Glasgow Math. J. **49** (2007) 23–28. © 2007 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089507003382. Printed in the United Kingdom

ISOPERIMETRIC FUNCTIONS OF GROUPS ACTING ON L_δ-SPACES

JON CORSON
University of Alabama, Tuscaloosa, Alabama 35487, USA
e-mail: jcorson@bama.ua.edu

and DOHYOUNG RYANG
Talladega College, Talladega, Alabama 35160, USA
e-mail: dryang@talladega.edu

(Received 9 December, 2005; revised 18 September, 2006; accepted 22 November, 2006)

Abstract. A finitely generated group acting properly, cocompactly, and by isometries on an L_δ-metric space is finitely presented and has a sub-cubic isoperimetric function.

2000 *Mathematics Subject Classification.* 20F65.

1. Introduction. Recently the L_δ-property has been introduced by Chatterji [**2**]. In [**3**], spaces with the L_δ-property are shown to have applications to group C^*-algebras. This property is used to define L_δ-groups, which are a generalization of hyperbolic groups. The precise definitions are given in Section 2.

Hyperbolic groups are characterized as the groups with a linear Dehn function [**5**]. Elder showed that if a Cayley graph $\Gamma(G, A)$ enjoys the L_δ-property, then G has a sub-cubic Dehn function [**4**]. This suggests the following question asked by I. Chatterji and K. Ruane (Albany conference talk, 2004): If a group G acts properly, cocompactly, and by isometries on an L_δ-space, then what is a bound for the Dehn function of G?

In this paper we give an answer to this question by showing that Elder's result generalizes to groups that are quasi-isometric to an L_δ-metric space (Theorem 3.2). It should be noted that it is unknown whether or not such a group always admits a finite generating set for which the Cayley graph is an L_δ-metric space.

2. Preliminary results. Let G be a group with finite presentation $\langle A \mid R\rangle$ and let Δ be a connected graph in $\mathbb{R}^2$ whose edges are oriented and labeled by elements in A. The graph Δ is said to be a *van Kampen Diagram* for $w \in A^*$ if reading the labels around the boundary of Δ gives w, and reading the labels on the boundary of each *region* gives a relator in $R^{\pm}$. A word w has a van Kampen diagram if and only if $\overline{w} = 1$, and the *area* $\mathcal{A}(w)$ is equal to the minimum number of regions in a van Kampen diagram for w.

The function $\mathcal{D}(n) = \max\{\mathcal{A}(w) : |w|_A \le n,\ \overline{w} = 1\}$ is called the *Dehn function* for the group presentation $\langle A \mid R\rangle$. An *isoperimetric function* for this presentation is any function satisfying $\mathcal{D}(n) \le f(n)$. To make the Dehn function independent of the presentation, we define an equivalence relation on functions. The notation $f \preceq g$ means that there are positive constants A, B, C, D, E such that $f(n) \le Ag(Bn + C) + Dn + E$.

Two functions f and g are said to be *equivalent*, denoted $f \sim g$, if $f \preceq g \preceq f$. If two finitely presented groups G and H are quasi-isometric, then their Dehn functions are equivalent; see for example [**7**]. In particular, the Dehn function of G is independent of its presentations up to this equivalence.

Let (X, d) be a metric space and let $\delta \geq 0$ be a constant. A finite sequence $(x_1, x_2, \ldots, x_n)$ of points $x_1, x_2, \ldots, x_n$ in X is said to be a δ-*path*, if $d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n) \leq d(x_1, x_n) + \delta$. Choose $x, y, z \in X$. If there exists a point $t \in X$ so that the paths (x, t, y), (y, t, z), and (z, t, x) are all δ-paths, t is called a δ-*center* for a triple x, y, z. We say that a geodesic metric space (X, d) has the L_δ-*property* and call it an L_δ-*metric space*, or an L_δ-*space* for short, if every triple $x, y, z \in X$ has a δ-center in X. Of course the L_δ-property makes sense for metric spaces in general, but here we are only interested in geodesic metric spaces.

