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Asymptotic estimates for the number of integer solutions to decomposable form inequalities

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Abstract

For homogeneous decomposable forms $F(\mathbf{X})$ in n variables with integer coefficients, we consider the number of integer solutions $\mathbf{x} \in \mathbb{Z}^n$ to the inequality $|F(\mathbf{x})| \leqslant m$ as $m \to \infty$. We give asymptotic estimates which improve on those given previously by the author in Ann. of Math. (2) **153** (2001), 767–804. Here our error terms display desirable behaviour as a function of the height whenever the degree of the form and the number of variables are relatively prime.

Introduction

In this paper we consider homogeneous polynomials $F(\mathbf{X})$ in $n > 1$ variables with integer coefficients which factor completely into a product of linear terms over $\mathbb{C}$. Such polynomials are called *decomposable forms*. We are concerned here with the integer solutions to the Diophantine inequality

$$|F(\mathbf{x})| \leqslant m. \tag{1}$$

Let $V(F)$ denote the n-dimensional volume of the set of all real solutions $\mathbf{x} \in \mathbb{R}^n$ to the inequality $|F(\mathbf{x})| \leqslant 1$, so that by homogeneity $m^{n/d}V(F)$ is the measure of the set of $\mathbf{x} \in \mathbb{R}^n$ which satisfy (1). Denote the number of integral solutions to (1) by $N_F(m)$.

In a previous paper [Thu01] we answered several open questions regarding (1). For example, $N_F(m)$ is finite for all m if and only if F is of *finite type*: $V(F)$ is finite, and the same is true for F restricted to any non-trivial subspace defined over $\mathbb{Q}$. Also proven in [Thu01] was the following asymptotic estimate.

Theorem [Thu01, Theorem 3]. *Let F be a decomposable form of degree d in n variables with integer coefficients. If F is of finite type, then there are $a(F), c(F) \in \mathbb{Q}$ satisfying*

$$1 \leqslant a(F) \leqslant \frac{d}{n} - \frac{1}{n(n-1)}$$

and

$$\frac{(d-n)}{d} \leqslant c(F) < \binom{d}{n}(d-n+1)$$

such that

$$|N_F(m) - m^{n/d}V(F)| << m^{(n-1)/[d-a(F)]}(1+\log m)^{n-2}\mathcal{H}(F)^{c(F)},$$

where the implicit constant depends only on n and d. In particular,

$$|N_F(m) - m^{n/d}V(F)| << m^{[n(n-1)^2]/[d(n-1)^2+1]}(1+\log m)^{n-2}\mathcal{H}(F)^{\binom{d}{n}(d-n+1)}.$$

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The quantity $\mathcal{H}(F)$ appearing here is defined as follows. Write $F(\mathbf{X}) = \prod_{i=1}^{d} L_i(\mathbf{X})$ where the $L_i(\mathbf{X}) \in \mathbb{C}[\mathbf{X}]$ are linear forms in n variables. Denote the coefficient vector of $L_i(\mathbf{X})$ by $\mathbf{L}_i$ and let $\|\cdot\|$ denote the L^2 norm. Then

$$\mathcal{H}(F) = \prod_{i=1}^{d} \|\mathbf{L}_i\|.$$

It is useful to note how the quantities $N_F(m)$, $V(F)$ and $\mathcal{H}(F)$ vary with the form F. In this regard, an important concept is the notion of *equivalent forms*. Two forms $F, G \in \mathbb{Z}[\mathbf{X}]$ are said to be equivalent if $F = G \circ T$ for some $T \in \mathrm{GL}_n(\mathbb{Z})$. This is useful since the quantities $N_F(m)$ and $V(F)$ are clearly unchanged when F is replaced by an equivalent form. On the other hand, the height $\mathcal{H}(F)$ is certainly not such a quantity. With this in mind, we define

$$\mathcal{M}(F) = \inf_{T \in \mathrm{GL}_n(\mathbb{Z})} \{\mathcal{H}(F \circ T)\}.$$

One may then replace the $\mathcal{H}(F)$ occurring in the theorem above with $\mathcal{M}(F)$. In a subsequent paper [Thu03] we showed how the main term in the estimate above, $m^{n/d}V(F)$, is dependent on $\mathcal{M}(F)$.

Theorem [Thu03, Theorem 2]. *Let $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ be a decomposable form of degree d in n variables which does not vanish on $\mathbb{Z}^n \setminus \{\mathbf{0}\}$. Suppose $V(F)$ is finite. Then*

$$\mathcal{M}(F)^{-n/d} << V(F) << \mathcal{M}(F)^{-1/d}(1 + (\log \mathcal{M}(F))^{n-1}),$$

where the implicit constants depend only on n and d.

Note that any form F of finite type satisfies the hypotheses of this theorem. This result points out a weakness in the asymptotic estimate above. To wit, $N_F(m)$ is estimated by a quantity $m^{n/d}V(F)$ which decreases as $\mathcal{M}(F)$ increases, exactly opposite the behaviour of the error term of the estimate.

Ideally, one would like an asymptotic estimate for the number of solutions $N_F(m)$ which could be usefully applied uniformly for all forms F of finite type, i.e. where the error term is always dominated by the estimate $m^{n/d}V(F)$. Unfortunately, such cannot be the case. For example, suppose F is a binary form of the kind $F(X, Y) = X^d + \cdots$. Then $N_F(m) \geqslant 2[m^{1/d}]$, where $[\ \cdot\ ]$ is the greatest integer function. But $\mathcal{M}(F)$ cannot be bounded above (and whence $V(F)$ cannot be bounded below) merely because the leading coefficient is 1. In general, simply knowing $\mathcal{M}(F)$ is large does not rule out the possibility of a great many (roughly $m^{(n-1)/d}$) integer solutions to (1) lying in an $(n-1)$-dimensional subspace.

Our goal here is to improve the dependence on F in the error term. Specifically, we aim to derive an error term which, in so much as possible, decreases as the 'height' of F increases. To accomplish this, we introduce the following more geometric 'height', one which has no arithmetic encumbrances and which is closely connected to the volume $V(F)$. Define

$$\mathfrak{m}(F) = \inf\{\mathcal{H}(F \circ T)\},$$

where the infimum is over all $T \in \mathrm{GL}_n(\mathbb{R})$ with $|\det(T)| = 1$. In [Thu01] the quantity $a(F)$ plays an important role. Here we use a quantity $a'(F) \geqslant a(F)$ which will play an analogous role. Like $a(F)$, the precise definition of $a'(F)$ is somewhat complicated (we give the definition after Lemma 4 below). For the present, we simply note that it satisfies the same inequalities as $a(F)$: $1 \leqslant a'(F) \leqslant d/n$, and if it is smaller than d/n, then

$$a'(F) \leqslant \frac{d}{n} - \frac{1}{n(n-1)}.$$

Further, $a'(F) < d/n$ if n and d are relatively prime and $V(F)$ is finite.

By [Thu01, Proposition], $V(F)$ is finite if $a(F) < d/n$. It turns out that $V(F)$ is controlled by the height $\mathfrak{m}(F)$ if $a'(F) < d/n$. Further, we can estimate $N_F(m)$ more precisely when $a'(F) < d/n$.

Theorem 1. *Let $F(\mathbf{X}) \in \mathbb{R}[\mathbf{X}]$ be a decomposable form of degree d in n variables and suppose $V(F)$ is finite. Then $\mathfrak{m}(F)$ is an attained positive minimum and $V(F) \geqslant (2/n)^n \mathfrak{m}(F)^{-n/d}$. If $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ and F does not vanish at any non-trivial rational point, then $\mathfrak{m}(F) \geqslant n^{-d(n+\frac{1}{2})/n}$. If $a'(F) < d/n$ (in particular, if n and d are relatively prime), then $V(F) << \mathfrak{m}(F)^{-n/d}$, where the implicit constant depends only on n and d.*

By Theorem 1, one does not expect as many integer solutions to (1) when $\mathfrak{m}(F)$ is large in terms of m. Specifically, if $a'(F) < d/n$ and $\mathfrak{m}(F) \geqslant m^{1/n}$, then $m^{n/d}V(F) << m^{(n-1)/d}$. Yet it is possible to have N solutions in a proper subspace with $N >> m^{(n-1)/d}$, hence one can only expect a useful asymptotic estimate when $m^{n/d}V(F)$ is larger than $m^{(n-1)/d}$; in particular, when $\mathfrak{m}(F) \geqslant m^{1/n}$ (if $a'(F) < d/n$). This explains the hypotheses for our asymptotic estimate of $N_F(m)$ below.

Theorem 2. *Let $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ be a decomposable form of finite type of degree d in n variables and suppose $\mathfrak{m}(F) \leqslant m^{1/n}$. If $a'(F) < d/n$ (in particular, if n and d are relatively prime) then*

$$1 \leqslant a'(F) \leqslant \frac{d}{n} - \frac{1}{n(n-1)}$$

and

$$|N_F(m) - m^{n/d}V(F)| << \left(\frac{m}{\mathfrak{m}(F)^{na'(F)/d}}\right)^{(n-1)/[d-a'(F)]}(1+\log m)^{n-2}.$$

The implicit constant here depends only on n and d. In particular,

$$|N_F(m) - m^{n/d}V(F)| << \left(\frac{m}{\mathfrak{m}(F)^{1-1/[d(n-1)]}}\right)^{[n(n-1)^2]/[d(n-1)^2+1]}(1+\log m)^{n-2}.$$

We note that the main term is (almost) larger than the error term in Theorem 2 when $\mathfrak{m}(F) \leqslant m^{1/n}$ (the 'almost' being due to the logarithmic term). We can improve our estimate for $N_F(m)$ when $\mathfrak{m}(F)$ is close to or larger than $m^{1/n}$ by abandoning our goal of an asymptotic one and instead striving for a simple upper bound.

Theorem 3. *Let $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ be a decomposable form of degree d in n variables of finite type. If $a'(F) < d/n$ (in particular, if n and d are relatively prime), then*

$$N_F(m) << \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} + m^{(n-1)/d},$$

where the implicit constant depends only on n and d.

By Theorem 3, $N_F(m) << m^{(n-1)/d}$ if $\mathfrak{m}(F) \geqslant m^{1/n}$. When $a'(F) < d/n$, this improves on [Thu03, Theorem 4], and (up to the implicit constants) [Győ01, Theorem 2] in the case $n = 2$; it is the best one can say in general.

Combining Theorems 1–3 gives the following asymptotic estimate.

Corollary. *Let $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ be a decomposable form of degree d in n variables of finite type. If n and d are relatively prime, then*

$$|N_F(m) - m^{n/d}V(F)| << m^{n/[d+1/(n-1)^2]}(1+\log m)^{n-2},$$

where the implicit constant depends only on n and d.

Note that we have $N_F(m) << m^{n/d}V(F) + m^{(n-1)/d}$ for $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ of finite type, provided that $a'(F) < d/n$. We would like to remove the hypothesis on $a'(F)$ here. Unfortunately, the volume $V(F)$ is not controlled solely by $\mathfrak{m}(F)$ in general; see the examples in § 5 below. However, a simple modification of the proof of [Thu03, Theorem 1] yields the following theorem.

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Theorem 4. *Let $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ be a decomposable form of degree d in n variables. If $V(F)$ is finite and F does not vanish at any non-trivial rational point, then*

$$V(F) << \mathfrak{m}(F)^{-n/d}(1+|\log \mathfrak{m}(F)|)^{n-1},$$

where the implicit constant depends only on n and d.

In view of Theorem 1, this result would only be useful in the case $a'(F) = d/n$.

Theorem 5. *Let $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ be a decomposable form of degree d in n variables of finite type. Suppose $a'(F) = d/n$ and $\mathfrak{m}(F) \geqslant 1$. If*

$$\mathfrak{m}(F)^{-n/d}(1+\log \mathfrak{m}(F))^{n-1} \leqslant m^{-1/d},$$

then

$$N_F(m) << m^{(n-1)/d}.$$

The implicit constant here depends only on n and d.

