19

Neutrino masses and mixing

In this chapter we introduce the phenomenology of neutrino masses and mixing, and show how the phenomenology can be made to be consistent with the $SU(2) \times U(1)$ broken gauge symmetry of the Standard Model. We take it that neutrinos and antineutrinos are distinct Dirac fermions, setting aside, until Chapter 21, the suggestions that neutrinos are Majorana fermions.

The phenomenology arose from the observations that the number of electron neutrinos arriving at the Earth from the Sun is only about half of the number expected from our knowledge of the nuclear reactions that occur in the Sun, and the physics of the Sun's interior. These observations are now explained as the result of some electron neutrinos turning into muon neutrinos and tau neutrinos during their transit between their creation in the interior of the Sun and their observation on Earth. These transitions violate the conservation laws of Section 9.3. We will show that they occur because the e, μ and τ neutrinos are not massless but, as conceived by Pontecorvo (1968) they do not have a definite mass, i.e., they are not eigenstates of the mass operator.

19.1 Neutrino masses

The most general Lorentz invariant neutrino mass term that can be introduced into the Lagrangian density of the Standard Model is

$$\mathcal{L}^{\nu}_{\text{mass}}(x) = -\sum_{\alpha,\beta} \nu^{\dagger}_{\alpha\text{L}}(x)\, m_{\alpha\beta}\, \nu_{\beta\text{R}}(x) \; + \text{Hermitian conjugate}, \tag{19.1}$$

where $m_{\alpha\beta}$ is an arbitrary 3×3 complex matrix, α and β run over the three neutrino types e, μ, τ, and $\nu_{\alpha\text{L}}(x)$, $\nu_{\alpha\text{R}}(x)$ are left-handed and right-handed two-component spinor fields. (Spinor indices are omitted here.)

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An arbitrary complex matrix can be put into real diagonal form with the help of two unitary matrices (see Problem A.4). We can write

$$m_{\alpha\beta} = \sum_i U^{L*}_{\alpha i} m_i U^{R}_{\beta i}, \tag{19.2}$$

where m_i are three real and positive masses; $\mathbf{U}^{L}$ and $\mathbf{U}^{R}$ are unitary matrices. It is evident that $U^{L}_{\alpha i}$ and $U^{R}_{\beta i}$ can be replaced by $U^{L}_{\alpha i}e^{-i\delta_i}$ and $U^{R}_{\beta i}e^{-i\delta_i}$, where the δ_i are three arbitrary phases.

If we now define the fields

$$\begin{aligned} \nu_{iL}(x) &= \sum_\alpha U^{L}_{\alpha i}\nu_{\alpha L}(x), \\ \nu_{iR}(x) &= \sum_\alpha U^{R}_{\alpha i}\nu_{\alpha R}(x), \end{aligned} \tag{19.3}$$

the mass term takes the standard Dirac form (5.12)

$$\mathcal{L}^{\nu}_{\mathrm{mass}}(x) = -\sum_i \mathrm{m}_i \left(\nu^{\dagger}_{iL}\nu_{iR} + \nu^{\dagger}_{iR}\nu_{iL}\right). \tag{19.4}$$

It is easy to show that the transformations given by equations (19.3) retain the Dirac form of the dynamical terms:

$$\begin{aligned} \mathcal{L}^{\nu}_{\mathrm{dyn}} &= \sum_\alpha \mathrm{i}\left[\nu^{\dagger}_{\alpha L}\tilde{\sigma}^{\mu}\partial_\mu \nu_{\alpha L} + \nu^{\dagger}_{\alpha R}\sigma^{\mu}\partial_\mu\nu_{\alpha R}\right] \\ &= \sum_i \mathrm{i}\left[\nu^{\dagger}_{iL}\tilde{\sigma}^{\mu}\partial_\mu\nu_{iL} + \nu^{\dagger}_{iR}\sigma^{\mu}\partial_\mu\nu_{Ri}\right]. \end{aligned} \tag{19.5}$$

