all real numbers r and integers m. Hence for rationals m/n we have

$$F\left(\frac{m}{n}x\right) = \frac{m}{n}F(x).$$

Since F is continuous F is linear on $[x]$. If $(Tx, x) = -(Ty, y)$, since $x \perp y$, $x + y, x - y$ are isotropic vectors. Thus for any real number λ, $F(\lambda(x + y)) = \lambda(F(x) + F(y))$ and $F(\lambda(x - y)) = \lambda[F(x) - F(y)]$. Hence $F(\lambda x) + F(\lambda y) = \lambda(F(x) + F(y))$ and $F(\lambda x) - F(\lambda y) = \lambda[F(x) - F(y)]$. Thus $F(\lambda x) = \lambda F(x)$. Hence F is linear on all 1-dimensional subspaces of E.

We proceed to show that F is indeed linear on E. Since F is linear on each line in E and orthogonally additive it is enough to show that in any two dimensional subspace $[x, y]$ there are two linearly independent orthogonal vectors. Let x, y be two linearly independent vectors. If $x \perp y$ we have two orthogonal vectors in $[x, y]$. If $x \not\perp y$, but $(Tx, x) \neq 0$ $((Ty, y) \neq 0)$ the pair $x, x + ay(y, y + ax)$ where $a = -(Tx, x)/(Tx, y)(a = -(Ty, y)(Tx, x))$ is verified to be a pair of of the required type in the subspace (x, y). If $(Tx, x) = 0 = (Ty, y)$ then the pair $x + y$, $x - y$ is one such since T is symmetric. This completes the proof of linearity of F. Thus $F \in E^*$.

Before proceeding to the case when every vector is T-isotropic let us recall that according to Lemma 2, if the rank of T is an odd integer $\geqq 3$ then there is at least one non-isotropic vector. We start with a preliminary result dealing with the case when rank $T = 4$.

PROPOSITION 3. *If* dim $E = 4$ *and* $T: E \to E^*$ *is a symmetric linear isomorphism and if every vector is isotropic, then every odd orthogonally additive continuous real valued function on* E *is linear.*

Proof. Let $e_1 \in E \sim \{0\}$. Since $Te_1 \neq 0$, the subspace $M = Te_1^{-1}(0)$ is 3-dimensional. Let e_2 be a vector in $Te_1^{-1}(0)$ such that e_1, e_2 are linearly independent. Since Te_2 and Te_1 are linearly independent there is a vector e_3 such that $e_1 \perp e_3$ and $(Te_2, e_3) = 1$ and a vector e_4 such that $e_2 \perp e_4$ and $(Te_1, e_4) = 1$. It is verified that $\{e_1, e_2, e_3, e_4\}$ is a base for E and representing linear functionals f on E with vectors in E by the isomorphism

$$f \leftrightarrow (f(e_1), f(e_2), f(e_3), f(e_4)).$$

It follows from the properties that every vector is isotropic, and orthogonality is symmetric, that

$$Te_1 = e_4,\ Te_2 = e_3,\ Te_3 = -e_2, \text{ and } Te_4 = -e_1.$$

Since for every $x \in E$, $x \perp x$ it follows that F is linear on $[x]$ for every $x \in E$. Thus if $x \perp y$ then F is linear on the subspace $[x, y]$. Since $e_1 \perp [e_1, e_2, e_3]$, $e_2 \perp [e_1, e_2, e_4]$, $e_3 \perp [e_1, e_3, e_4]$, $e_4 \perp [e_2, e_3, e_4]$ and $[e_2, e_3] \perp [e_1, e_4]$ it is enough to verify that F is linear on the subspaces $[e_2, e_3]$ and $[e_1, e_4]$. Consider a typical vector, say $\lambda e_2 + \mu e_3$ in $[e_2, e_3]$. It is verified that $e_1 + \lambda e_2 \perp \mu e_3 - \lambda\mu e_4$ and $e_1 - \lambda\mu e_4 \perp \lambda e_2 + \mu e_3$. Thus

$$F(e_1 + \lambda e_2 + \mu e_3 - \lambda\mu e_4) = F(e_1 + \lambda e_2) + F(\mu e_3 - \lambda\mu e_4).$$

Since $e_1 \perp e_2$ and $e_3 \perp e_4$,

$$(1) \quad F(e_1 - \lambda\mu e_4) + F(\lambda e_2 + \mu e_3) = F(e_1) + F(\lambda e_2) + F(\mu e_3) - F(\lambda\mu e_4).$$