Theorem 5 sharpens [Thu03, Theorem 4] and [Győ01, Theorem 2] (for the $n = 2$ case). In these results, the hypothesis was $\mathcal{M}(F)^{1-\varepsilon} \geqslant m^n$ for some positive ε and the conclusion was that the solutions are contained in N proper subspaces where $N << \varepsilon^{1-n}$. We shall show that $\mathcal{M}(F) << \mathfrak{m}(F)^n$ (see Lemma 7); thus Theorem 5 represents a true improvement on these results. A reasonable conjecture, in view of our results here, is that $N_F(m) << m^{(n-1)/d}$ whenever $V(F) << m^{-1/d}$ and F is of finite type.

It is possible to determine bounds explicitly for the implicit constants in the above results. Frankly, they would not be very 'good', as our proofs ultimately rely on quantitative versions of the subspace theorem. We have attempted to keep some track of constants depending on n and d in the following two sections and, to the extent where relatively painless, in the proofs of our theorems. In general, very little effort has been expended trying to get good bounds for these constants. For the remainder of this paper, all implicit constants depend only on n and d.

1. Preparatory lemmas

Throughout this section, $F(\mathbf{X}) \in \mathbb{R}[\mathbf{X}]$ is assumed to be a decomposable form of degree d in n variables. Also, all vectors are assumed to be row vectors.

Lemma 1. *Let $a > 0$ and $T \in \mathrm{GL}_n(\mathbb{R})$. Then*

$$\mathcal{H}(aF) = a\mathcal{H}(F), \quad \mathfrak{m}(aF) = a\mathfrak{m}(F), \quad \text{and} \quad \mathfrak{m}(F \circ T) = |\det(T)|^{d/n}\mathfrak{m}(F).$$

Further, if $V(F)$ is finite, then

$$V(aF) = a^{-n/d}V(F) \quad \text{and} \quad V(F \circ T) = |\det(T)|^{-1}V(F).$$

Proof. The first two equations are clear from the definitions. As for the third, write $T = DS$, where D is the diagonal matrix with entries $a = |\det(T)|^{1/n}$ and $S \in \mathrm{GL}_n(\mathbb{R})$ with $|\det(S)| = 1$. Then $F \circ D = a^dF$ and $\mathfrak{m}(F \circ T) = \mathfrak{m}(a^dF \circ S) = \mathfrak{m}(a^dF) = a^d\mathfrak{m}(F)$. For the last equation, let $\mathcal{S}$ be the set of all $\mathbf{x} \in \mathbb{R}^n$ with $|F(\mathbf{x})| \leqslant 1$, so that $V(F)$ is the volume of $\mathcal{S}$. Then $V(F \circ T)$ is the volume of $T^{-1}(\mathcal{S})$, which is $|\det(T)|^{-1}V(F)$. Finally, the fourth equation can be viewed as a special case of the last (write T as we did above), or as a simple consequence of the homogeneity of F. □

Lemma 2. *Suppose $V(F)$ is finite. Then $\mathfrak{m}(F)$ is an attained positive minimum and $V(F) \geqslant (2/n)^n\mathfrak{m}(F)^{-n/d}$.*

Proof. For a $T \in \mathrm{GL}_n(\mathbb{R})$ with $|\det T| = 1$, let $P(T)$ be the parallelepiped defined by

$$P(T) = \{a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n : |a_j| \leqslant 1 \text{ for all } 1 \leqslant j \leqslant n\},$$

where $\mathbf{x}_1^{\mathrm{tr}}, \ldots, \mathbf{x}_n^{\mathrm{tr}}$ are the columns of T. Note that the volume of $P(T)$ is 2^n and that

$$\prod_{i=1}^{d} \max_{1\leqslant j\leqslant n} \{|L_i(\mathbf{x}_j)|\} \leqslant \mathcal{H}(F \circ T).$$

In particular, $|F(\mathbf{x})| \leqslant n^d\mathcal{H}(F \circ T)$ for all $\mathbf{x}$ in $P(T)$.

Let $\mathcal{C}$ be the set of all $T \in \mathrm{GL}_n(\mathbb{R})$ with $|\det(T)| = 1$ and $\mathcal{H}(F \circ T) \leqslant 2\mathfrak{m}(F)$. Suppose $\mathcal{C}$ is unbounded (viewed as a subset of $\mathbb{R}^{n^2}$ in the usual way). Then for some $1 \leqslant j_0 \leqslant n$ there is an infinite sequence $T_1, \ldots \in \mathcal{C}$ where, letting $\mathbf{x}_{i,1}^{\mathrm{tr}}, \ldots, \mathbf{x}_{i,n}^{\mathrm{tr}}$ denote the columns of T_i, we have $\|\mathbf{x}_{i+1,j_0}\| \geqslant 2\|\mathbf{x}_{i,j_0}\|$ for all $i \geqslant 1$. But this implies the existence of an infinite sequence of parallelepipeds $P(T_1), \ldots$, all of which are contained in the set

$$\{\mathbf{x} \in \mathbb{R}^n \; |F(\mathbf{x})| \leqslant n^d 2\mathfrak{m}(F)\},$$

and also satisfying

$$\mathrm{Vol}\biggl(P(T_{i+1}) \Big\backslash \bigcup_{l\leqslant i} P(T_l)\biggr) \geqslant 2^{-1}\,\mathrm{Vol}(P(T_{i+1})) = 2^{n-1}.$$

This contradicts the hypothesis that $V(F)$ is finite, thus $\mathcal{C}$ is bounded.

The map $T \mapsto \mathcal{H}(F \circ T)$ is clearly continuous. Since $\mathcal{C}$ is bounded (and certainly closed), $\mathfrak{m}(F)$ is an attained minimum. Let $\mathfrak{m}(F) = \mathcal{H}(F \circ T)$ for some $T \in \mathcal{C}$. Then $|F(\mathbf{x})| \leqslant n^d\mathfrak{m}(F)$ for all $\mathbf{x}$ in $P(T)$, which implies that $V(n^{-d}\mathfrak{m}(F)^{-1}F) \geqslant 2^n$ and $V(F) \geqslant (2/n)^n\mathfrak{m}(F)^{-n/d}$ by Lemma 1. □

Lemma 3. *Suppose $V(F)$ is finite. Write $F(\mathbf{X}) = \prod_{i=1}^d L_i(\mathbf{X})$ where the $L_i(\mathbf{X})$ are real linear forms for $i \leqslant r$ and complex for $i = r+1, \ldots, d = r+2s$, with $\mathbf{L}_{i+s} = \overline{\mathbf{L}_i}$ for $i = r+1, \ldots, r+s$. Suppose $a_1, \ldots, a_d$ are positive real numbers whose product is 1, so that $F(\mathbf{X}) = \prod_{i=1}^d a_iL_i(\mathbf{X})$. Then*

$$\sum_{1\leqslant i_1,\ldots,i_n\leqslant d} |\det(a_{i_1}\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots a_{i_n}\mathbf{L}_{i_n}^{\mathrm{tr}})|^2 \geqslant \frac{n!}{n^n} d^n \mathfrak{m}(F)^{2n/d}.$$

Proof. Let σ be the permutation of $\{1, \ldots, d\}$ induced by complex conjugation, i.e.

$$\sigma(i) = \begin{cases} i, & \text{if } i \leqslant r, \\ i+s, & \text{if } r < i \leqslant r+s, \\ i-s, & \text{if } r+s < i \leqslant d. \end{cases}$$

Let b_i be the geometric mean of a_i and $a_{\sigma(i)}$. Then the product $\prod_{i=1}^d b_i = \prod_{i=1}^d a_i$ and for any n-tuple $(i_1, \ldots, i_n)$ we have $(a_{i_1} \cdots a_{i_n})^2 + (a_{\sigma(i_1)} \cdots a_{\sigma(i_n)})^2 \geqslant 2(b_{i_1} \cdots b_{i_n})^2$. Since

$$|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|^2 = |\det(\mathbf{L}_{\sigma(i_1)}^{\mathrm{tr}} \cdots \mathbf{L}_{\sigma(i_n)}^{\mathrm{tr}})|^2,$$

we see that

$$\begin{aligned} 2\sum_{1\leqslant i_1,\ldots,i_n\leqslant d} |\det(a_{i_1}\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots a_{i_n}\mathbf{L}_{i_n}^{\mathrm{tr}})|^2 &= \sum_{1\leqslant i_1,\ldots,i_n\leqslant d} (a_{i_1} \cdots a_{i_n})^2 |\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|^2 \\ &\quad + \sum_{1\leqslant i_1,\ldots,i_n\leqslant d} (a_{\sigma(i_1)} \cdots a_{\sigma(i_n)})^2 |\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|^2 \\ &\geqslant 2\sum_{1\leqslant i_1,\ldots,i_n\leqslant d} |\det(b_{i_1}\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots b_{i_n}\mathbf{L}_{i_n}^{\mathrm{tr}})|^2. \end{aligned}$$

The outcome is that we may replace a_i and $a_{\sigma(i)}$ with b_i, i.e. we may assume $a_i = a_{\sigma(i)}$. But if this is the case, then it suffices to prove the lemma under the assumption that $a_i = 1$ for all i.

Let $\mathbb{E}^d \subset \mathbb{R}^r \oplus \mathbb{C}^{2s}$ be the set of $\mathbf{x} = (x_1, \dots, x_d)$ where $x_{i+s} = \overline{x_i}$ for $r+1 \leqslant i \leqslant r+s$. Then $\mathbb{E}^d$ is d-dimensional Euclidean space via the usual hermitian inner product on $\mathbb{C}^d$. If M is the $d \times n$ matrix with rows $\mathbf{L}_1, \dots, \mathbf{L}_d$, then the columns of M are in $\mathbb{E}^d$. Moreover, the rank of M must be n since $V(F)$ is finite (see [Thu03]). Applying Gram–Schmidt to the matrix M, we see that there is an upper triangular $T \in \mathrm{GL}_n(\mathbb{R})$ such that MT is a matrix with orthonormal columns (in $\mathbb{E}^d$). Denote the rows of MT by $\mathbf{L}'_1, \dots, \mathbf{L}'_d$ and the columns by $\mathbf{m}_1^{\mathrm{tr}}, \dots, \mathbf{m}_n^{\mathrm{tr}}$.

Using the inequality between the arithmetic and geometric means and also Lemma 1, we have

$$\begin{aligned}
\frac{n}{d} = \frac{1}{d}\sum_{j=1}^{n} \|\mathbf{m}_j\|^2 &= \frac{1}{d}\sum_{i=1}^{d} \|\mathbf{L}'_i\|^2 \\
&= \frac{1}{d}\sum_{i=1}^{d} \|\mathbf{L}_i T\|^2 \\
&\geqslant \left(\prod_{i=1}^{d} \|\mathbf{L}_i T\|^2\right)^{1/d} \\
&= \mathcal{H}(F \circ T)^{2/d} \\
&\geqslant \mathfrak{m}(F \circ T)^{2/d} \\
&= \mathfrak{m}(F)^{2/d} |\det T|^{2/n}.
\end{aligned}$$

Hence $|\det T|^2 \leqslant (n/d)^n \mathfrak{m}(F)^{-2n/d}$. On the other hand, since the $\mathbf{m}_j$ are orthonormal,

$$\begin{aligned}
1 = \|\mathbf{m}_1 \wedge \cdots \wedge \mathbf{m}_n\| &= \sum_{1 \leqslant i_1 < \cdots < i_n \leqslant d} |\det((\mathbf{L}'_{i_1})^{\mathrm{tr}} \cdots (\mathbf{L}'_{i_n})^{\mathrm{tr}})|^2 \\
&= \sum_{1 \leqslant i_1 < \cdots < i_n \leqslant d} |\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|^2 |\det T|^2 \\
&\leqslant \sum_{1 \leqslant i_1 < \cdots < i_n \leqslant d} |\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|^2 \left(\frac{n}{d}\right)^n \mathfrak{m}(F)^{-2n/d}.
\end{aligned}$$

This inequality suffices to prove the lemma. □

Let

$$c_1 = \frac{(d/n)^n}{\binom{d}{n}}.$$

Lemma 4. *Suppose $V(F)$ is finite and $\mathcal{H}(F) = \mathfrak{m}(F)$. Let $A < 1$ and $1 \leqslant j < n$ and suppose there is an $S \subset \{1, \dots, d\}$ with cardinality $|S| = [jd/n] + 1$ such that*

$$\frac{\|\mathbf{L}_{i_1} \wedge \cdots \wedge \mathbf{L}_{i_{j+1}}\|}{\|\mathbf{L}_{i_1}\| \cdots \|\mathbf{L}_{i_{j+1}}\|} \leqslant A$$

for all $i_1, \dots, i_{j+1} \in S$. Then there is a factorization $F(\mathbf{X}) = \prod_{i=1}^{d} a_i L_i(\mathbf{X})$ as in Lemma 3 with

$$\sum_{1 \leqslant i_1, \dots, i_n \leqslant d} |\det(a_{i_1}\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots a_{i_n}\mathbf{L}_{i_n}^{\mathrm{tr}})|^2 \leqslant n! \binom{d}{n} A^{\{n([jd/n]+1) - jd\}/((n-j)d)} \mathfrak{m}(F)^{2n/d}.$$

In particular,

$$A \geqslant c_1^{(n-j)d/\{n([jd/n]+1) - jd\}}.$$

Proof. Without loss of generality we may assume $\mathfrak{m}(F)=1$ and $\|\mathbf{L}_i\|=1$ for all $i=1,\dots,d$.