($\mathcal{L}^{\nu}_{\mathrm{dyn}} + \mathcal{L}^{\nu}_{\mathrm{mass}}$) is the Lagrangian density of free neutrinos of masses m_1, m_2, m_3. Since $\mathbf{U}^{L}$ and $\mathbf{U}^{R}$ are unitary matrices, and a unitary matrix $\mathbf{U}$ satisfies $\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I}$, we can invert equations (19.3) to give

$$\begin{aligned} \nu_{\alpha L}(x) &= \sum_i U^{L*}_{\alpha i}\nu_{iL}(x), \\ \nu_{\alpha R}(x) &= \sum_i U^{R*}_{\alpha i}\nu_{iR}(x). \end{aligned} \tag{19.6}$$

The e, μ and τ neutrinos are mixtures of the neutrinos having definite mass. We shall see that this leads to the phenomenon of neutrino oscillations.

19.2 The weak currents

Neutrinos interact with each other and with other particles through the weak currents. The charged weak current (9.2), expressed in terms of the neutrino mass eigenfields using (19.6), becomes

$$j^{\mu} = \sum_\alpha \alpha^{\dagger}_{L}\tilde{\sigma}^{\mu}\nu_{\alpha L} = \sum_{\alpha,i}\alpha^{\dagger}_{L}\tilde{\sigma}^{\mu}U^{L*}_{\alpha i}\nu_{iL} \tag{19.7}$$

α_L are the charged lepton fields $\alpha = e,\ \mu,\ \tau$.

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The neutral weak current (9.17) keeps the same form: since $\mathbf{U}^{\mathrm{L}}$ is unitary, we have

$$\sum_\alpha (\nu_{\alpha\mathrm{L}})^\dagger \tilde{\sigma}^\mu \nu_{\alpha\mathrm{L}} = \sum_i (\nu_{i\mathrm{L}})^\dagger \tilde{\sigma}^\mu \nu_{i\mathrm{L}}. \tag{19.8}$$

As an example of how these modifications influence the physics discussed in earlier chapters, consider our effective pion interaction (9.1):

$$\mathcal{L}_{\mathrm{int}} = \alpha_\pi \left[j^\mu \partial_\mu \Phi_\pi + j^{\mu\dagger} \partial_\mu \Phi_\pi^\dagger \right].$$

The β decay rate formula (9.3) for $\pi^- \to \mathrm{e}^- + \bar{\nu}_\mathrm{e}$ becomes three decay rates:

$$\frac{1}{\tau\left(\pi^- \to \mathrm{e}^-\bar{\nu}_i\right)} = \frac{\alpha_\pi^2}{4\pi}\left(1 - \frac{v_\mathrm{e}}{c}\right) p_\mathrm{e}^2 E_\mathrm{e} \left|U_{\mathrm{e}i}^{\mathrm{L}}\right|^2, \; i = 1, 2, 3.$$

In the derivation of this result the effects of small neutrino masses have been neglected. Because neutrino masses are small (see Table 1.2), it is not possible with present technology to discern differences in energy between these decay modes. The total decay rate is measured, and since $\sum_i U_{\mathrm{e}i}^{\mathrm{L}} U_{\mathrm{e}i}^{\mathrm{L}*} = 1$ we recover the expression (9.3) for this. A similar conclusion can be drawn about the processes $\pi^- \to \mu^- + \bar{\nu}_\mu$ and $\tau^- \to \pi^- + \nu_\tau$, described in Section 9.2 by the same effective Lagrangian, and about the results on muon decay of Section 9.4.

19.3 Neutrino oscillations

The Lagrangian density (19.1) with (19.5) for a free neutrino yields the equations

$$\begin{aligned} \mathrm{i}\tilde{\sigma}^\mu \partial_\mu \nu_\alpha \mathrm{L} - m_{\alpha\beta} \nu_{\beta\mathrm{R}} &= 0, \\ \mathrm{i}\sigma^\mu \partial_\mu \nu_{\alpha\mathrm{R}} - m_{\beta\alpha}^* \nu_{\beta\mathrm{L}} &= 0. \end{aligned} \tag{19.9}$$

These equations are a generalisation of the Dirac equations (5.11), and in this section we shall interpret their solutions as neutrino wave functions for the three types $\alpha = e,\ \mu,\ \tau$, not as neutrino fields. We shall look for energy eigenfunctions with time dependence $\mathrm{e}^{-\mathrm{i}Et}$.

