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THE RELATIVE SCHOENFLIES THEOREM

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The relative Schoenflies theorem says that a locally flat embedding $e : S^{n-1} \to \mathbb{R}^n$ for which $e^{-1}(\mathbb{R}^k) = S^{k-1}$ extends to a homeomorphism of the pair $(\mathbb{R}^n, \mathbb{R}^k)$ provided the local collars respect $\mathbb{R}^k$. In this note it is shown that the proviso is essential, at least when $k = 3$.

Let $\mathbb{R}^n$ denote euclidean n-space and embed $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$ by adjoining the nth coordinate 0. Denote by S^{n-1} the unit sphere in $\mathbb{R}^n$ and by B^n the unit ball in $\mathbb{R}^n$. Thus for $k \le n$, $S^{k-1} \subset B^k \subset \mathbb{R}^n$.

In [1] and [3] it is observed that a collared embedding of S^{n-1} in $\mathbb{R}^n$ which respects $\mathbb{R}^k$ extends to a homeomorphism of the pair $(\mathbb{R}^n, \mathbb{R}^k)$. More precisely, the following relative version of the Schoenflies theorem holds.

THEOREM. *Let* $e : N \to \mathbb{R}^n$ *be an embedding, where* N *is a neighbourhood of* S^{n-1} *in* $\mathbb{R}^n$, *for which* $e(N \cap \mathbb{R}^k) = e(N) \cap \mathbb{R}^k$. *Then* $e|S^{n-1}$ *extends to a homeomorphism of the pair* $(\mathbb{R}^n, \mathbb{R}^k)$.

By taking care in the proof of the collaring theorem in [2], we can improve this result to the following.

THEOREM. *Let* $e : S^{n-1} \to \mathbb{R}^n$ *be an embedding so that*

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$e^{-1}(\mathbb{R}^k) = S^{k-1}$. *Suppose that there is an open cover* $\mathcal{V}$ *of* S^{n-1} *in* $\mathbb{R}^n$ *so that for each* $V \in \mathcal{V}$, e *extends to an embedding* $e_V : V \to \mathbb{R}^n$ *with* $e_V^{-1}(\mathbb{R}^k) = V \cap \mathbb{R}^k$. *Then* e *extends to a homeomorphism of the pair* $(\mathbb{R}^n, \mathbb{R}^k)$.

One might ask whether this result can be taken further. For example, does a locally flat embedding $e : S^{n-1} \to \mathbb{R}^n$ for which $e^{-1}(\mathbb{R}^k) = S^{k-1}$ necessarily extend to a homeomorphism of the pair $(\mathbb{R}^n, \mathbb{R}^k)$? The purpose of this note is to show that the answer is no, at least when $k = 3$.

EXAMPLE. Let $h : B^3 \to \mathbb{R}^3$ be an embedding so that $h(S^2)$ is the Fox-Artin sphere, for example the embedding illustrated on page 68 of [4]. Define $e : S^3 \to \mathbb{R}^4$ by

$$e(w, x, y, z) = (h(w, x, y), z) \quad \text{for} \quad (w, x, y, z) \in S^3 .$$

Now e is an embedding satisfying $e^{-1}(\mathbb{R}^3) = S^2$. Clearly e is locally flat except possibly at the point where $h|S^2$ is not locally flat. By Cantrell's almost locally flat theorem, page 100 of [4], e is actually locally flat. However, e cannot extend to a homeomorphism of the pair $(\mathbb{R}^4, \mathbb{R}^3)$ since this would imply the flatness of the Fox-Artin sphere.

Repeating the procedure for constructing e enables us to construct counterexamples for $k = 3$ and any $n \geq 4$.

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