Let $a=A^{\{[jd/n]+1-d\}/((n-j)d)}$, which is greater than 1 since $A<1$. Let $b=a^{(-[jd/n]-1)/(d-[jd/n]-1)}$, so that $b<1<a$ and $a^{[jd/n]+1}b^{d-[jd/n]-1}=1$. We note that $a^jb^{n-j}=a^nA=A^{\{n([jd/n]+1)-jd\}/((n-j)d)}$.

For our factorization, let $a_i=a$ if $i\in S$ and $a_i=b$ otherwise. Given an n-tuple $(i_1,\dots,i_n)$ with l of the indices in S, we have

$$|\det(a_{i_1}\mathbf{L}_{i_1}^{\mathrm{tr}}\cdots a_{i_n}\mathbf{L}_{i_n}^{\mathrm{tr}})|\leqslant a^lb^{n-l}\leqslant a^jb^{n-j}\quad \text{if } l<j+1,$$

and

$$|\det(a_{i_1}\mathbf{L}_{i_1}^{\mathrm{tr}}\cdots a_{i_n}\mathbf{L}_{i_n}^{\mathrm{tr}})|\leqslant a^lb^{n-l}A\leqslant a^nA\quad \text{if } l\geqslant j+1.$$

Lemma 4 follows from this and Lemma 3. □

We can now state with more clarity exactly what the quantity $a'(F)$ is. Suppose $\mathfrak{m}(F)=\mathcal{H}(F)$. For $1\leqslant j\leqslant n-1$, let $s_j(F)$ be the cardinality of the largest subset $S\subset\{1,\dots,d\}$ where

$$\frac{\|\mathbf{L}_{i_1}\wedge\cdots\wedge\mathbf{L}_{i_{j+1}}\|}{\|\mathbf{L}_{i_1}\|\cdots\|\mathbf{L}_{i_{j+1}}\|}<c_1^{(n-j)d/\{n([jd/n]+1)-jd\}}$$

for all $i_1,\dots,i_{j+1}\in S$. Note that we do not demand that the i_j be distinct, so any set with just one element will vacuously satisfy this criterion. By Lemma 4, $s_j(F)\leqslant[jd/n]$. Let $s(F)$ be the maximum of $s_j(F)/j$ over all $1\leqslant j\leqslant n-1$. Then $1\leqslant s(F)\leqslant d/n$. Moreover,

$$s(F)\leqslant\frac{d}{n}-\frac{1}{n(n-1)}$$

if $s(F)<d/n$.

For an arbitrary F, we define

$$a'(F):=\max\{s(F\circ T)\},$$

where the maximum is over all $T\in\mathrm{GL}_n(\mathbb{R})$ with $|\det(T)|=1$ and $\mathfrak{m}(F)=\mathcal{H}(F\circ T)$. Then $1\leqslant a'(F)\leqslant d/n$, and

$$a'(F)\leqslant\frac{d}{n}-\frac{1}{n(n-1)}$$

if $a'(F)<d/n$. Moreover, $a'(F)<d/n$ if $n\nmid jd$ for all $j=1,\dots,d$, i.e. if n and d are relatively prime.

Lemma 5a. *Let $M,N\geqslant1$ and fix $\mathbf{K}_1,\dots,\mathbf{K}_N\in\mathbb{C}^M$. Then for any $\mathbf{L}_1,\dots,\mathbf{L}_{N+1}\in\mathbb{C}^M$ with $\|\mathbf{L}_i\|=1$ for all i, we have*

$$\sum_{j=1}^{N+1}\|\mathbf{K}_1\wedge\cdots\wedge\mathbf{K}_n\wedge\mathbf{L}_j\|^2\geqslant\|\mathbf{K}_1\wedge\cdots\wedge\mathbf{K}_N\|^2\cdot\|\mathbf{L}_1\wedge\cdots\wedge\mathbf{L}_{N+1}\|^2.$$

Proof. This is trivial if the $\mathbf{L}_i$ are linearly dependent, so assume otherwise. After possibly applying a unitary transformation, we may assume that the span of $\mathbf{L}_1,\dots\mathbf{L}_j$ is equal to the span of the first j canonical basis vectors $\mathbf{e}_1,\dots,\mathbf{e}_j$ for $j=1,\dots,N+1$. Let $a_j=\mathbf{L}_j\cdot\mathbf{e}_j$ for $j=1,\dots,N+1$ and write $\mathbf{K}_i=\sum_{l=1}^{M}k_{i,l}\mathbf{e}_l$ for $i=1,\dots,N$.

By [Sch91, ch. I, Lemma 5A], we see that

$$\|\mathbf{K}_1\wedge\cdots\wedge\mathbf{K}_N\wedge\mathbf{L}_j\|^2\geqslant\sum_{\sigma}\left|\det_{\substack{1\leqslant i\leqslant N\\ l\in\sigma}}(k_{i,l})\right|^2\cdot|a_j|^2$$

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for all $j = 1, \dots, N+1$, where the sum is over all N-tuples $\sigma = (i_1, \dots, i_N)$ with $i_1 < \dots < i_N$ and $j \notin \sigma$. On the other hand

$$\|\mathbf{K}_1 \wedge \dots \wedge \mathbf{K}_N\|^2 \cdot \|\mathbf{L}_1 \wedge \dots \wedge \mathbf{L}_{N+1}\|^2 = \sum_{\sigma} \left| \det_{\substack{1 \leqslant i \leqslant N \\ l \in \sigma}} (k_{i,l}) \right|^2 |a_1|^2 \cdots |a_{N+1}|^2,$$

where the sum is over all N-tuples $\sigma = (i_1, \dots, i_N)$ with $i_1 < \dots < i_N$. But for any such σ, there is a $j \in \{1, \dots, N+1\}$ with $j \notin \sigma$. Further, since $\|\mathbf{L}_j\| = 1$ for all j, $|a_j|^2 \geqslant |a_1|^2 \cdots |a_{N+1}|^2$ for all j. This proves the lemma. □

Lemma 5b. *Suppose $V(F)$ is finite and $\mathfrak{m}(F) = \mathcal{H}(F)$. Let $B > 1$, $1 \leqslant j < n$, and fix linearly independent $\mathbf{L}_{i_1}, \dots, \mathbf{L}_{i_j}$. Suppose $S \subset \{1, \dots, d\}$ with cardinality $|S| = [jd/n] + 1$ such that*

$$\frac{\|\mathbf{L}_{i_1} \wedge \dots \wedge \mathbf{L}_{i_j} \wedge \mathbf{L}_l\|}{\|\mathbf{L}_{i_1} \wedge \dots \wedge \mathbf{L}_{i_j}\| \cdot \|\mathbf{L}_l\|} \leqslant B$$

for all $l \in S$. Then

$$\sqrt{j+1}\, B \geqslant c_1^{(n-j)d/\{n([jd/n]+1)-jd\}}.$$

Proof. Without loss of generality, we may assume $\mathfrak{m}(F) = 1$ and $\|\mathbf{L}_i\| = 1$ for all i. Let $l_1, \dots, l_{j+1} \in S$. Then by Lemma 5a and the hypotheses,

$$\|\mathbf{L}_{l_1} \wedge \dots \wedge \mathbf{L}_{l_{j+1}}\|^2 \leqslant \sum_{k=1}^{j+1} \frac{\|\mathbf{L}_{i_1} \wedge \dots \wedge \mathbf{L}_{i_j} \wedge \mathbf{L}_{l_k}\|^2}{\|\mathbf{L}_{i_1} \wedge \dots \wedge \mathbf{L}_{i_j}\|^2} \leqslant (j+1) B^2.$$

Lemma 5b follows from this and Lemma 4. □

We now come to our fundamental inequality. This result will be used as an alternative to [Thu01, Lemma 5]. (It can also be used in place of [Thu01, Lemma 6] in the case when $a'(F) = d/n$.)

Lemma 6. *Suppose $V(F)$ is finite and $\mathfrak{m}(F) = \mathcal{H}(F)$. Then for any $\mathbf{x} \in \mathbb{R}^n$, there are n linearly independent linear factors $L_{i_1}(\mathbf{X}), \dots, L_{i_n}(\mathbf{X})$ of $F(\mathbf{X})$ satisfying*

$$\left(\frac{\prod_{j=1}^n |L_{i_j}(\mathbf{x})|}{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|} \right)^{a'(F)} \leqslant c_2 \frac{|F(\mathbf{x})|}{\|\mathbf{x}\|^{d - na'(F)} \mathfrak{m}(F)},$$

where

$$c_2 = n^{n(d-na'(F))/2} \left((n!)^{1/2} \prod_{j=1}^{n-1} c_1^{(j-n)d/\{n([jd/n]+1)-jd\}} \right)^{d-(n-1)a'(F)}.$$

If $V(F)$ is finite and $T \in \mathrm{GL}_n(\mathbb{R})$ satisfies $|\det T| = 1$ and $\mathfrak{m}(F) = \mathcal{H}(F \circ T)$, then for any $\mathbf{x} \in \mathbb{R}^n$, there are n linearly independent linear factors $L_{i_1}(\mathbf{X}), \dots, L_{i_n}(\mathbf{X})$ of $F(\mathbf{X})$ satisfying

$$\left(\frac{\prod_{j=1}^n |L_{i_j}(\mathbf{x})|}{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|} \right)^{a'(F)} \leqslant c_2 \frac{|F(\mathbf{x})|}{\|T^{-1}(\mathbf{x})\|^{d - na'(F)} \mathfrak{m}(F)}.$$

Proof. We first note how the second part follows directly from the first. Given such a T and $\mathbf{x}$, $F \circ T(T^{-1}(\mathbf{x})) = F(\mathbf{x})$, and similarly for each linear factor $L_i(\mathbf{x})$. Apply the first part of the lemma to $F \circ T$ and $T^{-1}(\mathbf{x})$ to obtain the second part.

As remarked in the proof of Lemma 3, there are n linearly independent factors of F if $V(F)$ is finite. So if $F(\mathbf{x}) = 0$, the lemma is trivially true. Suppose now that $F(\mathbf{x}) \neq 0$. By homogeneity and Lemma 1, we may assume without loss of generality that $\mathcal{H}(F) = 1$, and further that $\|\mathbf{L}_i\| = 1$ for

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all $i = 1, \ldots, d$. For notational convenience, set

$$c_{1,j} = \frac{c_1^{(n-j)d/\{n([jd/n]+1)-jd\}}}{\sqrt{j+1}}$$

for $j = 1, \ldots, n-1$.