Zero mass neutrinos would have plane wave solutions of negative helicity (see Section 6.6). For a wave in the z direction

$$\nu_{\alpha\mathrm{L}}(z, t) = \mathrm{e}^{-\mathrm{i}E(t-z)} f_\alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \nu_{\alpha\mathrm{R}} = 0,$$

where the f_α are constants.

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The introduction of neutrino masses modifies these solutions by allowing the f_α to depend on z:

$$\begin{aligned} \nu_{\alpha \mathrm{L}}(z,t) &= \mathrm{e}^{-\mathrm{i}E(t-z)} f_\alpha(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \nu_{\alpha \mathrm{R}}(z,t) &= \mathrm{e}^{-\mathrm{i}E(t-z)} g_\alpha(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{19.10}$$

Substituting in the Dirac equations gives

$$\begin{aligned} &\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z} f_\alpha(z) - m_{\alpha\beta} g_\beta(z) = 0, \\ &\left(2E - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\right) g_\gamma(z) - m^*_{\alpha\gamma} f_\alpha(z) = 0. \end{aligned} \tag{19.11}$$

$$\left(\text{Note that } \tilde{\sigma}^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \sigma^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}.\right)$$

For neutrino energies much greater that their mass we can neglect $-\mathrm{i}\,\mathrm{d}g_\gamma/\mathrm{d}z$ compared with $2Eg_\gamma$ (see Problem 19.1) to obtain

$$g_\gamma(z) = m^*_{\alpha\gamma} f_\alpha(z)/2E, \tag{19.12}$$

and hence by substitution three coupled equations for $f_\alpha(z)$:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z} f_\beta(z) = m_{\beta\gamma} m^*_{\alpha\gamma} f_\alpha(z)/2E.$$

Diagonalising the mass matrices $m_{\beta\gamma}$ and $m_{\alpha\gamma}$ gives

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z} f_\beta(z) = U^{\mathrm{L}*}_{\beta i} U^{\mathrm{L}}_{\alpha i} f_\alpha(z) m_i^2/2E. \tag{19.13}$$

The right-handed U^{R} do not now appear, so that the label L is now redundant and we shall put $U^{\mathrm{L}}_{\alpha i} = U_{\alpha i}$ for the remainder of this section.

To solve these equations we construct linear combinations

$$f_i(z) = U_{\alpha i} f_\alpha(z); \qquad i = 1,\ 2,\ 3. \tag{19.14}$$

which satisfy, using (19.13),

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z} f_i(z) &= \mathrm{i}U_{\alpha i}\frac{\mathrm{d}}{\mathrm{d}z} f_\alpha(z) = U_{\alpha i} U^*_{\alpha j} U_{\beta j} m_j^2 f_\beta(z)/2E \\ &= \delta_{ij} U_{\beta j} m_j^2 f_\beta(z)/2E = \left(m_i^2/2E\right) f_i(z). \end{aligned} \tag{19.15}$$

These uncoupled equations have the simple solutions

$$f_i(z) = \mathrm{e}^{-\mathrm{i}\left(m_i^2/2E\right)z} f_i(0).$$

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Inserting the factor $e^{-iE(t-z)}$, the ν_i neutrino wave function is

$$\nu_i(z,t) = e^{-iEt+i\left(E-m_i^2/2E\right)z} f_i(0). \tag{19.16}$$

This state has energy E and momentum $p_i = E - m_i^2/2E$. For $m_i^2 \ll E^2$, $p_i^2 = E^2 - m_i^2$, which is the relativistic relationship for a particle of mass m_i. Thus the neutrino ν_i carries mass m_i. $\nu_i(z,t)$ are the left-handed wavefunctions of (19.3).

Suppose that at $z = 0$ a neutrino of type α is born. The ν_α wavefunction is a linear superposition of mass eigenstates ν_i with $f_i(0) = U_{\alpha i} f_\alpha(0)$. Different mass eigenstates propagate with different phases so that the neutrino type changes with z:

$$f_\beta(z) = U^*_{\beta i} f_i(z) = U^*_{\beta i} e^{-i\left(m_i^2/2E\right)z} U_{\alpha i} f_\alpha(0). \tag{19.17}$$

To be exact a neutrino is born as a wave packet in some localised region of space time around some point $z=0, t=0$. A realistic treatment of its propagation requires the construction of the appropriate wave packet. We take it that the packet travels with almost the speed of light and with little distortion so that having travelled a distance $z = D$ the probability amplitude for finding a neutrino type β will be $e^{-iE(t-D)} f_\beta(D)$.