Definition 2.1 (L_δ-*group*). An L_δ-*group* is a finitely generated group G that acts properly, cocompactly, and by isometries on an L_δ-space for a constant $\delta \geq 0$.

Next we introduce the Rips graph of a geodesic metric space (X, d). Let $s > 0$ be a constant. Construct a metric graph $\Gamma_s(X)$ by requiring that $\mathcal{V}(\Gamma_s(X)) = X$ and $[x, y] \in \mathcal{E}(\Gamma_s(X))$ if and only if $0 < d(x, y) \leq s$. By d_s denote the path metric obtained by making each edge isometric to the unit interval $[0, 1]$. If γ is an edge path in $\Gamma_s(X)$, then $\ell(\gamma)$ is the number of edges in γ. That is, $\Gamma_s(X)$ is the 1-skeleton of the Rips complex for (X, d) with parameter s. It is easy to see that $(\Gamma_s(X), d_s)$ is a geodesic space. Moreover, the Rips graph is a generalization of the Cayley graph: Taking $(X, d) = (G, d_A)$, where G is a group generated by a finite set A and d_A is the corresponding word metric, $\Gamma_s(X)$ is the Cayley graph $\Gamma(G, A)$.

Lemma 2.2. *Let (X, d) be a geodesic space and $\Gamma_s(X)$ be its associated Rips graph. Then for all $s \geq 1$,*

(1) $\frac{1}{s}d(x, y) \leq d_s(x, y) < \frac{1}{s}d(x, y) + 1$ *for all $x, y \in X$,*
(2) *(X, d) and $(\Gamma_s(X), d_s)$ are quasi-isometric.*

Proof. (1) For the first inequality, let $d_s(x, y) = n$. Then there is a geodesic path $\gamma = [x_0, x_1][x_1, x_2] \ldots [x_{n-1}, x_n]$ where $\gamma(0) = x_0 = x$, $\gamma(1) = x_n = y$. Note that each $[x_i, x_{i+1}]$ is an edge in $\Gamma_s(X)$, i.e., $d_s(x_i, x_{i+1}) = 1$ and $d(x_i, x_{i+1}) \leq s$. Thus

$$d(x, y) \leq d(x_0, x_1) + d(x_1, x_2) + \ldots + d(x_{n-1}, x_n) \leq s \cdot n = s \cdot d_s(x, y).$$

For the second inequality, let γ be a geodesic path from x to y. Choose a partition $\mathcal{P} : t_0 < t_1 < \cdots < t_n$ on $[0, 1]$ where $t_0 = 0$, $t_n = 1$, $d(\gamma(t_{i-1}), \gamma(t_i)) = s$ for all $1 \leq i \leq n - 1$, and $0 < d(\gamma(t_{n-1}), \gamma(t_n)) \leq s$. Let $x_i = \gamma(t_i)$, $x = x_0$, and $y = x_n$. Then there is an edge path $[x_0, x_1][x_1, x_2] \cdots [x_{n-1}, x_n]$ in $\Gamma_s(X)$ from x to y. Thus,

$$d_s(x, y) \leq \frac{1}{s}\sum_{i=1}^{n-1} d(x_{i-1}, x_i) + 1 \leq \frac{1}{s}\ell(\gamma) + 1 = \frac{1}{s}d(x, y) + 1.$$

(2) By the above fact (1), the identity $\iota : (X, d) \to (\Gamma_s(X), d_s)$ is a quasi-isometric embedding. And every point in $\Gamma_s(X)$ is less than one edge apart from some vertex in $X \subset \Gamma_s(X)$. □

https://doi.org/10.1017/S0017089507003382 Published online by Cambridge University Press

It is an open question whether or not the L_δ-property is invariant under quasi-isometries. Nevertheless, in the next section we reduce to the case of a metric graph by way of the following lemma.