Let $|L_{i_1}(\mathbf{x})| = \min_{1 \leqslant i \leqslant d}\{|L_i(\mathbf{x})|\}$ and let $S_1 \subset \{1, \ldots, d\}$ be the subset of indices l such that

$$\|\mathbf{L}_{i_1} \wedge \mathbf{L}_l\| < c_{1,1}.$$

Continue recursively in the following manner: for $j > 1$ let $|L_{i_j}(\mathbf{x})|$ be the minimum of $|L_i(\mathbf{x})|$ over all i not in S_{j-1} and let $S_j \subset \{1, \ldots, d\}$ be the subset of indices l with

$$\frac{\|\mathbf{L}_{i_1} \wedge \cdots \wedge \mathbf{L}_{i_j} \wedge \mathbf{L}_l\|}{\|\mathbf{L}_{i_1} \wedge \cdots \wedge \mathbf{L}_{i_j}\|} < c_{1,j}.$$

By definition, $|S_j| \leqslant ja'(F) \leqslant [jd/n] < d$ for each $j = 1, \ldots, n-1$, allowing us to continue up to a choice for $L_{i_n}(\mathbf{x})$. Note that $|S_n| = d$.

By construction, we have $|L_{i_1}(\mathbf{x})| \leqslant \cdots \leqslant |L_{i_n}(\mathbf{x})|$ and

$$|F(\mathbf{x})| = \prod_{i=1}^{d} |L_i(\mathbf{x})| \geqslant \prod_{j=1}^{n} |L_{i_j}(\mathbf{x})|^{a_j},$$

where $a_1 = |S_1|$ and $a_j = |S_j| - |S_{j-1}|$ for $j > 1$. Note in particular that $a_1 + \cdots + a_j = |S_j| \leqslant ja'(F)$ for all $j < n$. Letting $s = (n-1)a'(F) - |S_{n-1}|$, we have

$$|F(\mathbf{x})| \geqslant |L_{i_1}(\mathbf{x})|^{a_1} \cdots |L_{i_{n-2}}(\mathbf{x})|^{a_{n-2}} \cdot |L_{i_{n-1}}(\mathbf{x})|^{a_{n-1}+s} \cdot |L_{i_n}(\mathbf{x})|^{a_n - s}.$$

Since $a_1 + \cdots + a_{n-2} + a_{n-1} + s = (n-1)a'(F)$ and $a_n - s = d - (n-1)a'(F)$, then [Thu01, Lemma 1] implies that

$$|F(\mathbf{x})| \geqslant \prod_{j=1}^{n} |L_{i_j}(\mathbf{x})|^{a'(F)} \cdot |L_{i_n}(\mathbf{x})|^{d - na'(F)}.$$

By [Thu01, Lemma 4],

$$|L_{i_n}(\mathbf{x})| \geqslant n^{-n/2}\|\mathbf{x}\|\,|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|,$$

so that

$$\left(\frac{\prod_{j=1}^{n} |L_{i_j}(\mathbf{x})|}{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|}\right)^{a'(F)} \leqslant \frac{n^{n(d-na'(F))/2}|F(\mathbf{x})|}{\|\mathbf{x}\|^{d-na'(F)}|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|^{d-(n-1)a'(F)}}.$$

Finally, we have

$$\begin{aligned}\|\mathbf{L}_{i_1} \wedge \cdots \wedge \mathbf{L}_{i_n}\| &\geqslant c_{1,n-1}\|\mathbf{L}_{i_1} \wedge \cdots \wedge \mathbf{L}_{i_{n-1}}\| \\ &\geqslant c_{1,n-1}c_{1,n-2}\|\mathbf{L}_{1_1} \wedge \cdots \wedge \mathbf{L}_{i_{n-2}}\| \\ &\;\;\vdots \\ &\geqslant c_{1,n-1} \cdots c_{1,1}.\end{aligned}$$

The lemma follows. □

Lemma 7. *Suppose $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ and that F does not vanish on $\mathbb{Z}^n \setminus \{\mathbf{0}\}$. Let $\mathcal{H}(F \circ T) = \mathfrak{m}(F)$ with $T \in \mathrm{GL}_n(\mathbb{R})$, $|\det(T)| = 1$. Then there is an $S \in \mathrm{GL}_n(\mathbb{Z})$ with $\mathcal{H}(F \circ S) \leqslant n^{d(n+1/2)}\mathfrak{m}(F)^n$ such that*

$$\mathfrak{m}(F)^{-1/d}n^{-3/2}(n!)^{-2}\|\mathbf{y}\| \leqslant \|T^{-1}S(\mathbf{y})\| \leqslant n^{n+1/2}\mathfrak{m}(F)^{(n-1)/d}\|\mathbf{y}\|$$

for all $\mathbf{y} \in \mathbb{R}^n$. For such an F, $\mathcal{M}(F) \leqslant n^{d(n+1/2)}\mathfrak{m}(F)^n$, and in particular $\mathfrak{m}(F) \geqslant n^{-(n+1/2)d/n}$.

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Proof. Let $\mathbf{x}_1^{\mathrm{tr}}, \dots, \mathbf{x}_n^{\mathrm{tr}}$ denote the columns of T and let $P(T)$ be the parallelepiped defined in the proof of Lemma 2 above. Let $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ be the successive minima of $P(T)$ with respect to the integer lattice $\mathbb{Z}^n$. Since the volume of $P(T)$ is 2^n, Minkowski's theorem says that

$$(n!)^{-1} \leqslant \lambda_1 \cdots \lambda_n \leqslant 1. \tag{2}$$

Choose a basis $\mathbf{z}_1, \dots, \mathbf{z}_n$ for $\mathbb{Z}^n$ satisfying $\mathbf{z}_i \in i\lambda_i P$ for $1 \leqslant i \leqslant n$ and let S be the matrix with columns $\mathbf{z}_1^{\mathrm{tr}}, \dots, \mathbf{z}_n^{\mathrm{tr}}$. Write

$$T^{-1}S = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix},$$

so that $\sum_{i=1}^n a_{i,j}\mathbf{x}_i^{\mathrm{tr}} = \mathbf{z}_j^{\mathrm{tr}}$ for all $1 \leqslant j \leqslant n$. In particular, since $\mathbf{z}_j \in j\lambda_j P$, we have

$$|a_{i,j}| \leqslant j\lambda_j, \quad 1 \leqslant i,j \leqslant n. \tag{3}$$

Similarly, writing $S^{-1}T = (b_{i,j})$ and using Cramer's rule, we see that

$$|b_{i,j}| \leqslant (n-1)! \prod_{l \neq i} l\lambda_l < n! \prod_{l \neq i} l\lambda_l, \quad 1 \leqslant i,j \leqslant n. \tag{3$'$}$$

As before, write $F(\mathbf{X}) = \prod_{i=1}^d L_i(\mathbf{X})$. By (3),

$$\|\mathbf{L}_i S\|^2 = \|\mathbf{L}_i T T^{-1} S\|^2 \leqslant n(n\lambda_n)^2 \|\mathbf{L}_i T\|^2,$$

which implies that

$$\mathcal{H}(F \circ S) \leqslant n^{3d/2} \lambda_n^d \mathfrak{m}(F). \tag{4}$$

Also by (3), for any $\mathbf{y} \in \mathbb{R}^n$,

$$\|T^{-1}S(\mathbf{y})\|^2 \leqslant n(n\lambda_n)^2 \|\mathbf{y}\|^2. \tag{5}$$

Using (3$'$) in a similar manner yields

$$\|T^{-1}S(\mathbf{y})\|^2 n(n!)^2 \left(2\lambda_2 \cdots n\lambda_n\right)^2 \geqslant \|\mathbf{y}\|^2. \tag{5$'$}$$

As seen in the proof of Lemma 2, $|F(\mathbf{y})| \leqslant n^d \mathfrak{m}(F)$ for all $\mathbf{y} \in P$, and by homogeneity $|F(\mathbf{z}_1)| \leqslant (n\lambda_1)^d \mathfrak{m}(F)$. But since $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ and since F does not vanish on $\mathbb{Z}^n \setminus \{\mathbf{0}\}$, we conclude that $|F(\mathbf{z}_1)| \geqslant 1$. Hence $\lambda_1 \geqslant n^{-1}\mathfrak{m}(F)^{-1/d}$. By (2), this implies that $\lambda_2 \cdots \lambda_n \leqslant n\mathfrak{m}(F)^{1/d}$ and $\lambda_n \leqslant n^{n-1}\mathfrak{m}(F)^{(n-1)/d}$. Lemma 7 follows from these estimates, (4), (5), (5$'$) and [Thu01, Lemma 2] (where it is shown that $\mathcal{M}(F) \geqslant 1$ if $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$). ☐

We will also need the following result from [Thu01].

LEMMA 8 [Thu01, Lemma 7]. *Let $K_1(\mathbf{X}), \dots, K_n(\mathbf{X}) \in \mathbb{C}[\mathbf{X}]$ be n linearly independent linear forms in n variables. Denote the corresponding coefficient vectors by $\mathbf{K}_1, \dots, \mathbf{K}_n$. Let $A, B, C > 0$ with $C > B$ and let $D > 1$. Consider the set of $\mathbf{x} \in \mathbb{R}^n$ satisfying*

$$\frac{\prod_{i=1}^n |K_i(\mathbf{x})|}{|\det(\mathbf{K}_1^{\mathrm{tr}} \cdots \mathbf{K}_n^{\mathrm{tr}})|} \leqslant A$$

and also $B \leqslant \|\mathbf{x}\| \leqslant C$. If $BC^{n-1} \geqslant D^{n-1} n! n^{n/2} A$, then this set lies in the union of fewer than

$$n^3 [\log_D(BC^{n-1}/n! n^{n/2} A)]^{n-2}$$

convex sets of the form

$$\begin{gathered} \{\mathbf{y} \in \mathbb{R}^n \; |K_i'(\mathbf{y})| \leqslant a_i \text{ for } i = 1, \dots, n\}, \\ |\det((\mathbf{K}_1')^{\mathrm{tr}} \cdots (\mathbf{K}_n')^{\mathrm{tr}})| = 1, \\ \|\mathbf{K}_i'\| = 1, \quad i = 1, \dots, n, \end{gathered} \tag{6}$$

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with

$$\prod_{i=1}^{n} a_i < D^n n! n^{n/2} \frac{CA}{B}.$$

If $BC^{n-1} < D^{n-1} n! n^{n/2} A$*, then this set lies in the union of no more than* $n!$ *convex sets of this form.*

2. Intermediate results

Proposition 1. *Let* $F(\mathbf{X}) \in \mathbb{R}[\mathbf{X}]$ *be a decomposable form of degree* d *in* n *variables with* $V(F)$ *finite. Suppose that* $\mathcal{H}(F) = \mathfrak{m}(F)$*. Let* $1 \leqslant B < C$ *and* $D > 1$ *and let* Λ *be a lattice of rank* n*. Then the* $\mathbf{x} \in \mathbb{R}^n$ *with*

$$\left(\frac{m}{\mathfrak{m}(F)}\right)^{1/d} B \leqslant \|\mathbf{x}\| \leqslant \left(\frac{m}{\mathfrak{m}(F)}\right)^{1/d} C$$

satisfying (1) lie in no more than

$$\binom{d}{n} \max\{n!,\ n^3[\log_D(B^{\{d-(n-1)a'(F)\}/a'(F)} C^{n-1})]^{n-2}\}$$

convex sets of the form (6) with

$$\prod_{i=1}^{n} a_i < c_3 n^{n/2} n! \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} D^n \frac{C}{B^{\{d-(n-1)a'(F)\}/a'(F)}},$$

where $c_3 = \max\{1, c_2^{1/a'(F)}\}$*. Further, such a set has volume no greater than*

$$c_3 2^n n^{n/2} (n!)^2 \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} D^n \frac{C}{B^{\{d-(n-1)a'(F)\}/a'(F)}}$$

and either all lattice points in such a set lie in a sublattice of smaller rank, or the convex set contains no more than

$$\frac{c_3 3^n 2^{n(n-1)/2} n^{n/2} (n!)^2}{\det \Lambda} \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} D^n \frac{C}{B^{\{d-(n-1)a'(F)\}/a'(F)}}$$

lattice points.