The probability of a transition $P_D\left(\nu_\alpha \to \nu_\beta\right)$ is

$$P_D(\nu_\alpha \to \nu_\beta) = \left|U^*_{\beta i} e^{-i\left(m_i^2/2E\right)z} U_{\alpha i}\right|^2 = \sum_{ij} U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j} e^{-i\left(\Delta m_{ij}^2 D/2E\right)}. \tag{19.18}$$

$\mathrm{Re}(U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j})$ is symmetric and $\mathrm{Im}\,(U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j})$ antisymmetric under the interchange of i and j, from this and the unitarity of U we can write

$$\begin{aligned} P_D(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} &- 4 \sum_{i>j} \mathrm{Re}\left(U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j}\right) \sin^2\left(\frac{\Delta m_{ij}^2 D}{4E}\right) \\ &+ 2 \sum_{i>j} \mathrm{Im}\left(U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j}\right) \sin\left(\frac{\Delta m_{ij}^2 D}{2E}\right) \end{aligned} \tag{19.19}$$

where $\Delta m_{ij}^2 = m_i^2 - m_j^2$.

These expressions describe the phenomena of *neutrino oscillations.* We note that experiments designed to observe and measure neutrino oscillations (Chapter 20) can only give values for the differences Δm_{ij}^2, and cannot give values for the individual masses m_i. The differences must satisfy the condition

$$\Delta m_{12}^2 + \Delta m_{23}^2 + \Delta m_{31}^2 = 0.$$

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Restoring factors of c and $\hbar$, it will be useful to write

$$\frac{\Delta m_{ij}^2 D}{4E} = \Delta m_{ij}^2 c^4 \left(\frac{D}{\hbar c}\right)\frac{1}{4E} = 1.27\left(\frac{\Delta m_{ij}^2 c^4}{\text{leV}^2}\right)\left(\frac{D}{1\text{ km}}\right)\left(\frac{1\text{ GeV}}{E}\right). \tag{19.20}$$

By considering the equations for the charge conjugate wave functions ν_α^c (see Section 7.4), similar formulae result, but with $U_{\alpha i}$ replaced by its complex conjugate $U_{\alpha i}^*$. If Im $\{U_{\beta i}^* U_{\alpha i} U_{\beta j} U_{\alpha j}^*\}$ is not zero it changes sign for antineutrinos and $P_D(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \neq P_D(\nu_\alpha \to \nu_\beta)$. The lepton sector joins the quark sector in displaying matter–antimatter asymmetry.

19.4 The MSW effect

In many experiments that investigate oscillations the neutrinos are not completely free, but pass through matter on their journey from source to detector. This modifies the free wave functions discussed in the previous sections. In particular, matter contains electrons that interact with neutrinos through the charged weak currents. The effective interaction Lagrangian for this process is given by (9.8):

$$\mathcal{L}_{\text{int}} = -2\sqrt{2} G_{\text{F}} g_{\mu\nu} j^\mu j^{\nu\dagger},$$

where, from (9.2), $j^\mu = e_{\text{L}}^\dagger \tilde{\sigma}^\mu \nu_{\text{eL}}$, $j^{\nu\dagger} = \nu_{\text{eL}}^\dagger \tilde{\sigma}^\mu e_{\text{L}}$, giving

$$\begin{aligned}\mathcal{L}_{\text{int}} &= -2\sqrt{2} G_{\text{F}} g_{\mu\nu} \left(e_{\text{L}}^\dagger \tilde{\sigma}^\mu \nu_{\text{eL}}\right)\left(\nu_{\text{eL}}^\dagger \tilde{\sigma}^\nu e_{\text{L}}\right) \\ &= -2\sqrt{2} G_{\text{F}} g_{\mu\nu} \left(e_{\text{L}}^\dagger \tilde{\sigma}^\mu e_{\text{L}}\right)\left(\nu_{\text{eL}}^\dagger \tilde{\sigma}^\nu \nu_{\text{eL}}\right).\end{aligned} \tag{19.21}$$

