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THE COMPLEXITY OF NOWHERE DIFFERENTIABLE CONTINUOUS FUNCTIONS

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Introduction. It was not always clear that there could exist a continuous function which was differentiable at no point. (Such functions are now known as nowhere differentiable continuous functions. By "differentiable" we mean having a finite derivative.) In fact in 1806 M. Ampere [**2**] even tried to show that no such function could exist but his reasonings were later discovered to be fallacious. Of the early attempts at constructing a nowhere differentiable continuous function mention must be made of B. Bolzano. In a manuscript dated around 1830, (see [**21**]) he constructed a continuous function on an interval and showed that it was not differentiable on a dense set of points. (It was later shown by K. Rychlik [**21**] that this function was in fact nowhere differentiable.)

Around 1873 K. Weierstrass gave the first legitimate example of a nowhere differentiable continuous function. This discovery was published by Du Bois-Reymond [**6**] and prior to this no such function was ever published. Another example published in 1890 (see [**5**]) was thought to have been discovered by C. Cellerier as early as 1850 but of this there is much doubt. Also a function studied by B. Riemann around 1860 and very often thought of as being nowhere differentiable turned out to be differentiable at certain points (see [**7**], [**8**] or [**25**]). So Weierstrass holds sole claim for the first discovery.

Later many more examples of nowhere differentiable continuous functions were constructed and it became fashionable to ask that more stringent requirements be satisfied (for instance, instead of being nowhere differentiable, the function might be required to have no derivative, finite or infinite). In 1925 A. Besicovitch [**4**] constructed a continuous function with no one-sided derivative, finite or infinite. Such functions are called Besicovitch functions in honour of their discoverer. There was, however, much controversy about Besicovitch's example because the construction was rather complicated and the reasoning could not be readily followed. E. D. Pepper [**19**] later examined the same example but there were still doubts in the minds of some as to the existence of such functions. These doubts were put to rest by A. P. Morse [**17**] who gave an example which satisfied even more stringent requirements than those of Besicovitch functions.

The studies on nowhere differentiable continuous functions took a different twist when in 1931 S. Mazurkiewicz [**16**] showed that the set of all such functions is a comeager subset of the set of all continuous functions of period 1. At about the same time Banach [**3**] found the same result except that in this case the functions were defined on the set [0,1]. So in the sense of Baire Category the continuous functions which are not nowhere differentiable are exceptional. This

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provided an abstract proof of the existence of nowhere differentiable continuous functions. A little later S. Saks [**22**] showed that the set of all Besicovitch functions was a meager subset of the set of all continuous functions. So we cannot get an abstract existence proof as before. However J. Malý [**13**] recently showed that the Besicovitch functions were co-meager in a certain restricted class of continuous functions, thus retrieving the situation.

Let C be the Polish space of all real valued continuous functions on [0,1] with the metric obtained from the sup norm. Let ND and BF be the sets of all nowhere differentiable functions and Besicovitch functions in C respectively. It is easy to show that ND and BF are coanalytic subsets of C. Mauldin [**14**], [**15**] showed that ND was not a Borel subset, and in a communication with Kechris (see [**9**]) indicated that he also had a proof that BF was not Borel. Kechris [**9**] later showed that ND and BF were both complete coanalytic subsets (and hence they can't be Borel).

In this paper we shall investigate a natural rank function on ND, the definition of which is essentially due to Kechris and Woodin [**10**]. With each f in ND we associate a well-founded tree $T(f)$ and show that the height of $T(f)$ is always a limit ordinal. The rank of f is then defined as the unique ordinal such that the height of $T(f)$ is $\omega.\alpha$. This rank function provides a natural measure of the complexity of the functions in ND in the sense that functions of small rank are easily seen to be nowhere differentiable and vice versa. In fact the functions of rank 1 are precisely the set BC of the Banach functions in C. (A Banach function is one such that at each point, at least one of the Dini derivates is infinite. What Banach [**3**] had essentially shown in his proof of the comeagerness of ND was that BC was comeager, hence the name.) All of the classical examples of functions in ND also turn out to have rank 1 or 2.