Proof. First assume $m = \mathfrak{m}(F) = 1$. By Lemma 6, if $|F(\mathbf{x})| \leqslant 1$ and $\|\mathbf{x}\| \geqslant 1$, then there are n linearly independent linear factors $L_{i_1}(\mathbf{X}), \ldots, L_{i_n}(\mathbf{X})$ of $F(\mathbf{X})$ such that

$$\frac{\prod_{j=1}^{n} |L_{i_j}(\mathbf{x})|}{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|} \leqslant c_3 \|\mathbf{x}\|^{\{-d+na'(F)\}/a'(F)}.$$

In particular, if $\|\mathbf{x}\| \geqslant B$, then

$$\frac{\prod_{j=1}^{n} |L_{i_j}(\mathbf{x})|}{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|} \leqslant c_3 B^{\{-d+na'(F)\}/a'(F)}. \tag{7}$$

We now invoke Lemma 8, using $A = c_3 B^{\{-d+na'(F)\}/a'(F)}$. Accordingly, the $\mathbf{x} \in \mathbb{R}^n$ with $B \leqslant \|\mathbf{x}\| \leqslant C$ and satisfying (7) lie in the union of no more than

$$\begin{aligned}
&\max\{n!,\ n^3[\log_D(BC^{n-1}/n! n^{n/2} A)]^{n-2}\} \\
&\quad = \max\{n!,\ n^3[\log_D(B^{\{d-(n-1)a'(F)\}/a'(F)} C^{n-1}/n! n^{n/2} c_3)]^{n-2}\} \\
&\quad \leqslant \max\{n!,\ n^3[\log_D(B^{\{d-(n-1)a'(F)\}/a'(F)} C^{n-1})]^{n-2}\}
\end{aligned}$$

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convex sets of the form (6) with

$$\prod_{i=1}^{n} a_i < n! n^{n/2} D^n \frac{CA}{B} = c_3 n! n^{n/2} D^n \frac{C}{B^{\{d-(n-1)a'(F)\}/a'(F)}}.$$

By [Thu01, Lemma 9], the volume of such a convex set is no greater than $2^n n! \prod_{i=1}^n a_i$ and the number of lattice points in such a set is no more than $[3^n 2^{n(n-1)/2} n!/\det(\Lambda)] \prod_{i=1}^n a_i$ if there are n linearly independent lattice points in the set.

Since there are at most $\binom{d}{n}$ possible n-tuples $L_{i_1}, \ldots, L_{i_n}$ to consider above, this proves Proposition 1 when $m = \mathfrak{m}(F) = 1$. For the general case, let $G = \mathfrak{m}(F)^{-1} F$. Then $\mathfrak{m}(G) = 1$ by Lemma 1 and $|F(\mathbf{x})| \leqslant m$ if and only if $|G(\mathbf{x})| \leqslant m/\mathfrak{m}(F)$. But this is true if and only if $|G(\mathbf{y})| \leqslant 1$, where $\mathbf{x} = (m/\mathfrak{m}(F))^{1/d}\mathbf{y}$. In this way, we see that the general case follows from the case $m = \mathfrak{m}(F) = 1$ via a dilation by $(m/\mathfrak{m}(F))^{1/d}$. □

Proposition 2. *Let $F(\mathbf{X}) \in \mathbb{R}[\mathbf{X}]$ be a decomposable form of degree d in n variables with $V(F)$ finite. Suppose $a'(F) < d/n$ and $\mathcal{H}(F) = \mathfrak{m}(F)$. Let $B_0 \geqslant 1$ and $D > 1$, and let Λ be a lattice of rank n. Then the volume of all $\mathbf{x}$ satisfying (1) and with $\|\mathbf{x}\| \geqslant B_0 (m/\mathfrak{m}(F))^{1/d}$ is smaller than*

$$c_4 \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} \frac{(1+\log_D B_0)^{n-2}}{B_0^{\{d-na'(F)\}/a'(F)}} D^{n+1} \sum_{l=0}^{\infty} (l+1)^{n-2} D^{\{-dl+nla'(F)\}/a'(F)},$$

where $c_4 = \binom{d}{n} c_3 2^n n^{n/2} (n!)^3 n^3 d^{n-2}$. For any $l_1 \geqslant 0$, the lattice points $\mathbf{z} \in \Lambda$ with

$$B_0 D^{l_1+1} (m/\mathfrak{m}(F))^{1/d} \geqslant \|\mathbf{z}\| \geqslant B_0 (m/\mathfrak{m}(F))^{1/d}$$

and satisfying (1) lie in the union of a set of cardinality less than

$$c_5 \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} \frac{(1+\log_D B_0)^{n-2}}{B_0^{\{d-na'(F)\}/a'(F)} \det\Lambda} D^{n+1} \sum_{l=0}^{l_1} (l+1)^{n-2} D^{\{-dl+nla'(F)\}/a'(F)},$$

where $c_5 = \binom{d}{n} c_3 3^n 2^{n(n-1)} n^{n/2} (n!)^3 n^3 d^{n-2}$, and fewer than

$$\binom{d}{n} n! n^3 d^{n-2} (1+\log_D B_0)^{n-2} (l_1+1)^{n-1}$$

sublattices of smaller rank.

Proof. For a given index $l \geqslant 0$ let $B_l = D^l B_0$ and $C_l = D B_l$. According to Proposition 1, the $\mathbf{x} \in \mathbb{R}^n$ with $B_l (m/\mathfrak{m}(F))^{1/d} \leqslant \|\mathbf{x}\| \leqslant C_l (m/\mathfrak{m}(F))^{1/d}$ and satisfying (1) lie in no more than $\binom{d}{n} \max\{n!,\ n^3[\log_D(B_l^{\{d-(n-1)a'(F)\}/a'(F)} C_l^{n-1})]^{n-2}\}$ convex sets of the form (6) with volume no greater than

$$\begin{aligned} c_3 2^n n^{n/2} (n!)^2 \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} D^n \frac{C_l}{B_l^{\{d-(n-1)a'(F)\}/a'(F)}} &= c_3 2^n n^{n/2} (n!)^2 \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} D^{n+1} B_l^{\{-d+na'(F)\}/a'(F)} \\ &= c_3 2^n n^{n/2} (n!)^2 \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} B_0^{\{-d+na'(F)\}/a'(F)} D^{n+1} D^{\{-dl+nla'(F)\}/a'(F)}. \end{aligned}$$

A quick estimate shows that

$$\begin{aligned} \log_D(B_l^{\{d-(n-1)a'(F)\}/a'(F)} C_l^{n-1}) &= \log_D[(B_0 D^l)^{d/a'(F)} D^{n-1}] \\ &\leqslant \log_D[(B_0 D^l)^d D^{n-1}] \\ &< d(l+1)(1+\log_D B_0). \end{aligned}$$

Thus, the volume of all solutions $\mathbf{x}$ to (1) with $B_0(m/\mathfrak{m}(F))^{1/d} \leqslant \|\mathbf{x}\|$ is smaller than

$$\binom{d}{n} c_3 2^n n^{n/2}(n!)^2 n^3 d^{n-2} \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} \frac{(1+\log_D B_0)^{n-2}}{B_0^{\{d-na'(F)\}/a'(F)}} D^{n+1} \sum_{l=0}^{\infty} (l+1)^{n-2} D^{\{-dl-nla'(F)\}/a'(F)}.$$

For the statement about the lattice points, we have fewer than

$$\binom{d}{n} n! n^3 d^{n-2} (1+\log_D B_0)^{n-2} \sum_{l=0}^{l_1} (l+1)^{n-2} \leqslant \binom{d}{n} n! n^3 d^{n-2} (1+\log_D B_0)^{n-2} (l_1+1)^{n-1}$$

convex sets of the form (6), and for those containing n linearly independent lattice points, together they contain fewer than

$$c_5 \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} \frac{(1+\log_D B_0)^{n-2}}{B_0^{\{d-na'(F)\}/a'(F)} \det \Lambda} D^{n+1} \sum_{l=0}^{l_1} (l+1)^{n-2} D^{\{-dl+nla'(F)\}/a'(F)}$$

lattice points by Proposition 1. □

Proposition 3. *Let $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ be a decomposable form of finite type in n variables and degree d. Suppose $a'(F) < d/n$. Then for any $D \geqslant e$, the integral solutions to (1) lie in the union of a set of cardinality no greater than $c_6 D^{n+1}(m/\mathfrak{m}(F))^{n/d}$ and c_7 proper subspaces, where $c_6 = 3^n 2^{n(n-1)/2} n! + c_5 \sum_{l=0}^{\infty} (l+1)^{n-2} e^{-ln/d}$ and $c_7 << (1+\log_D m + \log_D \mathfrak{m}(F))^{n-1}$.*

Proof. Let $T \in \mathrm{GL}_n(\mathbb{R})$ and $S \in \mathrm{GL}_n(\mathbb{Z})$ be as in the statement of Lemma 7, and write $T = S^{-1}T'$. Consider the equivalent form $G = F \circ S$. Then $N_G(m) = N_F(m)$, $V(G) = V(F)$ and $\mathfrak{m}(G) = \mathfrak{m}(F) = \mathcal{H}(G \circ T')$. By Lemma 7, $\mathcal{H}(G) \leqslant n^{d(n+1/2)} \mathfrak{m}(G)^n$ and, for every $\mathbf{x} \in \mathbb{R}^n$,

$$\mathfrak{m}(G)^{-1/d} n^{-3/2} (n!)^{-2} \|\mathbf{x}\| \leqslant \|(T')^{-1}(\mathbf{x})\| \leqslant n^{n+1/2} \mathfrak{m}(G)^{(n-1)/d} \|\mathbf{x}\|.$$

In other words, we may assume without loss of generality that

$$\begin{aligned} c_8 \mathfrak{m}(F)^{-1/d} \|\mathbf{x}\| \leqslant \|T^{-1}(\mathbf{x})\| &\leqslant c_9 \mathfrak{m}(F)^{(n-1)/d} \|\mathbf{x}\|, \\ \mathfrak{m}(F) &= \mathcal{H}(F \circ T), \\ \mathcal{H}(F) &\leqslant c_9^d \mathfrak{m}(F)^n, \end{aligned} \tag{8}$$

where $c_8 = n^{-3/2}(n!)^{-2}$ and $c_9 = n^{n+1/2}$.

We will apply Proposition 2 to the lattice $\Lambda = T^{-1}(\mathbb{Z}^n)$ of determinant 1, using $B_0 = 1$.

Let l_1 be minimal such that

$$\begin{aligned} [D^{l_1+1}(m/\mathfrak{m}(F))^{1/d}]^{1/2} \geqslant \max\{ &(c_9 \mathfrak{m}(F)^{(n-1)/d})^{1/2} c_8^{-1} \mathfrak{m}(F)^{1/d}, \\ &(c_9 \mathfrak{m}(F)^{(n-1)/d})^{1/2} c_9^{(d/2)} m^{1/(2d)} \mathfrak{m}(F)^{n/2}, \\ &(c_2 m/\mathfrak{m}(F))^{1/(d-na'(F))} \}. \end{aligned}$$

Clearly $l_1 << 1 + \log_D m + \log_D \mathfrak{m}(F)$, where the implicit constant depends only on n and d. Moreover, if $\|T^{-1}(\mathbf{z})\| \geqslant D^{l_1+1}(m/\mathfrak{m}(F))^{1/d}$, then by (8) we have

$$\|\mathbf{z}\|^{1/2} \geqslant \max\{c_8^{-1} \mathfrak{m}(F)^{1/d},\ c_9^{d/2} m^{1/(2d)} \mathfrak{m}(F)^{n/2}\},$$

and using (8) once more,

$$m^{1/d} \mathcal{H}(F) \leqslant \|\mathbf{z}\| \leqslant \|T^{-1}(\mathbf{z})\|^2. \tag{9}$$

By Lemma 1, (8), (9) and our choice for l_1, if $\mathbf{z} \in \mathbb{Z}^n$ is a solution to (1) with $\|T^{-1}(\mathbf{z})\| \geqslant D^{l_1+1}(m/\mathfrak{m}(F))^{1/d}$, then there are n linearly independent factors $L_{i_1}(\mathbf{X}), \ldots, L_{i_n}(\mathbf{X})$ of $F(\mathbf{X})$ such

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that

$$\left(\frac{\prod_{j=1}^{n}|L_{i_j}(\mathbf{z})|}{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}}\cdots\mathbf{L}_{i_n}^{\mathrm{tr}})|}\right)^{a'(F)} \leqslant \frac{c_2 m}{\|T^{-1}(\mathbf{z})\|^{d-na'(F)}\mathfrak{m}(F)} \leqslant \frac{1}{\|T^{-1}(\mathbf{z})\|^{(d-na'(F))/2}} \leqslant \frac{1}{\|\mathbf{z}\|^{(d-na'(F))/4}}.$$

In particular,

$$\frac{\prod_{j=1}^{n}|L_{i_j}(\mathbf{z})|}{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}}\cdots\mathbf{L}_{i_n}^{\mathrm{tr}})|} < \|\mathbf{z}\|^{\{-d+na'(F)\}/4a'(F)} < \|\mathbf{z}\|^{-1/4d}. \tag{10}$$

Here we used $\{d-na'(F)\}/a'(F) > 1/d$.