LEMMA 2.3. *If (X, d) is an L_δ-space, then $(\Gamma_s(X), d_s)$ is an $L_{\delta''}$-space where $\delta'' = \frac{\delta}{s} + 6$ and $s \geq 1$.*

Proof. First choose $x, y, z \in X \subset \Gamma_s(X)$ and let $t \in X$ be a δ-center of the triple x, y, z in (X, d). By Lemma 2.2.(1),

$$d_s(x, t) + d_s(t, y) \leq d_s(x, y) + \frac{\delta}{s} + 2.$$

Also, $d_s(y, t) + d_s(t, z) \leq d_s(y, z) + \frac{\delta}{s} + 2$ and $d_s(x, t) + d_s(t, z) \leq d_s(x, z) + \frac{\delta}{s} + 2$. So $t \in X$ is a δ'-center of the triple x, y, z in (X, d_s), and hence (X, d_s) is an $L_{\delta'}$-space for $\delta' = \frac{\delta}{s} + 2$.

Now let x, y, z be in $\Gamma_s(X)$. Choose $x', y', z' \in X = \mathcal{V}(\Gamma_s(X))$ such that $d_s(x, x') < 1$, $d_s(y, y') < 1$, and $d_s(z, z') < 1$. Let $t \in X \subset \Gamma_s(X)$ be a δ'-center for x', y', z' in (X, d_s). A simple calculation shows that

$$\begin{aligned} d_s(x, t) + d_s(t, y) &\leq d_s(x, x') + d_s(x', t) + d_s(t, y') + d_s(y', y) \\ &\leq d_s(x', y') + \delta' + 2 \\ &\leq d_s(x', x) + d_s(x, y) + d_s(y, y') + \delta' + 2 \\ &\leq d_s(x, y) + \delta' + 4. \end{aligned}$$

Similarly, $d_s(y, t) + d_s(t, z) \leq d_s(y, z) + \delta' + 4$ and $d_s(z, t) + d_s(t, x) \leq d_s(z, x) + \delta' + 4$. Take $\delta'' = \delta' + 4 = \frac{\delta}{s} + 6$. Then $t \in \Gamma_s(X)$ is a δ''-center for the triple $x, y, z \in \Gamma_s(X)$, and hence $(\Gamma_s(X), d_s)$ is an $L_{\delta''}$-space for $\delta'' = \frac{\delta}{s} + 6$. □

3. Main result.

We first observe a fact about polygons in $\mathbb{R}^2$. By a *polygon* in $\mathbb{R}^2$, we mean a simple closed curve consisting of a finite number of line segments, called *edges*. For each edge e of a polygon P, let H_e be the open half-plane on the side of the line through e determined by a P-inward pointing normal vector to e. Define the *convex core* of P by $\mathcal{C}(P) = \bigcap_{e \in \mathcal{E}(P)} H_e$. Being an intersection of half-planes, $\mathcal{C}(P)$ is convex, and in some bad cases it is empty.

Assume that $\mathcal{C}(P)$ is non-empty, and choose $c \in \mathcal{C}(P)$. Then for all $x \in P$, $[x, c] \cap P = \{x\}$, where $[x, c]$ is a straight line segment. Note, in particular, that if P is a convex polygon, then $\mathcal{C}(P)$ is the *inside* of P. The following lemma is obvious and easy to prove.

LEMMA 3.1. *Suppose that P is a polygon in $\mathbb{R}^2$ with non-empty convex core and let x, y, z be distinct vertices of P. If $c \in \mathcal{C}(P)$, then the three line segments $[x, c]$, $[y, c]$, and $[z, c]$ subdivide P into three polygons, each with non-empty convex core.*

Let (X, d) be an L_δ-space and $\Gamma_s(X)$ be the associated Rips graph with parameter $s \geq 1$. We now give a procedure for constructing a sequence of planar combinatorial graphs and combinatorial maps to $\Gamma_s(X)$ which we use in the proof of the main theorem. This is similar to the procedure used by Elder [**4**] in a Cayley graph. By Lemma 3.1, these combinatorial graphs can be constructed by vertices and *straight* edges.

https://doi.org/10.1017/S0017089507003382 Published online by Cambridge University Press

Let a convex n-gon Δ_0 in $\mathbb{R}^2$ with vertices $v_0, v_1, \ldots, v_{n-1}$ in this order and a combinatorial map $\varphi_0 : \Delta_0 \to \Gamma_s(X)$ be given. Put $x_i = \varphi_0(v_i)$. Note that $d(x_i, x_{i+1}) \leq s$ since $[x_i, x_{i+1}]$ is an edge in $\Gamma_s(X)$.