The last step uses a Fierz transformation (Appendix A),

For matter at rest, the expectation value of $e_{\text{L}}^\dagger \tilde{\sigma}^{\text{o}} e_{\text{L}} = e_{\text{L}}^\dagger e_{\text{L}} = \frac{1}{2} N_{\text{e}}(x)$ where $N_{\text{e}}(x)$ is the total electron density at x. The factor of $1/2$ stems from the involvement of the left-handed electron field components only. Also, apart possibly from ferromagnetic effects, we can expect that the expectation value of $e_{\text{L}}^\dagger \tilde{\sigma}^i e_{\text{L}} = 0$. The neutrino Lagrangian density acquires an additional term $-\sqrt{2} G_{\text{F}} N_{\text{e}}(x)\, \nu_{e\text{L}}^\dagger \nu_{\text{eL}}$. This results in the modified equations for $f(z)$:

$$\mathrm{i}\frac{\mathrm{d} f_\beta(z)}{\mathrm{d}z} - m_{\beta\gamma} m_{\alpha\gamma}^* f_\alpha(z) \big/ 2E - V(z)\, \delta_{\beta e} f_{\text{e}}(z) = 0,$$

or equivalently (see equation 19.15)

$$\mathrm{i}\frac{\mathrm{d} f_i(z)}{\mathrm{d}z} = \frac{m_i^2}{2E} f_i(z) + V(z)\, \mathrm{U}_{\text{e}i} \mathrm{U}_{\text{e}j}^* f_j(z) \tag{19.22}$$

where $V(z) = \sqrt{2} N_{\text{e}}(z)\, G_{\text{F}}$.

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The influence of matter on the propagation of neutrinos was pointed out by Wolfenstein (1978), and further elaborated by Mikheyev and Smirnov (1986). It is known as the MSW effect.

The neutral weak currents also contribute to the Lagrangian density of all neutrino types and result in an additional common phase factor on the wave functions of all types, which has no influence on neutrino oscillations.

19.5 Neutrino masses and the Standard Model

In the Weinberg–Salam electroweak theory for leptons of Chapter 12 we introduced three left-handed lepton doublet fields:

$$\mathbf{L}_{\mathrm{e}} = \begin{pmatrix} \mathrm{L}_{\mathrm{eA}} \\ \mathrm{L}_{\mathrm{eB}} \end{pmatrix} = \begin{pmatrix} \nu_{\mathrm{eL}} \\ e_{\mathrm{eL}} \end{pmatrix}, \quad \mathbf{L}_{\mu} = \begin{pmatrix} \nu_{\mu\mathrm{L}} \\ \mu_{\mathrm{L}} \end{pmatrix}, \quad \mathbf{L}_{\tau} = \begin{pmatrix} \nu_{\tau\mathrm{L}} \\ \tau_{\mathrm{L}} \end{pmatrix},$$

and three right-handed singlets $e_{\mathrm{R}}, \mu_{\mathrm{R}}, \tau_{\mathrm{R}}$. Under an $SU(2)$ transformation,

$$\mathbf{L}_{\alpha} \to \mathbf{L}'_{\alpha} = \mathbf{U}\mathbf{L}_{\alpha}, \;\; \alpha_{\mathrm{R}} \to \alpha'_{\mathrm{R}} = \alpha_{\mathrm{R}}.$$

Dirac neutrinos having mass implies the existence of right-handed neutrino fields. In the Standard Model the right-handed neutrino fields, like the right-handed fields of the charged leptons, must be $SU(2)$ singlets. Neutrino masses are introduced into the model in the same way as the u, c and t quarks by coupling to the Higgs field. An $SU(2)$ invariant coupling of the Higgs field to neutrinos is then (equation (14.9) and Problem 14.3.)