Take such a $\mathbf{z}$ and write it as a multiple of a primitive point $\mathbf{z}'$; say $\mathbf{z} = g\mathbf{z}'$, for some positive integer g. Since $|F(\mathbf{z}')| \geqslant 1$, we see that $g \leqslant m^{1/d}$, so that $\|\mathbf{z}'\| \geqslant \mathcal{H}(F)$ by (9). Moreover, we may replace $\mathbf{z}$ in (10) with $\mathbf{z}'$. By [Eve96, Corollary] and [Thu01, Lemma 2], such primitive $\mathbf{z}'$ lie in c_7' proper subspaces, where c_7' depends only on n and d.

We thus see that all lattice points $T^{-1}(\mathbf{z})$ with $\|T^{-1}(\mathbf{z})\| \geqslant D^{l_1+1}(m/\mathfrak{m}(F))^{1/d}$ lie in c_7' proper subspaces. By Proposition 2, those with $(m/\mathfrak{m}(F))^{1/d} \leqslant \|T^{-1}(\mathbf{z})\| \leqslant D^{l_1+1}(m/\mathfrak{m}(F))^{1/d}$ lie in the union of a set of cardinality no greater than $D^{n+1}(m/\mathfrak{m}(F))^{n/d}c_5\sum_{l=0}^{\infty}(l+1)^{n-2}e^{ln/d}$ and no more than $\binom{d}{n}n!n^3d^{n-2}(l_1+1)^{n-1}$ proper subspaces. (Here we used $\{d-na'(F)/a'(F)\} > n/d$ again and $D \geqslant e$.)

It remains to deal with those lattice points with $\|T^{-1}(\mathbf{z})\| \leqslant (m/\mathfrak{m}(F))^{1/d}$. It is simpler instead to estimate the number of lattice points with supnorm no greater than $(m/\mathfrak{m}(F))^{1/d}$. Such lattice points will be in a convex set of the form (6) with $\prod_{i=1}^{n}a_i = (m/\mathfrak{m}(F))^{n/d}$. By [Thu01, Lemma 9], either all such lattice points lie in a proper subspace, or their number is fewer than $3^n2^{n(n-1)}n!(m/\mathfrak{m}(F))^{n/d}$. The proposition follows. □

Suppose $W \subseteq \mathbb{R}^n$ is a subspace defined over $\mathbb{Q}$. Let $T_W \in \mathrm{GL}_n(\mathbb{Z})$ be such that $T_W^{-1}(W)$ is the subspace spanned by the first $\dim W$ canonical basis vectors of $\mathbb{R}^n$. We will denote by $F|_W$ the decomposable form of degree d in $\dim W$ variables obtained by restricting $F \circ T_W$ to the subspace spanned by the first $\dim W$ canonical basis vectors of $\mathbb{R}^n$. Note that the solutions $\mathbf{x}$ to the inequality $|F|_W(\mathbf{x})| \leqslant m$ are in one-to-one correspondence to the solutions to (1) lying in W. Further, the same holds when we consider integral solutions.

Proposition 4. *Suppose $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ is a decomposable form of degree d in n variables of finite type. Let W be a subspace of $\mathbb{R}^n$ defined over $\mathbb{Q}$ of dimension $n-1$. There are positive constants c_{10}, c_{11}, c_{12} and c_{13}, depending only on n and d, such that $N_{F|_W}(m) \leqslant c_{10}m^{(n-1)/d}$, and if $\mathcal{M}(F|_W) \leqslant m^{c_{11}}$, then*

$$c_{12}m^{(n-1)/d}V(F|_W) \leqslant N_{F|_W}(m) \leqslant c_{13}m^{(n-1)/d}V(F|_W).$$

Proof. This follows directly from [Thu01, Theorem 3], Lemma 7 and [Thu03, Theorem 2] applied to the form $F|_W$ when $n > 2$, i.e. when $n-1 \geqslant 2$. In the case $n = 2$, the form $F|_W$ is a form in one variable and the result is trivially valid. □

Corollary. *Suppose $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ is a decomposable form of degree d in n variables of finite type and $a'(F) < d/n$. If $\mathfrak{m}(F) \geqslant m^{2/n}$, then $N_F(m) << m^{(n-1)/d}$.*

Proof. Set $D = \mathfrak{m}(F)^{n/\{2d(n+1)\}}$ in Proposition 3. The integral solutions to (1) then lie in a set of cardinality no greater than

$$c_6D^{n+1}(m/\mathfrak{m}(F))^{n/d} = c_6(m/\mathfrak{m}(F)^{1/2})^{n/d} \leqslant c_6m^{(n-1)/d}$$

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and in

$$c_7 << (1 + \log_D m + \log_D \mathfrak{m}(F))^{n-1} << 1$$

proper subspaces. The corollary follows from Proposition 4. □

Proposition 5. *Suppose $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ is a decomposable form of degree d in n variables of finite type. Let $B, m \geqslant 1$. Suppose $W_1, \dots, W_S$ are distinct subspaces of dimension $n-1$ defined over $\mathbb{Q}$ satisfying $\mathcal{M}(F|_{W_i}) \leqslant B$ for all i and*

$$\left(\frac{m}{B^{n-1}}\right)^{1/d} \geqslant c_{14}(S-1),$$

where

$$c_{14} = \frac{2c_{10}}{c_{12}} \left(\frac{n-1}{2}\right)^{n-1}.$$

Then the number of integer solutions $\mathbf{x} \in \bigcup_{i=1}^S W_i$ to (1) is at least $\frac{1}{2}\sum_{i=1}^S N_{F|_{W_i}}(m)$. In particular, we have

$$2N_F(m) \geqslant \sum_{i=1}^S N_{F|_{W_i}}(m).$$

Proof. A simple induction argument on S shows that the number of integer solutions $\mathbf{x} \in \bigcup_{i=1}^S W_i$ to (1) is at least

$$\sum_{i=1}^S N_{F|_{W_i}}(m) - \sum_{i=1}^{S-1} \left(\sum_{j=i+1}^S N_{F|_{W_i \cap W_j}}(m) \right).$$

By Proposition 4 (since $\dim(W_i \cap W_j) = n-2$),

$$\sum_{j=i+1}^S N_{F|_{W_i \cap W_j}}(m) \leqslant (S-1)c_{10} m^{(n-2)/d}$$

for all i, so that the number of integer solutions we are considering is at least

$$\sum_{i=1}^S (N_{F|_{W_i}}(m) - c_{10}(S-1)m^{(n-2)/d}).$$

By Proposition 4 and Lemma 2 (applied to $F|_{W_i}$),

$$\begin{aligned}
N_{F|_{W_i}}(m) &\geqslant c_{12} m^{(n-1)/d} V(F|_{W_i}) \\
&\geqslant c_{12} m^{(n-1)/d} \left(\frac{2}{n-1}\right)^{n-1} \mathfrak{m}(F|_{W_i})^{-(n-1)/d} \\
&\geqslant c_{12} m^{(n-1)/d} \left(\frac{2}{n-1}\right)^{n-1} \mathcal{M}(F|_{W_i})^{-(n-1)/d} \\
&\geqslant c_{12} m^{(n-1)/d} \left(\frac{2}{n-1}\right)^{n-1} B^{-(n-1)/d} \\
&\geqslant c_{12} m^{(n-2)/d} \left(\frac{2}{n-1}\right)^{n-1} \left(\frac{m}{B^{n-1}}\right)^{1/d} \\
&\geqslant 2c_{10}(S-1)m^{(n-2)/d}
\end{aligned}$$

for all i. This together with the above estimate completes the proof. □

Proposition 6. *Suppose $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ is a decomposable form of degree d in n variables of finite type. Let $\mathcal{S}$ be a set of subspaces of dimension $n-1$ defined over $\mathbb{Q}$ of cardinality S.*

Suppose $m^{1/2d} \geqslant \mathcal{M}(F)^{1/4nd} \geqslant c_{14}(S-1)$. *Then there is a positive constant* c_{15}, *depending only on* n *and* d, *such that*

$$\sum_{W\in\mathcal{S}} N_{F|_W}(m) << m^{(n-1)/d} + S(m^{(n-1)/d}\mathcal{M}(F)^{-c_{15}} + m^{(n-2)/d}(1+\log m)^{n-1}).$$

Proof. Set $B = \mathcal{M}(F)^{c'_{11}}$, where c'_{11} is the minimum of $c_{11}/2n$ and $1/4n(n-1)$. Write $\mathcal{S}$ as the union of three disjoint subsets, $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, where $\mathcal{S}_1$ consists of the subspaces W with $\mathcal{M}(F|_W) \leqslant B$, $\mathcal{S}_2$ consists of the subspaces W with $B < \mathcal{M}(F|_W) \leqslant m^{2n}$, and $\mathcal{S}_3$ consists of the remaining subspaces.

For the moment, set $m = \mathcal{M}(F)^{1/2n}$. Then since $B \leqslant \mathcal{M}(F)^{1/4n(n-1)}$, $(m/B^{n-1})^{1/d} \geqslant \mathcal{M}(F)^{1/4nd} \geqslant c_{14}(S-1)$. But $B \leqslant \mathcal{M}(F)^{c_{11}/2n} \leqslant m^{c_{11}}$ also, so by Propositions 4 and 5,

$$2N_F(m) \geqslant \sum_{W\in\mathcal{S}_1} N_{F|_W}(m) \geqslant c_{12} m^{(n-1)/d} \sum_{W\in\mathcal{S}_1} V(F|_W).$$

But by [Thu03, Theorem 4], the integer solutions to (1) lie in the union of c_{16} proper subspaces, where $c_{16} << 1$ if $m \leqslant \mathcal{M}(F)^{1/2n}$. By Proposition 4, there are no more than $c_{10}m^{(n-1)/d}$ solutions in a proper subspace, so that $N_F(m) << m^{(n-1)/d}$. Thus,

$$\sum_{W\in\mathcal{S}_1} V(F|_W) << 1.$$

Now let m be as in the statement of Proposition 6. Since $B \leqslant \mathcal{M}(F)^{c_{11}/2n} \leqslant m^{c_{11}}$, Proposition 4 and the above inequality give

$$\sum_{W\in\mathcal{S}_1} N_{F|_W}(m) \leqslant c_{13} m^{(n-1)/d} \sum_{W\in\mathcal{S}_1} V(F|_W) << m^{(n-1)/d}. \tag{11}$$

By [Thu03, Theorem 3], for any $W \in \mathcal{S}_2$ we have

$$\begin{aligned} N_{F|_W}(m) &<< m^{(n-1)/d}\mathcal{M}(F|_W)^{-1/d}\{1 + (\log \mathcal{M}(F|_W))^{n-1}\} \\ &\quad + m^{(n-2)/d}\{1 + (\log m)^{n-1} + (\log \mathcal{M}(F|_W))^{n-1}\} \\ &<< m^{(n-1)/d}B^{-1/2d} + m^{(n-2)/d}(1+\log m)^{n-1}. \end{aligned}$$

Thus,

$$\sum_{W\in\mathcal{S}_2} N_{F|_W}(m) << S(m^{(n-1)/d}B^{-1/2d} + m^{(n-2)/d}(1+\log m)^{n-1}). \tag{12}$$

Finally, $N_{F|_W}(m) << m^{(n-2)/d}$ for all $W \in \mathcal{S}_3$ by [Thu03, Theorem 4] and Proposition 4. Thus

$$\sum_{W\in\mathcal{S}_3} N_{F|_W}(m) << Sm^{(n-2)/d}. \tag{13}$$

Proposition 6 follows from (11)–(13), setting $c_{15} = c'_{11}/2d$. ☐

3. Proofs of Theorems 1, 2 and 3

Proof of Theorem 1. The lower bounds for $V(F)$ and $\mathfrak{m}(F)$ in Theorem 1 are contained in Lemmas 2 and 7.

