ON PARTITIONS OF N POINTS

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In a paper (**1**) by Harding there is a tacit invitation to seek the connection between the following two problems:

(i) Find the number, $\eta_k(N)$, of regions into which a k-dimensional space is partitioned by a set of N $(k-1)$-dimensional hyperplanes.

(ii) Find the number, $\nu_k(N)$, of distinct partitions of a given set of N points in a k-dimensional space E that can be induced by $(k-1)$-dimensional hyperplanes.

Schläfli (**2**) solved the first problem and Harding (**1**) solved the second. I wish to show that the first problem can be expressed as a dual of the second and thus provide an alternative derivation of Harding's result.

First, form the dual $\bar{E}$, of E in the following way. Let the dual of a point u in E be the half-space

$$\bar{U} = \{r \mid u \,.\, r \leqq 1\}$$

of $\bar{E}$; in particular let the dual of the origin of E be all of $\bar{E}$. Let the dual of the half-space

$$V = \{r \mid \bar{v} \,.\, r \leqq 1\}$$

of E be the point $\bar{v}$ of $\bar{E}$. Note that

$$u \in V \Rightarrow \bar{v} \,.\, u \leqq 1 \Rightarrow \bar{v} \in \bar{U},$$

so incidence properties of points and half-spaces are preserved in the transformation.

Now consider N points in E and place the origin at one of the points. The dual of the N points will be $(N-1)$ half-spaces and the whole space of $\bar{E}$, dividing $\bar{E}$ into $\eta_k(N-1)$ regions, each expressible as an intersection $\bar{I}$ of some of the half-spaces and the complements of the remaining half-spaces.

Let a $(k-1)$-dimensional hyperplane in E separate the N points into two parts—a set P containing the origin and another set P'. Let $\mathscr{P}$ denote the family of all half-spaces that contain P and are disjoint with P'. The dual of $\mathscr{P}$ is a set $\bar{\mathscr{P}}$ of points $\bar{E}$; and each point of $\bar{\mathscr{P}}$ will belong to the intersection $\bar{I}_P$ of the half-spaces which are duals of points of P and the complements of the half-spaces which are duals of points of P'. Conversely any point of $\bar{I}_P$ will be the dual of a half-space containing P but disjoint with P' and so must belong to $\bar{\mathscr{P}}$. It follows that $\bar{I}_P$ is identical with $\bar{\mathscr{P}}$.

We have therefore established a one-to-one correspondence between the partitions of the N points and the regions defined by $(N-1)$ half-spaces in $\bar{E}$. It follows that

$$\nu_k(N) = \eta_k(N-1).$$

REFERENCES

(1) E. F. Harding, The number of partitions of a set of N points in k dimensions by hyperplanes, *Proc. Edinburgh Math. Soc.* (2) **15** (1967), 285-289.

(2) L. Schläfli, Theorie der vielfachen Kontinuität (Berne (1852); *Ges. Math. Abh.* I (Basel, 1950), p. 209).

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