Construct Δ_1. If $n > 3\delta'' + 8$, then we subdivide Δ_0 as follows: Let $p = \lfloor \frac{n}{3} \rfloor$ and $q = \lfloor \frac{2n}{3} \rfloor$, where $\lfloor\ \rfloor$ is the greatest integer function. Then the three vertices v_0, v_p, v_q subdivide Δ_0 into three sub-paths, each of edge length less than or equal to $\lfloor \frac{n}{3} \rfloor + 1$. Let t be a δ-center for x_0, x_p, x_q in (X, d). Then by Lemma 2.3 and its proof, t is also a δ''-center for x_0, x_p, x_q in $(\Gamma_s(X), d_s)$, where $\delta'' = \frac{\delta}{s} + 6$. Since Δ_0 is convex, its convex core is non-empty. Choose a point $c \in \mathcal{C}(\Delta_0)$ so that the three line segments $[v_0, c]$, $[v_p, c]$ and $[v_q, c]$ intersect only at c. Then Δ_1 has 3 regions.

Define a map $\varphi_1 : \Delta_1 \to \Gamma_s(X)$ by requiring that $\varphi_1|_{\Delta_0} = \varphi_0$, $\varphi_1(c) = t$, and $\varphi_1[c, v_i]$ is a geodesic path in $\Gamma_s(X)$ from $t = \varphi_1(c)$ to $x_i = \varphi_1(v_i)$ where $i = 0, p, q$. Subdivide each of $[v_0, c]$, $[v_p, c]$, and $[v_q, c]$ so that φ_1 maps them combinatorially onto their images in $\Gamma_s(X)$. Define the combinatorial length $\ell(\gamma)$ of a path γ in Δ_1 to be the number of edges in γ. Then

$$\ell([v_p, c]) + \ell([c, v_q]) = d_s(x_p, t) + d_s(t, x_q) \leq d_s(x_p, x_q) + \delta'' \leq \frac{n}{3} + 1 + \delta''.$$

Similarly, $\ell([v_0, c]) + \ell([c, v_p]) \leq \frac{n}{3} + 1 + \delta''$ and $\ell([v_0, c]) + \ell([c, v_q]) \leq \frac{n}{3} + 1 + \delta''$. So the combinatorial perimeter of each region of Δ_1 is bounded by

$$\frac{n}{3} + 1 + \frac{n}{3} + 1 + \delta'' = \frac{2n}{3} + 2 + \delta''.$$

Recall that $n > 3\delta'' + 8$ or $\delta'' < \frac{n-8}{3}$. Thus it is shorter than $\frac{2n}{3} + 2 + \frac{n-8}{3} = n - \frac{2}{3} < n$. That is, the combinatorial perimeter of each new region in Δ_1 is strictly shorter than the combinatorial perimeter of Δ_0.

Repeat this trisection process on each region in Δ_1 whose perimeter is greater than $3\delta'' + 8$ to construct Δ_2 and $\varphi_2 : \Delta_2 \to \Gamma_s(X)$. Thus, the number of regions in Δ_2 is less than or equal to 3^2 and the combinatorial perimeter of each new region in Δ_2 is bounded by

$$\frac{2}{3}\left(\frac{2n}{3} + 2 + \delta''\right) + 2 + \delta'' = \left(\frac{2}{3}\right)^2 n + \frac{2}{3}(2 + \delta'') + (2 + \delta'').$$

Choose k so that $(\frac{3}{2})^k \leq n < (\frac{3}{2})^{k+1}$. After k iterations, we have Δ_k and $\varphi_k : \Delta_k \to \Gamma_s(X)$. Thus Δ_k has at most 3^k regions, and the combinatorial perimeter of each region in Δ_k is at most

$$\left(\frac{2}{3}\right)^k n + \left(\frac{2}{3}\right)^{k-1}(\delta'' + 2) + \cdots + \frac{2}{3}(2 + \delta'') + (2 + \delta'')$$

which is bounded by $(\frac{2}{3})^k(\frac{3}{2})^{k+1} + \frac{\delta''+2}{1-\frac{2}{3}} < 3\delta'' + 8$. In particular, our procedure terminates after at most k steps.