Suppose $a'(F) < d/n$. By Lemma 1, we may assume without loss of generality that $\mathcal{H}(F) = \mathfrak{m}(F) = 1$. Set $D = e$ and $m = B_0 = 1$ in Proposition 2. Then we see that the volume of all solutions $\mathbf{x} \in \mathbb{R}^n$ to (1) with $\|\mathbf{x}\| \geqslant 1$ is smaller than

$$c_4 e^{n+1} \sum_{l=0}^{\infty} (l+1)^{n-2} e^{\{-ld + lna'(F)\}/a'(F)} << 1.$$

Of course, the set of all $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\| \leqslant 1$ is no more than the volume of the unit ball in $\mathbb{R}^n$. This shows that $V(F) << m^{-n/d}$ when $a'(F) < d/n$, and completes the proof of Theorem 1. □

Proof of Theorem 2. As in the proof of Proposition 3, we will assume (8). Fix a $B_0 \geqslant 1$. Let N_0 denote the cardinality of the set of integral solutions to (1) with supnorm no greater than $c_8^{-1}m^{1/d}B_0$ and let V_0 denote the volume of all real solutions to (1) with supnorm no greater than $c_8^{-1}m^{1/d}B_0$. According to [Thu01, Lemma 14],

$$|N_0 - V_0| << 1 + (m^{1/d}B_0)^{n-1}. \tag{14}$$

By (8), if $\mathbf{x} \in \mathbb{R}^n$ is a solution to (1) with (sup)norm at least $c_8^{-1}m^{1/d}B_0$, then $T^{-1}(\mathbf{x})$ is a solution to $|F \circ T(\mathbf{X})| \leqslant m$ and $\|T^{-1}(\mathbf{x})\| \geqslant B_0(m/\mathfrak{m}(F))^{1/d}$. Setting $D = e$ in Proposition 2, we see that the total volume of all such $T^{-1}(\mathbf{x})$ is smaller than

$$\begin{aligned} c_4 \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} &\frac{(1+\log B_0)^{n-2}}{B_0^{\{d-na'(F)\}a'(F)}} e^{n+1} \sum_{l=0}^{\infty} (l+1)^{n-2} e^{\{-ld+lna'(F)\}/a'(F)} \\ &<< \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} \frac{(1+\log B_0)^{n-2}}{B_0^{\{d-na'(F)\}/a'(F)}}. \end{aligned}$$

In other words,

$$m^{n/d}V(F) - V_0 << \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} \frac{(1+\log B_0)^{n-2}}{B_0^{\{d-na'(F)\}/a'(F)}}. \tag{15}$$

Similar to the proof of Proposition 3, let l_1 be minimal such that

$$\begin{aligned} (e^{l_1+1}(m/\mathfrak{m}(F))^{1/d})^{1/2} \geqslant \max\{&(c_9\mathfrak{m}(F)^{(n-1)/d})^{1/2}c_8^{-1}\mathfrak{m}(F)^{1/d}, \\ &(c_9\mathfrak{m}(F)^{(n-1)/d})^{1/2}c_9^{(d/2)}m^{1/(2d)}\mathfrak{m}(F)^{n/2}, \\ &(c_2 m/\mathfrak{m}(F))^{1/(d-na'(F))}\}. \end{aligned}$$

Then $l_1 << 1+\log m+\log \mathfrak{m}(F) << 1+\log m$ (since $\mathfrak{m}(F) \leqslant m^{1/n}$). As in the proof of Proposition 3, if

$$\|T^{-1}(\mathbf{z})\| \geqslant B_0 e^{l_1+1}(m/\mathfrak{m}(F))^{1/d} \geqslant e^{l_1+1}(m/\mathfrak{m}(F))^{1/d},$$

then we have (9) again. Arguing exactly as in the proof of Proposition 3, the integer solutions $\mathbf{z} \in \mathbb{Z}^n$ to (1) with $\|T^{-1}(\mathbf{z})\| \geqslant B_0 e^{l_1+1}(m/\mathfrak{m}(F))^{1/d}$ lie in the union of c_7' proper subspaces. By Proposition 2, then, the set of integer solutions to (1) with (sup)norm greater than $c_8^{-1}m^{1/d}B_0$ lie in the union of a set of cardinality N satisfying

$$\begin{aligned} N < c_5 \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} &\frac{(1+\log B_0)^{n-2}}{B_0^{\{d-na'(F)\}/a'(F)}} e^{n+1} \sum_{l=0}^{l_1} (l+1)^{n-2} e^{\{-ld+lna'(F)\}/a'(F)} \\ << \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} &\frac{(1+\log B_0)^{n-2}}{B_0^{\{d-na'(F)\}/a'(F)}} \end{aligned} \tag{16}$$

and S proper subspaces, where $S << (1+\log B_0)^{n-2}(1+\log m)^{n-1}$.

We now choose B_0. Set

$$(B_0 m^{1/d})^{n-1} = \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} B_0^{\{-d+na'(F)\}/a'(F)}.$$

Then

$$
\begin{aligned}
B_0^{\{d-a'(F)\}/a'(F)} &= \left(\frac{m}{\mathfrak{m}(F)^n}\right)^{1/d} \\
B_0 &= \left(\frac{m}{\mathfrak{m}(F)^n}\right)^{a'(F)/\{d(d-a'(F))\}} \\
B_0 m^{1/d} &= \frac{m^{1/\{d-a'(F)\}}}{\mathfrak{m}(F)^{na'(F)/\{d(d-a'(F))\}}} \\
(B_0 m^{1/d})^{n-1} &= \frac{m^{(n-1)/\{d-a'(F)\}}}{\mathfrak{m}(F)^{n(n-1)a'(F)/\{d(d-a'(F))\}}}.
\end{aligned}
\tag{17}
$$

Note that, by the second equation in (15) and since $\mathfrak{m}(F) \leqslant m^{1/n}$, we indeed have $B_0 \geqslant 1$.

Consider the S subspaces above and let $\mathcal{S}$ denote this collection of proper subspaces. Without loss of generality, we may assume $\dim W = n-1$ for all $W \in \mathcal{S}$.

Suppose first that $\mathcal{M}(F) \geqslant m^{1/4n}$. If $m^{1/2d} \geqslant \mathcal{M}(F)^{1/4nd} \geqslant c_{14}(S-1)$, then by Proposition 6

$$
\begin{aligned}
\sum_{W\in\mathcal{S}} N_{F|_W}(m) &<< m^{(n-1)/d} + S(m^{(n-1)/d}\mathcal{M}(F)^{-c_{15}} + m^{(n-2)/d}(1+\log m)^{n-1}) \\
&<< m^{(n-1)/d},
\end{aligned}
$$

since $S << (1+\log m)^{2n-3}$. If $m^{1/2d} < \mathcal{M}(F)^{1/4nd}$, then by Lemma 7

$$m < \mathcal{M}(F)^{1/2n} << \mathfrak{m}(F)^{1/2} \leqslant m^{1/2n},$$

so $m << 1$. Also, if $\mathcal{M}(F)^{1/4nd} < c_{14}(S-1)$, then $m^{1/d} << (1+\log m)^{2n-3}$ and $m << 1$ again. Moreover, if $m << 1$, then $S << 1$ and by Proposition 4

$$\sum_{W\in\mathcal{S}} N_{F|_W}(m) \leqslant c_{10} S m^{(n-1)/d} << m^{(n-1)/d}.$$

Now suppose $\mathcal{M}(F) < m^{1/4n}$. Then $\mathfrak{m}(F) < m^{1/4n}$, too. In this case

$$\left(\frac{m}{\mathfrak{m}(F)^{na'(F)/d}}\right)^{(n-1)/\{d-a'(F)\}} > m^{(n-1)/d} m^{3a'(F)(n-1)/\{4d(d-a'(F))\}} >> m^{(n-1)/d}(1+\log m)^{2n-3}.$$

Again by Proposition 4,

$$\sum_{W\in\mathcal{S}} N_{F|_W}(m) \leqslant c_{10} S m^{(n-1)/d} << m^{(n-1)/d}(1+\log m)^{2n-3}.$$

Thus, in all cases

$$\sum_{W\in\mathcal{S}} N_{F|_W}(m) << \left(\frac{m}{\mathfrak{m}(F)^{na'(F)/d}}\right)^{(n-1)/\{d-a'(F)\}} (1+\log m)^{n-2}. \tag{18}$$

Theorem 2 follows from (14)–(18). □

Proof of Theorem 3. By the corollary to Propositions 3 and 4, we only need to deal with the case where $\mathfrak{m}(F) \leqslant m^{2/n}$. Set $D = e$ in Proposition 3. Since $\log \mathfrak{m}(F) \leqslant (2/n)\log m$, Proposition 3 shows that the integral solutions to (1) lie in the union of a set of cardinality N and S proper subspaces, where $N << (m/\mathfrak{m}(F))^{n/d}$ and $S << (1+\log m)^{n-1}$. Without loss of generality we may assume these S proper subspaces are of dimension $n-1$. Denote the number of solutions in these proper subspaces by N'.

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If $\mathcal{M}(F) \geqslant m^{1/4n}$, then we argue exactly as in the proof of Theorem 2 above and conclude that $N' << m^{(n-1)/d}$. If $\mathcal{M}(F) < m^{1/4n}$, then by Proposition 4, $N' << m^{(n-1)/d}(1+\log m)^{n-1}$. But now

$$m^{(n-1)/d}(1+\log m)^{n-1} << \frac{m^{n/d}}{m^{1/4d}} \leqslant \left(\frac{m}{\mathcal{M}(F)}\right)^{n/d} \leqslant \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d}. \qquad \square$$

4. Proofs of Theorems 4 and 5

We need some notation from [Thu03]. Let $F(\mathbf{X}) = \prod_{i=1}^{d} L_i(\mathbf{X})$ be a factorization of F as in the statement of Lemma 3. Consider the d^n-dimensional vector with components $\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})$. The quantity $Q(F)$ is defined to be the infimum over all such factorizations of the L^2 norms of these vectors. Let $\mathcal{NS}(F)$ denote the normalized semi-discriminant of F:

$$\mathcal{NS}(F) = \prod_{(i_1,\ldots,i_n)}{}' \frac{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|}{\|\mathbf{L}_{i_1}\| \cdots \|\mathbf{L}_{i_n}\|}.$$

Here the restricted product is over those $(i_1, \ldots, i_n)$ where $\mathbf{L}_{i_1}, \ldots, \mathbf{L}_{i_n}$ are linearly independent.