We show that the rank function has the definability properties summarized in the concept of a coanalytic norm and that the it is unbounded in ω_1 on BF. (With a little more effort we can actually show that for each $\alpha < \omega_1$ there is an f in BF with rank exactly α.) By using the Boundedness theorem [**18**, p. 196] we obtain proofs of the non-Borelness of BF and ND that are different from those of Mauldin and Kechris. Finally we give an alternative definition of the rank function by associating a transfinite sequence of nested closed sets with each f in ND. The rank of f is defined there as the least ordinal for which the sequences stabilizes at the empty set (in much the same spirit as the Cantor-Bendixson analysis of closed sets).

1. Preliminaries. *Coanalytic subsets and coanalytic norms.* A *Polish space* is a complete, separable metric space and in this section X will always be a Polish space. A subset A of X is a *Borel subset* of X if it belongs to the smallest σ-algebra of subsets of X which contains all the open sets of X. Let Y be a Polish space and $f : Y \longrightarrow X$ be a function. We say that f is *Borel measurable* if for each open set E in X, the set of $f^{-1}[E]$ is a Borel subset of Y. A subset A of X is an *analytic subset* of X if there exists a Polish space Y and a Borel subset B of $X \times Y$ such that A is the projection of B onto X (i.e.,

$A = \{x \in X : \quad (\exists y \in Y)(\langle x, y \rangle \in B)\}$). A subset A of X is a coanalytic *subset* of X if its complement $X - A$ is an analytic subset of X.

A coanalytic subset A of X is said to be *complete* if for any Polish space Y and coanalytic subset B of Y, there is a Borel measurable function $f : Y \longrightarrow X$ such that $y \in B$ if and only if $f(y) \in A$. Since the Polish space of real numbers has a coanalytic subset which is not Borel it follows that no complete coanalytic subset of X can be Borel.

A *norm* on a set $A \subseteq X$ is just a map $\varphi : A \longrightarrow \mathrm{ORD}$, where ORD is the class of all ordinals. (A norm is also referred to as a *rank function*.) The map induces a pre-wellordering $\leqq_\varphi$ on A which is defined by

$$x \leqq_\varphi y \Leftrightarrow \varphi(x) \leqq \varphi(y).$$

Two norms are said to be equivalent if they induce the same pre-wellordering. Now let A be a coanalytic subset of X. A norm $\varphi : A \longrightarrow \mathrm{ORD}$ is a *coanalytic norm* if there are analytic and coanalytic subsets B and C of X^2 such that

$$y \in A \Rightarrow (\forall x)\big[\{x \in A \& \varphi(x) \leqq \varphi(y)\} \Leftrightarrow \langle x, y \rangle \in B \Leftrightarrow \langle x, y \rangle \in C\big].$$

It is known that every coanalytic subset has at least one coanalytic norm. Moreover this coanalytic norm is always equivalent to one which takes values in ω_1 (see [**18**]).

For the sake of convenience we list the main descriptive set theoretic results that we will need.

Proposition A. (Boundedness theorem) [**18** p. 196] *Suppose A is a coanalytic subset of X and $\varphi : A \longrightarrow \omega_1$ is a coanalytic norm. Then A is Borel if an only if $\varphi[A]$ is countable.*

From Proposition A we immediately see that to show A is not Borel, it will suffice to show that φ is unbounded in ω_1 on A. Such a proof of the non-Borelness of A is usually referred to as a *rank argument.* There is a slight extension of this rank argument which depends on the following result.

Proposition B. [**10**] *Suppose A is a coanalytic subset of X and $\varphi : A \longrightarrow \omega_1$ is a norm on A such that*

(i) *there is an analytic subset of B of X^2 such that for all $x, y \in A$ we have $\varphi(x) < \varphi(y)$ if and only if $\langle x, y \rangle \in B$, and*

(ii) *φ is unbounded in ω_1 on A. Then A is not a Borel subset of X.*

Proposition C. [**10**] *Let X, Y be Polish spaces and $A \subseteq X, B \subseteq Y$ be coanalytic subsets. If $\varphi : B \longrightarrow \omega_1$ is a coanalytic norm and there is a Borel measurable function $f : X \longrightarrow Y$ with $f^{-1}[B] = A$ then the map $\psi : A \longrightarrow \omega_1$ defined by $\psi(x) = \varphi(f(x))$ is also a coanalytic norm.*

Well-founded trees and their heights. Let A be any non-empty set. We define A^* to be the set of all finite sequences from A (including the empty sequence $\emptyset$).