THEOREM 3.2. *If a finitely generated group G is quasi-isometric to an L_δ-space for some $\delta \geq 0$, then G is finitely presented and has a sub-cubic Dehn function.*

Proof. Let A be a finite generating set for G which is inverse closed and use the word metric d_A for G. Suppose (G, d_A) is quasi-isometric to an L_δ-space (X, d). Choose

https://doi.org/10.1017/S0017089507003382 Published online by Cambridge University Press

quasi-isometries $\alpha : G \to X$ and $\beta : X \to G$ such that for all $g \in G$ and $x \in X$, $d_A(g, (\beta \circ \alpha)(g)) \leq C$ and $d(x, (\alpha \circ \beta)(x)) \leq C$, where C is a constant. We may assume that α and β are both (λ, ε)-quasi-isometries with the same constants $\lambda \geq 1$ and $\varepsilon \geq 0$.

Choose w in A^* so that $w = a_1 a_2 \ldots a_n$ where $a_i \in A$ and $\overline{w} = 1$. Let $g_i = \overline{a_1 \ldots a_i} \in G$. We want to construct a van Kampen diagram for w. Start with a convex n-gon Δ_0 in $\mathbb{R}^2$ with vertices $v_0, \ldots, v_{n-1}$ in this order.

Put $s = \lambda + \varepsilon$ and define $\varphi_0 : \Delta_0 \to \Gamma_s(X)$ by $\varphi_0(v_i) = \alpha(g_i) = x_i$, say. Note that $[x_i, x_{i+1}]$ is an edge in $\Gamma_s(X)$, since

$$d(x_i, x_{i+1}) = d(\alpha(g_i), \alpha(g_{i+1})) \leq \lambda d_A(g_i, g_{i+1}) + \varepsilon = \lambda + \varepsilon = s.$$

So the map φ_0 is a closed path in $\Gamma_s(X)$ with combinatorial length n.

If $n = |w|_A$ is greater than $3\delta'' + 8$, then trisect Δ_0 to construct Δ_1 and $\varphi_1 : \Delta_1 \to \Gamma_s(X)$. Iterate the trisection process for each region whose combinatorial perimeter is greater than $3\delta'' + 8$ until all regions have perimeter shorter than $3\delta'' + 8$. Suppose this is achieved after k-iteration. Thus we have Δ_k and $\varphi_k : \Delta_k \to \Gamma_s(X)$, where Δ_k has at most 3^k regions and $(\frac{3}{2})^k \leq n < (\frac{3}{2})^{k+1}$.

In order to get an n-gon which is mapped to a closed path of A-length n in $\Gamma(G, A)$, we inflate Δ_k a bit to form the graph Δ. We put n vertices on the outside of Δ_k labeled by $y_0, \ldots, y_{n-1}$ and put $2n$ edges $[v_i, y_i]$ and $[y_i, y_{i+1}]$ where $i = 0, 1, \ldots, n - 1 \bmod n$. Thus Δ has n regions outside of Δ_k.

Define a map $\varphi : \Delta \to \Gamma(G, A)$ as follows: (1) φ is the composition $\mathcal{V}(\Delta_k) \xrightarrow{\varphi_k} X \xrightarrow{\beta} G \hookrightarrow \Gamma(G, A)$; (2) $\varphi(y_i) = g_i$; and (3) $\varphi([u, v])$ is a geodesic path from $\varphi(u)$ to $\varphi(v)$ in $\Gamma(G, A)$, for $[u, v] \in \mathcal{E}(\Delta)$. Then φ maps $\partial\Delta$ to a closed path labeled by w in $\Gamma(G, A)$ of A-length n.

