ON CONVERGENCE OF PROJECTIONS IN LOCALLY CONVEX SPACES

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This note is concerned with the extension to locally convex spaces of a theorem of J.Y. Barry [1]. The basic assumptions are as follows. E is a separated locally convex topological vector space, henceforth assumed to be barreled. E' is its strong dual. For any subset A of E, we denote by w(A) the closure of A in the σ-(E, E')-topology. See [2] for further information about locally convex spaces. By a projection we shall mean a continuous linear mapping of E into itself which is idempotent. A net $\{P_\alpha : \alpha \in \Gamma\}$ of projections will be said to be increasing if $\alpha \geq \alpha'$ always implies $P_\alpha P_{\alpha'} = P_{\alpha'} P_\alpha = P_{\alpha'}$. The symbol $\bigvee_\alpha P_\alpha$ is used to denote that projection, if it exists, with the properties i) $(\bigvee_\alpha P_\alpha)(E) = \text{clm}\,\{P_\alpha(E)\}$, the smallest closed linear subspace of E which contains $\bigcup_\alpha P_\alpha(E)$, and ii) $(I - \bigvee_\alpha P_\alpha)(E) = \bigcap_\alpha [(I - P_\alpha)(E)]$.

Following Barry we now give the following definition.

Definition. Given an increasing net $\{P_\alpha\}$ of projections on E, we say that a point y_x of E is a weak x-cluster point of $\{P_\alpha\}$ if $y_x \in \bigcap_\alpha w(\{P_\beta x : \beta \geq \alpha\})$.

THEOREM. Let $\{P_\alpha\}$ be a bounded increasing net of projections on E. Then there is a projection P on E such

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that $P = \bigvee_\alpha P_\alpha$ and $Px = \lim_\alpha P_\alpha x$ for all x in E if and only if $\{P_\alpha\}$ has a weak x-cluster point for every x in E.

Proof:

I) When P exists, it is apparent that $Px = \lim_\alpha P_\alpha x$ is a weak x-cluster point of $\{P_\alpha\}$ for every x.

II) Conversely, letting y_x be a weak x-cluster point of $\{P_\alpha\}$, we first show that

(1) $$P_\alpha x = P_\alpha y_x \qquad \text{for every } \alpha .$$

To this end, let α be fixed. Then for every x' in E' and every $\varepsilon > 0$, let $N = N(y_x, {}^tP_\alpha x', \varepsilon) = \{z \in E' : |<z - y_x, {}^tP_\alpha x' >| < \varepsilon\}$. From the definition of an x-cluster point, it follows that there must be some $\beta \geq \alpha$ such that $P_\beta x \in N$. Hence $\varepsilon > |P_\beta x - y_x, {}^tP_\alpha x' >| = |<P_\alpha P_\beta x - P_\alpha y_x, x' >| = |< P_\alpha(x - y_x), x' >|$. Since ε and x' are arbitrary, (1) follows.

Now since $y_x \in w(\{P_\beta x : \beta \geq \alpha\})$ for every α, it follows from a classic theorem of Banach, (see [3], p. 422, Theorem V. 3. 13 for a convenient formulation) that y_x is in the closure, in the original topology on E, of the convex hull of $\{P_\beta x : \beta \geq \alpha\}$ for every α. Thus for every neighborhood S of zero in E there are finite sets of scalars $\{b_{k,S} : k = 1, 2, \ldots, n_S\}$ and of indices $\{\alpha_{k,S} : k = 1, 2, \ldots, n_S\}$ such that

(2) $$y_x - T_S x \in S ,$$

where $T_S = \sum_{k=1}^{n_s} b_{k,S} P_{\alpha_{k,S}}$. It is easily verified that if

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$\alpha \geq \alpha_{k,S}$ for each $k = 1, 2, \ldots, n_S$, then

(3) $$P_\alpha T_S = T_S .$$

Now let W be any neighborhood of zero in E, and find a neighborhood U of zero in E such that $U + U \subseteq W$. Since $\{P_\alpha\}$ is bounded and E is barreled, $\{P_\alpha\}$ is equicontinuous and there is a neighborhood V of zero such that $P_\alpha(V) \subseteq U$ for every α. Let $\alpha_o \geq \alpha k, U \cap V$ for every $k = 1, 2, \ldots, n_{U \cap V}$. Then, for all $\alpha \geq \alpha_o$,

(4) $y_x - T_{U \cap V} x \in U \cap V$ from (2),

(5) $T_{U \cap V} x - P_\alpha T_{U \cap V} x = 0$ from (3),

(6) $P_\alpha T_{U \cap V} x - P_\alpha y_x = P_\alpha [T_{U \cap V} x - y_x] \in U$ from (4), and

(7) $P_\alpha y_x - P_\alpha x = 0$ from (1).

From these statements it follows that $y_x - P_\alpha x \in U + U \subset W$. Hence $\lim_\alpha P_\alpha x = y_x$. Let $Px = \lim_\alpha P_\alpha x$. Since $\{P_\alpha\}$ is equicontinuous, P is in E'. Also, from 1), $P^2 x = Py_x = \lim_\alpha P_\alpha y_x = \lim_\alpha P_\alpha x = Px$. Hence P is a projection. Also, $PP_\alpha = P_\alpha P = P_\alpha$ for all α.

Finally, if $x \in \bigcap_\alpha (I - P_\alpha)(E)$ then $(I - P_\alpha)x = x$ for all α. Hence $(I - P)x = x - \lim_\alpha P_\alpha x = \lim_\alpha (x - P_\alpha x) = x$. Therefore, $\bigcap_\alpha (I - P_\alpha))(E) \subseteq (I - P)\ (E)$. At the same time, for each α, $(I - P_\alpha)\ (I - P) = I - P - P_\alpha + P_\alpha P = I - P$. Consequently, $(I - P)\ (E) \subseteq (I - P_\alpha)\ (E)$ for every α so that $(I - P)\ (E) \subseteq \bigcap_\alpha (I - P_\alpha)\ (E)$. Likewise, since $Px = \lim_\alpha P_\alpha x$ by definition, we have $P(E) \subseteq \mathrm{clm}\ \{P_\alpha(E)\}$, while from $PP_\alpha = P_\alpha$ we get

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$P_\alpha(E) \subseteq P(E)$ for every α, so that $P(E) = \text{clm}\{P_\alpha(E)\}$. Thus we are justified in using the notation $P = \bigvee_\alpha P_\alpha$.

REFERENCES

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2. N. Bourbaki, Espaces Vectoriels Topologiques, Paris, 1953-1955.

3. N. Dunford and J. Schwartz, Linear Operators, New York, 1958.

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