$$\mathcal{L}^{\nu}_{\mathrm{Higgs}} = -\sum_{\alpha\beta}\left[G^{\nu}_{\alpha\beta}\left(L^{\dagger}_{\alpha}\varepsilon\Phi^{*}\right)\nu_{\beta\mathrm{R}} - G^{\nu *}_{\alpha\beta}\nu^{\dagger}_{\beta\mathrm{R}}\left(\Phi^{\mathrm{T}}\varepsilon L_{\alpha}\right)\right] \tag{19.23}$$

where $G^{\nu}_{\alpha\beta}$ is a complex 3×3 matrix. On symmetry breaking this gives the neutrino mass term

$$\mathcal{L}^{\nu}_{\mathrm{mass}} = -\phi_o \sum_{\alpha,\beta}\left[G^{\nu}_{\alpha\beta}\nu^{\dagger}_{\alpha\mathrm{L}}\nu_{\beta\mathrm{R}} + G^{\nu *}_{\alpha\beta}\nu^{\dagger}_{\beta\mathrm{R}}\nu_{\alpha\mathrm{L}}\right]. \tag{19.24}$$

This is just the mass term of equation (19.1) if we identify $\phi_o G^{\nu}_{\alpha\beta}$ with $m_{\alpha\beta}$.

19.6 Parameterisation of U

We have taken the parameters $m_{\mathrm{e}}, m_{\mu}, m_{\tau}$ and g_2 to be real and positive, but this is in fact a phase convention: any phase on these parameters can be absorbed in phase factors multiplying the lepton fields, and such phase factors are of no physical significance. It is also the case that the definition of the mass matrix $m_{\alpha\beta}$ depends on a phase convention.

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Define the six neutrino fields $\nu'_{\alpha\mathrm{L}}, \nu'_{\alpha\mathrm{R}} (\alpha = \mathrm{e}, \mu, \tau)$ and the six charged lepton fields $\alpha'_{\mathrm{L}}, \alpha'_{\mathrm{R}}$ by

$$\nu_{\alpha\mathrm{L}} = \mathrm{e}^{\mathrm{i}\theta_\alpha}\nu'_{\alpha\mathrm{L}}, \quad \nu_{\alpha\mathrm{R}} = \mathrm{e}^{\mathrm{i}\gamma_\alpha}\nu'_{\alpha\mathrm{R}}, \quad \begin{pmatrix}\alpha_{\mathrm{L}}\\ \alpha_{\mathrm{R}}\end{pmatrix} = \mathrm{e}^{\mathrm{i}\theta_\alpha}\begin{pmatrix}\alpha'_{\mathrm{L}}\\ \alpha'_{\mathrm{R}}\end{pmatrix}.$$

The leptonic part of the electroweak Lagrangian density described in Chapter 12 (equation (12.12)), and the charged current (equation (12.16)) and neutral current (equation (12.23)) that give the neutrino coupling to the $\mathrm{W}^{\pm}$ and Z fields, are unchanged in form under these transformations. The neutrino mass matrix retains the same form but with $m_{\alpha\beta}$ replaced by

$$m'_{\alpha\beta} = \mathrm{e}^{-\mathrm{i}\theta_\alpha + \mathrm{i}\gamma_\beta} m_{\alpha\beta}.$$

We can redefine $m_{\alpha\beta}$ in this way, keeping the physical content of the theory unchanged.

The unitary matrix $\mathbf{U}^{\mathrm{L}}$ was defined by $m_{\alpha\beta} = \sum_i U^{\mathrm{L}*}_{\alpha\mathrm{i}} m_{\mathrm{i}} U^{\mathrm{R}}_{\beta\mathrm{i}}$. Hence we can redefine $U^{\mathrm{L}'}_{\alpha i} = \mathrm{e}^{\mathrm{i}(\theta_\alpha - \delta_i)} U^{\mathrm{L}}_{\alpha i}$, where the phase factors $\mathrm{e}^{\mathrm{i}\delta_i}$ were introduced in Section 19.1. As in our discussion of the KM matrix in Section 14.2, when the non-physical phase factors have been taken out, the resulting matrix depends on four physical parameters. We parameterise it in the same way as the KM matrix but replace θ_{1j} by $\theta_{\mathrm{e}j}$, θ_{2j} by $\theta_{\mu j}$ and θ_{3j} by $\theta_{\tau j}$, etc. It can be called the neutrino mass mixing matrix.