Once again since $e_1 + \lambda e_2 + \mu e_3 \perp \lambda e_2 + \lambda\mu e_4$ and $e_3 \perp e_1 - \lambda\mu e_4$ it follows that

$$\begin{aligned} F(e_1 + \mu e_3 - \lambda\mu e_4) &= F(\mu e_3) + F(e_1 - \lambda\mu e_4) \\ &= F(e_1 + \lambda e_2 + \mu e_3) - F(\lambda e_2 + \lambda\mu e_4) \\ &= F(e_1) + F(\lambda e_2 + \mu e_3) - [F(\lambda e_2) + F(\lambda\mu e_4)]. \end{aligned}$$

Thus

$$(2) \quad F(e_1 - \lambda\mu e_4) - F(\lambda e_2 + \mu e_3) = F(e_1) - F(\lambda\mu e_4) - F(\lambda e_2) - F(\mu e_3).$$

From equations (1) and (2) and from the linearity of F on each line in E it follows that

$$F(\lambda e_2 + \mu e_3) = F(\lambda e_2) + F(\mu e_3) = \lambda F(e_2) + \mu F(e_3)$$

and

$$F(e_1 - \lambda\mu e_4) = F(e_1) - \lambda\mu F(e_4).$$

Thus F is verified to be linear on the subspaces $[e_2, e_3]$ and $[e_1, e_4]$. Hence F is a linear functional on E.

THEOREM 4. *Let E be an arbitrary topological vector space, and let $T:E \to E^*$ be a linear mapping such that* rank $T \geqq 3$ *and* $(Tx, x) = 0$ *for all $x \in E$, and T-orthogonality is symmetric. If F is a continuous orthogonally additive functional on E, then F is linear.*

Proof. Let e_1, e_4 be an arbitrary pair of linearly independent vectors. If

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$e_1 \perp e_4$ then since F is linear on $[x]$ for each $x \in E$, F is linear on the subspace $[e_1, e_4]$. Next let $e_1 \not\perp e_4$. Since $e_1 \not\perp e_4$, $e_4 \perp e_4$ and $Te_1 \neq 0 \neq Te_4$ it is verified that Te_1, Te_2 are linearly independent. Since $x \perp x$ for all $x \in E$ and dim $T \geqq 3$, it follows from the remarks preceding Proposition 3 that dim $T \geqq 4$. Thus there exists a vector ξ, say $\xi = \lambda e_4 + h$, where $h \in Te_1^{-1}(0)$ such that $T\xi \notin [Te_1, Te_4]$. Now let $h = \mu e_1 + e_2$ where $e_2 \perp e_1$. Then it is verified that $Te_2 \notin [Te_1, Te_4]$ and $e_1 \perp e_2$, $e_4 \perp e_2$.

Now let e_3 be a vector in $Te_1^{-1}(0) \cap Te_4^{-1}(0)$ such that $e_2 \not\perp e_3$. It follows that $Te_3 \notin [Te_1, Te_2, Te_4]$. Further it is verified that the rank of $T_1 = T|E^4$ is 4, where $E^4 = [e_1, e_2, e_3, e_4]$ and the T-orthogonality restricted to E^4 coincides with T_1-orthogonality. Thus applying the preceding proposition, it is inferred that $F|E^4$ is linear. Hence F is linear on $[e_1, e_2]$, completing the proof of the theorem.

Before summarizing the results we discuss an example showing that the preceding theorem cannot be improved.

Example. Consider $E = R^2$. Let $\{e_1, e_2\}$ be a base of E. Let T be the operator defined by $Te_1 = e_2$ and $Te_2 = -e_1$. Then it is verified that $(Tx, x) = 0$ for $x \in R^2$. Let $F:R^2 \to R$ be defined by,

$$F(ae_1 + be_2) = (a^3 + b^3)^{1/3}.$$

It is verified that F is a continuous T-orthogonally additive odd functional on R^2. Thus in the preceding theorem rank $T \geqq 3$ cannot be replaced by rank $T \geqq 2$.

Since every orthogonally additive functional F is the sum of an even and an odd orthogonally additive functional we can summarize the results of this section as follows.

THEOREM 5. *Let $T:E \to E^*$ be a linear mapping such that* dim $T \geqq 2$. *If T-orthogonality is symmetric and if there is at least one non-isotropic vector, then a continuous function $F:E \to R$ is orthogonally additive only if there are a real number c and a functional $l \in E^*$ such that*

$$F(x) = c(Tx, x) + l(x)$$

for all $x \in E$. If T is as above except that every vector in E is isotropic, then if dim $T \geqq 3$ *every continuous orthogonally additive functional is linear.*

In conclusion it might be remarked that if the quadratic form associated with the linear mapping T is not continuous on E, then $c = 0$ in Theorems 2 and 5.

Some applications of the concept of T-orthogonality to harmonic analysis will be indicated elsewhere.

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