We now show that each region in $\Gamma(G, A)$ has a perimeter bounded by a constant. If $[u, v] \in \mathcal{E}(\Delta_k)$, then

$$|\varphi([u, v])|_A = d_A((\beta \circ \varphi_k)(u), (\beta \circ \varphi_k)(v)) \leq \lambda d(\varphi_k(u), \varphi_k(v)) + \varepsilon \leq \lambda s + \varepsilon.$$

Recall that the combinatorial perimeter of each region of Δ_k is bounded by $3\delta'' + 8$. So, for each region D in Δ_k, $\varphi(\partial D)$ is a closed path in $\Gamma(G, A)$ of length at most $(3\delta'' + 8)(\lambda s + \varepsilon)$. And for each outer region M in $\Delta \setminus \Delta_k$, $\varphi(\partial M) = \varphi[v_i, v_{i+1}]\varphi[v_{i+1}, y_{i+1}]\varphi[y_{i+1}, y_i]\varphi[y_i, v_i]$ in $\Gamma(G, A)$, and

$$d_A(\varphi(y_i), \varphi(y_{i+1})) = 1; \; d_A(\varphi(v_i), \varphi(y_i)) \leq C; \; d_A(\varphi(v_i), \varphi(v_{i+1})) \leq \lambda s + \varepsilon.$$

So, $\varphi(\partial M)$ is the closed path in $\Gamma(G, A)$ of length at most $\lambda s + \varepsilon + 2C + 1$.

Let $K = \max\left\{(\lambda s + \varepsilon)(3\delta'' + 8), \lambda s + \varepsilon + 2C + 1\right\}$. Then the perimeter of every region in $\Gamma(G, A)$ is bounded by K, whence $\{w \in A^* \mid \overline{w} = 1 \text{ and } |w|_A \leq K\}$ is a finite set of defining relators for G, and so G is finitely presented.

The area $\mathcal{A}(w)$ is at most $3^k + n$. Remember $\left(\frac{3}{2}\right)^k \leq n$ or $k \leq \log_{1.5} n$. Hence

$$\mathcal{A}(w) \leq 3^k + n \leq 3^{\log_{1.5} n} + n = n^{\log_{1.5} 3} + n \sim n^{\log_{1.5} 3}.$$

If $|w|_A = n \leq 3\delta'' + 8$, then w is a relator so again $\mathcal{A}(w) = 1 \leq n^{\log_{1.5} 3}$. $\square$

We obtain the following statement which is an answer to the question posed in the introduction.

https://doi.org/10.1017/S0017089507003382 Published online by Cambridge University Press

COROLLARY 3.3. *If a group G acts properly, cocompactly, and by isometries on an L_δ-space for some $\delta \geq 0$, then G is finitely presented and has a sub-cubic Dehn function.*

Proof. Suppose G acts properly, cocompactly, and by isometries on an L_δ-space X. By the Švarc-Milnor Theorem [**1**, Proposition 8.19], G is finitely generated and quasi-isometric to X. The result follows from Theorem 3.2. □

REFERENCES

1. M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature* (Springer-Verlag, 1999).

2. I. Chatterji, *On property RD for certain discrete groups*, Ph.D. Thesis, 2001. available at www.math.ohio-state.edu/∼indira.

3. I. Chatterji and K. Ruane, Some geometric groups with rapid decay, *Geometric and Functional Analysis* **15** (2005), 311–339.

4. M. Elder, L_δ groups are almost convex and have a sub-cubic Dehn function, *Algebraic and Geometric Topology* **4** (2004), 23–29.

5. M. Gromov, Hyperbolic groups, in *Essays in Group Theory* (S. M. Gersten, editor), (Springer-Verlag, 1987), 75–263.

6. P. de la Harpe, *Topics in geometric group theory* (The University of Chicago Press, 2000).

7. M. Neumann and M. Shapiro, *A short course in geometric group theory* (Notes for the ANU Workshop, 1996).

https://doi.org/10.1017/S0017089507003382 Published online by Cambridge University Press