The term exhibiting matter–antimatter asymmetry in $P_D(\nu_\alpha \to \nu_\beta)$ is (see Problem 19.2)

$$2\sum_{i>j} \mathrm{Im}\left(U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j}\right) \sin\frac{\Delta m^2_{ij} D}{2E}$$
$$= \begin{cases} 0 \ \ \text{if}\, \alpha = \beta \\ \pm 8J \sin\left(\dfrac{\Delta m^2_{21} D}{4E}\right) \sin\left(\dfrac{\Delta m^2_{32} D}{4E}\right) \sin\left(\dfrac{\Delta m^2_{31} D}{4E}\right), \text{ otherwise}\end{cases}$$

where $J = c_{\mathrm{e}2} c^2_{\mathrm{e}3} c_{\mu 3} s_{\mathrm{e}2} s_{\mathrm{e}3} s_{\mu 3} \sin\delta$, cf. (14.18, 14.19), the minus sign is taken for transitions $\mathrm{e} \to \mu$, $\mu \to \tau$, $\tau \to \mathrm{e}$, and the plus sign otherwise.

19.7 Lepton number conservation

Having defined the phase conventions that fix the parameters of the neutrino mixing matrix, the Lagrangian density has only one remaining global $U(1)$ symmetry. It is unchanged if all lepton fields, charged and neutral, left-handed and right-handed, are multiplied by the same phase factor $\mathrm{e}^{\mathrm{i}\delta}$. Following the method of Section 7.1, we consider an arbitrary small space- and time-dependent variation in δ, and conclude

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that we have one conserved current:

$$j^\mu(x) = \sum_\alpha \Big[\alpha_L^\dagger(x)\tilde{\sigma}^\mu\alpha_L(x) + \alpha_R^\dagger(x)\sigma^\mu\alpha_R(x) + \nu_{\alpha L}^\dagger(x)\tilde{\sigma}^\mu\nu_{\alpha L}(x) + \nu_{\alpha R}^\dagger(x)\sigma^\mu\nu_{\alpha R}(x)\Big]. \tag{19.25}$$

The quantity $\int j^o(x)\, d^3\mathbf{x}$ counts the number of leptons minus the number of antileptons, and this number is conserved.

19.8 Sterile neutrinos

We will see in the next chapter that there is some experimental indication that there are more than three neutrino mass eigenstates. If these indications are confirmed then we will be obliged to introduce a fourth neutrino type (perhaps more), say ν_w. Since there is no indication of another charged lepton to partner ν_{wL} in an $SU(2)$ doublet, and since the decays of the Z (Section 13.6) confirm that only three neutrino types participate in the weak interaction, both ν_{wL} and ν_{wR} must be $SU(2)$ singlets and have no electroweak interactions except through the mass eigenstate. Such a neutrino is known as a sterile neutrino.

Problems

19.1 Neglect the term $i(dg_\gamma/dz)$ in (19.11) and show that $g_\gamma(z) = m^*_{\alpha\gamma} {}^{f_\alpha(z)}\!/\!_{2E}$. Show that an estimate of $i(dg_\gamma/dz)$ is then $i\,dg_\gamma(z)/dz = S_{\gamma\beta}(2Eg_\beta(z))$ with $S_{\gamma\beta} = m^*_{\alpha\gamma}m_{\alpha\beta}/4E^2$, very small for E much greater than the masses.

19.2 Define $F_{\beta\alpha ij} = \mathrm{Im}\,(U^*_{\beta i}U_{\alpha i}U_{\beta j}U^*_{\alpha j})$

(a) Show that $F_{\beta\alpha ij} = -F_{\beta\alpha ji}$ and that $\sum_i F_{\beta\alpha ij} = 0$, and hence that $F_{\beta\alpha 12} = F_{\beta\alpha 23} = F_{\beta\alpha 31}$. Define $J = F_{\mu e12}$ (this conforms with (14.18) and (19.25)). Using the trigonometric identity $\sin(x) + \sin(y) - \sin(x+y) = 4\sin(x/2)\sin(y/2)\sin((x+y)/2)$.

(b) verify the matter–antimatter asymmetry term in (19.25) for $P_D(\nu_\alpha \to \nu_\beta)$.

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