A *tree* on A is any subset T of A^* such that for all $a_1, \dots, a_{n+1}$ in A, $\langle a_1, \dots, a_n \rangle$ is in T whenever $\langle a_1, \dots, a_n, a_{n+1} \rangle$ is in T. The elements of T are called *nodes*. By definition $\emptyset$ is always a node of any non-empty tree. We call $\emptyset$ the *root* of the tree. A subset of S and T which is also a tree on A is called a *subtree* of T.

Let $\mathbf{u}$ and $\mathbf{v}$ be finite sequences from A and let T be a tree on A. We denote the concatenation of $\mathbf{u}$ and $\mathbf{v}$ by $\mathbf{u}\hat{}\mathbf{v}$. We define $T[\mathbf{u}]$ to be the set of all finite sequences $\mathbf{v}$ in A such that $\mathbf{u}\hat{}\mathbf{v}$ is in T. It is easy to verify that $T[\mathbf{u}]$ is always a tree. (Note that if $\mathbf{u}$ is not in T then $T[\mathbf{u}]$ is empty.) When $\mathbf{u}$ is in T we shall refer to $T[\mathbf{u}]$ as the *tree at the node* $\mathbf{u}$ in T. A tree T on A is said to be *well-founded* provided there is no sequence $\langle a_n \rangle$ $(n = 1, 2, 3, \dots)$ from A such that $\langle a_1, \dots, a_n \rangle$ is in T for each n.

Let T be a well-founded tree on A. We define a sequence of trees as follows. Put $T^0 = T$, $T^\lambda = \cap\{T^\alpha : \alpha < \lambda\}$ for λ a limit ordinal, and

$$T^{\alpha+1} = \{\mathbf{v} \in T : (\exists a \in A)(\mathbf{v}\hat{}\langle a \rangle \in T\}.$$

Observe that the sequence $\langle T^\alpha \rangle$ is strictly decreasing so for sufficiently large α, T^α is empty. Note also that if T is non-empty, then the least α such that T^α is empty must be a successor ordinal. We define the *height* $|T|$ of the tree T be the least α such that $T^{\alpha+1}$ is empty. (If T is the empty tree we adopt the convention that $|T| = -1$ and if T is not well-founded we put $|T| = \infty$.) If $\mathbf{v}$ is a node in T we define the *rank* $|\mathbf{v}; T|$ of $\mathbf{v}$ in T to be the height of the tree at $\mathbf{v}$ in T (i.e., $|\mathbf{v}; T| = |T[\mathbf{v}]|$). It is easy to see that for any finite sequence $\mathbf{u}$ from A that

$$T^\alpha[\,\mathbf{u}\,] = (T[\,\mathbf{u}\,])^\alpha \quad \text{for each } \alpha.$$

Using this we get

$$|T| = \sup\{|\mathbf{v}; T| + 1 : \mathbf{v} \in T, \mathbf{v} \neq \emptyset\}.$$

It turns out that the height is a coanalytic norm if we view the set of all well-founded trees on $P = \{1, 2, 3, \dots\}$ as a subset of a certain Polish space. A tree on P is a subset of P^* and so can be identified with its characteristic function

$$\chi_T : P^* \longrightarrow \{0, 1\} = 2.$$

So a tree on P can be viewed as an element of the Polish space $F(P^*, 2)$ of all functions from P^* to 2. Let WF be the set of all well-founded trees on P viewed as a subset of $F(P^*, 2)$. Then we have the following precise result.

PROPOSITION D. **[10]** WF *is a coanalytic subset of* $F(P^*, 2)$ *and* $|\cdot| : \mathrm{WF} \longrightarrow \omega_1$ *is a coanalytic norm.*

2. Basic properties of the rank function. In this section we define the rank function and investigate its basic properties. But first some notation. Let f be in

C and I be any interval with endpoints a and b (where $a \neq b$). We define the *difference quotient* of f on I by

$$\Delta f(I) = (f(b) - f(a))/(b - a).$$

PROPOSITION 1. ND *and* BF *are coanalytic subsets of* C.