Proof of Theorem 4. Lemma 3 shows that

$$Q(F)^2 \geqslant (d/n)^n n! \mathfrak{m}(F)^{2n/d}. \tag{19}$$

If $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$, then $|\mathcal{NS}(F)|^{-1} \leqslant \mathcal{H}(F)^{\binom{d}{n}}$ by [Thu01, Lemma 3]. If we assume $\mathcal{H}(F) = \mathcal{M}(F)$ and further that F does not vanish at any non-trivial rational point, then this together with Lemma 7 shows that

$$|\log \mathcal{NS}(F)| << |\log(\mathfrak{m}(F))|. \tag{20}$$

By [Thu03, Theorem 1],

$$V(F) << Q(F)^{-1}(1 + |\log \mathcal{NS}(F)|)^{n-1}.$$

Theorem 4 follows from this, (19) and (20). $\square$

To prove Theorem 5, we note that if $a'(F) = d/n$ in the proof of Proposition 1, then (7) becomes

$$\frac{\prod_{j=1}^{n} |L_{i_j}(\mathbf{x})|}{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|} \leqslant c_3. \tag{7$'$}$$

Moreover, the hypothesis $\mathcal{H}(F) = \mathfrak{m}(F)$ used to obtain (7) is not necessary here. Thus, the hypothesis $\mathcal{H}(F) = \mathfrak{m}(F)$ in Proposition 1 is unnecessary when $a'(F) = d/n$.

Proof of Theorem 5. By [Thu03, Lemma 4], any solution $\mathbf{x}$ to (1) satisfies an inequality of the form

$$\frac{\prod_{j=1}^{n} |L_{i_j}(\mathbf{x})|}{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|} \leqslant \left(\frac{c_{17} m}{\|\mathbf{x}\|^{d-na(F)} \mathcal{H}(F) \mathcal{NS}(F)^{d-(n-1)a(F)}}\right)^{1/a(F)}, \tag{21}$$

where c_{17} is a constant which depends only on n and d and $a(F)$ is the same as in [Thu01, Theorem 3] above.

We first handle the case where $\mathcal{M}(F) \leqslant m^2$.

Choose l_1 minimal such that

$$(e^{l_1+1}(m/\mathfrak{m}(F))^{1/d})^{(d-na(F))/2} \geqslant \max\left\{(m^{1/d}\mathcal{H}(F))^{(d-na(F))/2}, \frac{c_{17} m}{\mathcal{H}(F)\mathcal{NS}(F)^{d-(n-1)a(F)}}\right\}.$$

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By (8) and (20), we see that $l_1 << 1 + \log \mathfrak{m}(F)$ if $\mathfrak{m}(F)^{-n/d}(1+\log \mathfrak{m}(F))^{n-1} \leqslant m^{-1/d}$. For any solution $\mathbf{x} \in \mathbb{R}^n$ to (1) which satisfies $\|\mathbf{x}\| > e^{l_1+1}(m/\mathfrak{m}(F))^{1/d}$, we have

$$\frac{\prod_{j=1}^{n} |L_{i_j}(\mathbf{x})|}{|\det(\mathbf{L}_{i_1}^{\mathrm{tr}} \cdots \mathbf{L}_{i_n}^{\mathrm{tr}})|} < \|\mathbf{x}\|^{-(d-na(F))/2a(F)}$$

and also $\|\mathbf{x}\| \geqslant m^{1/d}\mathcal{H}(F)$. As in the proof of Proposition 3, such $\mathbf{x} \in \mathbb{Z}^n$ lie in the union of c_7' proper subspaces.

For an index $l \geqslant 0$, set $D = e$, $B = e^l$ and $C = eB$ in Proposition 1. Then

$$\binom{d}{n} \max\{n!, n^3[\log_D(BC^{n-1})]^{n-2}\} < \binom{d}{n} n^{n+1}(1+l)^{n-2}.$$

All integral solutions $\mathbf{x}$ to (1) with $(m/\mathfrak{m}(F))^{1/d}e^l \leqslant \|\mathbf{x}\| \leqslant (m/\mathfrak{m}(F))^{1/d}e^{l+1}$ thus lie in the union of a set of cardinality N_l and S_l proper subspaces, where $N_l << (m/\mathfrak{m}(F))^{n/d}(1+l)^{n-2}$ and $S_l << (1+l)^{n-2}$.

Let N_{-1} denote the number of solutions $\mathbf{x} \in \mathbb{Z}^n$ to (1) with $\|\mathbf{x}\| \leqslant (m/\mathfrak{m}(F))^{1/d}$. Trivially N_{-1} is no greater than the total number of all $\mathbf{x} \in \mathbb{Z}^n$ with $\|\mathbf{x}\| \leqslant (m/\mathfrak{m}(F))^{1/d}$, so $N_{-1} << (m/\mathfrak{m}(F))^{n/d}$. Since $\mathfrak{m}(F) \geqslant 1$ we have $\mathfrak{m}(F)^{-n/d} \leqslant m^{-n/d}(1+\log \mathfrak{m}(F))^{n-1} \leqslant m^{-1/d}$, so that $N_{-1} << m^{(n-1)/d}$.

We also have

$$\begin{aligned}\sum_{l=0}^{l_1} N_l << \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} \sum_{l=0}^{l_1}(1+l)^{n-2} &\leqslant \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d}(1+l_1)^{n-1} \\ &<< \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d}(1+\log \mathfrak{m}(F))^{n-1} \\ &\leqslant m^{(n-1)/d}.\end{aligned}$$

In this way we see that the solutions $\mathbf{x} \in \mathbb{Z}^n$ to (1) lie in the union of a set of cardinality N and S proper subspaces, where $N << m^{(n-1)/d}$ and $S << (1+\log \mathfrak{m}(F))^{n-1}$. As before, we may assume these S proper subspaces are of dimension $n-1$. Denote this collection of subspaces by $\mathcal{S}$.

Recall that we are assuming $m \geqslant \mathcal{M}(F)^{1/2}$. If $\mathcal{M}(F)^{1/4nd} \geqslant c_{14}(S-1)$, then by Proposition 6

$$\sum_{W \in \mathcal{S}} N_{F|_W}(m) << m^{(n-1)/d}.$$

If $\mathcal{M}(F)^{1/4nd} < c_{14}(S-1)$, then $\mathcal{M}(F) << 1$ and we have $S << 1$. In this case, Proposition 4 gives

$$\sum_{W \in \mathcal{S}} N_{F|_W}(m) << m^{(n-1)/d}.$$

It remains to deal with the case where $\mathcal{M}(F) > m^2$. In this case, Lemma 7 shows that $\mathfrak{m}(F) >> m^{2/n}$; we set $D = \mathfrak{m}(F)^{n/\{2d(n+1)\}}$ and choose l_1 minimal such that

$$(D^{l_1+1}(m/\mathfrak{m}(F))^{1/d})^{(d-na(F))/2} \geqslant \max\left\{(m^{1/d}\mathcal{H}(F))^{(d-na(F))/2},\ \frac{c_{17}m}{\mathcal{H}(F)\mathcal{NS}(F)^{d-(n-1)a(F)}}\right\}.$$

Now $l_1 << 1$ and, as above, the solutions $\mathbf{x} \in \mathbb{Z}^n$ with $\|\mathbf{x}\| \geqslant D^{l_1+1}(m/\mathfrak{m}(F))^{n/d}$ lie in c_7' proper subspaces.

For an index $l \geqslant 0$ let $B_l = D^l$ and $C_l = D^{l+1}$ in Proposition 1. The solutions $\mathbf{x} \in \mathbb{Z}^n$ to (1) with $(m/\mathfrak{m}(F))^{1/d}B_l \leqslant \|\mathbf{x}\| \leqslant (m/\mathfrak{m}(F))^{1/d}C_l$ lie in the union of a set of cardinality N_l and S_l proper

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subspaces, where $S_l << (1+l)^{n-1}$ and

$$N_l \leqslant c_3 3^n 2^{n(n-1)/2} n^{n/2} (n!)^2 \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} D^n \frac{C_l}{B_l} << \left(\frac{m}{\mathfrak{m}(F)}\right)^{n/d} D^{n+1}$$
$$= \frac{m^{n/d}}{\mathfrak{m}(F)^{n/2d}}$$
$$<< m^{(n-1)/d}.$$

As above, the number N_{-1} of solutions $\mathbf{x} \in \mathbb{Z}^n$ to (1) with $\|\mathbf{x}\| \leqslant (m/\mathfrak{m}(F))^{1/d}$ satisfies $N_{-1} << m^{(n-1)/d}$. Thus, since $l_1 << 1$, the solutions $\mathbf{x} \in \mathbb{Z}^n$ to (1) lie in the union of a set of cardinality N and S proper subspaces, where $N << m^{(n-1)/d}$ and $S << 1$. Proposition 4 applied to the subspaces completes the proof of this case, and thus the proof of Theorem 5. □

5. Some examples

Fix an even $d \geqslant 4$ and $0 < \epsilon \leqslant 1/3$. Let

$$F_\epsilon(X,Y) = (X^l - (\epsilon Y)^l)((\epsilon X)^l - Y^l),$$

where $l = d/2$. Then

$$F_\epsilon(X,Y) = \prod_{i=1}^{l} (X - \rho^i \epsilon Y)(\rho^i \epsilon X - Y),$$

where ρ is a primitive lth root of unity.

Suppose $1 \leqslant y \leqslant (3\epsilon)^{-1/2}$ and $\epsilon y \leqslant x \leqslant 1/(3y)$. Then

$$|x - \rho^i \epsilon y| \leqslant x + \epsilon y \leqslant 2x,$$
$$|\rho^i \epsilon x - y| \leqslant \epsilon x + y \leqslant 1/(9y) + y \leqslant 10y/9,$$

for any i. In particular, $|F_\epsilon(x,y)| \leqslant (2x)^l (10y/9)^l \leqslant (2/3)^l (10/9)^l < 1$. From this, we see that

$$V(F_\epsilon) > \int_1^{(3\epsilon)^{-1/2}} \int_{\epsilon y}^{(3y)^{-1}} dx\, dy$$
$$= \int_1^{(3\epsilon)^{-1/2}} (3y)^{-1} - \epsilon y \, dy$$
$$> \frac{-\log 3 - \log \epsilon - 1}{6}.$$

Thus, if $\epsilon < (3e)^{-2}$ we have

$$V(F_\epsilon) > \frac{-\log \epsilon}{12}. \tag{22}$$

Now suppose $T \in \mathrm{GL}_n(\mathbb{R})$ with $|\det T| = 1$ and write $F_\epsilon(X,Y) = \prod_{i=1}^d L_i(X,Y)$, where

$$L_i(X,Y) = \begin{cases} X - \rho^i \epsilon Y, & \text{if } i \leqslant l, \\ \rho^{d-i} \epsilon X - Y, & \text{if } l < i \leqslant d. \end{cases}$$

By Hadamard's inequality, for $1 \leqslant i \leqslant l$ we have

$$\|\mathbf{L}_i T\| \cdot \|\mathbf{L}_{i+l} T\| \geqslant |\det(\mathbf{L}_i^{\mathrm{tr}} \mathbf{L}_{i+l}^{\mathrm{tr}})| = 1 - \epsilon^2$$

since $|\det T| = 1$. Thus, $\mathcal{H}(F_\epsilon \circ T) \geqslant (1-\epsilon^2)^l$ and

$$(1+\epsilon^2)^l = \mathcal{H}(F_\epsilon) \geqslant \mathfrak{m}(F_\epsilon) \geqslant (1-\epsilon^2)^l. \tag{23}$$

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Since we may choose ϵ arbitrarily small, (22) and (23) show that $V(F)$ cannot be bounded above by a function of $\mathfrak{m}(F)$ in the case where d is even and $n = 2$.

Now let $\epsilon = p^{-1/l}$ for a large prime p, for example. Then $p^2 F_\epsilon(X, Y) = (pX^l - Y^l)(X^l - pY^l) \in \mathbb{Z}[X, Y]$ is of finite type, even. Moreover, by Lemma 1, (22) and (23), we have

$$p^{-4/d} \log p << V(p^2 F_\epsilon),$$
$$p^{-4/d} >><< \mathfrak{m}(p^2 F_\epsilon)^{-2/d},$$

with absolute implicit constants. In particular,

$$V(p^2 F_\epsilon) >> \mathfrak{m}(p^2 F_\epsilon)^{-2/d} \log \mathfrak{m}(p^2 F_\epsilon).$$

This shows that the upper bound for $V(F)$ in Theorem 4 can be attained.

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