Proof. It will suffice to show that $C-$ ND and $C-$ BF are analytic. Now $C-$ ND is the set of all f in C such that there exists an x such that f is differentiable at x. But f is differentiable at x if and only iff for all n there is an m such that

(*) for all h_i, h_2 with $0 < |h_1|, h_2| < 1/m$ and $x+h_1, x+h_2$ in [0, 1] we have

$$|\Delta f(x+h_1, x) - \Delta f(x+h_2, x)| \leqq 1/n.$$

Let $E(n,m)$ be the set of all $\langle f, x\rangle$ such that (*) holds. Then it is easy to see that $E(n,m)$ is closed and consequently $\bigcap_n \bigcup_m E(n,m)$ is Borel. $C-$ ND is the projection of this Borel set onto C and so it is analytic.

Also $C-$ BF is the set of all f in C such that there exists an x such that f has a one-sided derivative (possible infinite) at x. Now this is so if and only if for all n there is an m such that

(**) $\exists a \in \{-1, 1\}$ such that $\forall h_1, h_2$ with $0 < h_1, h_2 < 1/m$ and $x + ah_1$, $x + ah_2$ in [0,1] we have

$$|\Delta f(x+ah_1, x) - \Delta f(x+ah_2, x)| \leqq 1/n,$$

or $\forall h$ with $0 < h < 1/m$ with $x + ah$ in [0,1]

$$\Delta f(x+ah_1, x) \geqq n,$$

or $\forall h$ with $0 < h < 1/m$ with $x + ah$ in [0,1]

$$\Delta f(x+ah_1, x) \leqq -n.$$

Let $F(n,m)$ be the set of all $\langle f, x\rangle$ such that (**) holds and proceed as above to conclude that $C-$ BF is analytic.

Let $R[0, 1]$ be the set of all nonempty intervals which are open with respect to the topology of $[0, 1]$. Let also $Q[0, 1]$ be the set of all intervals in $R[0, 1]$ which have rational endpoints. If I is an interval we denote its closure by $\bar{I}$ and its length by $|I|$.

Definition Let f be in C and $M > 0$. We define the tree $T(M,f)$ on $Q[0, 1]$ as follows. $\langle I_1, \ldots, I_n\rangle$ is in $T(M,f)$ if and only if

(i) $I_1 = [0, 1]$, $\bar{I}_{i+1} \subseteq I_i$, $|I_i| \leqq 1/i$ and

(ii) for all K, L in $R[0,1]$ with $I_n \subseteq K, L \subseteq I_i$ we have

$$|\Delta f(K) - \Delta f(L)| \leqq M/i.$$

We define the tree $T(f)$ by

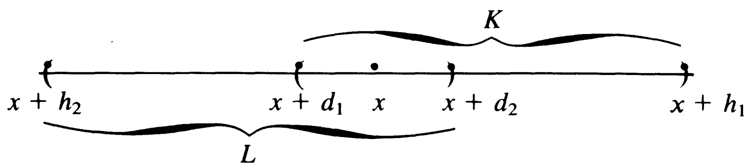
$$T(f) = \{\emptyset\} \cup \bigcup_N \{\langle N\rangle^\frown \mathbf{u} : \mathbf{u} \text{ is in } \mathrm{T(N, f)}\}.$$

PROPOSITION 2. *f is in* ND *if and only if $T(f)$ is well-founded.*

Proof. "$\Rightarrow$" It will suffice to show that $T(M,f)$ is well-founded for each $M > 0$. Suppose $T(M,f)$ is not well-founded. Then there is an infinite sequence $\langle I_n\rangle$ such that $\langle I_1, \ldots, I_n\rangle$ is in $T(M,f)$ for each $n \geqq 1$. Take x in $\cap\{\bar{I}_n : n \geqq 1\}$. We shall show that f is differentiable at x. Fix m and let $h_1, h_2 \neq 0$ be such that $x + h_1, x + h_2$ are in I_m. Then $0 < |h_1|, |h_2| < 1/m$. Let also K, L be intervals in $R[0,1]$ with endpoints $x + h_1$, $x + d_1$ and $x + h_2$, $x + d_2$ (respectively) where d_1 and d_2 are chosen so that

$$x \in K, L \subseteq I_m, \quad |d_1| < |h_1|, |d_2| < |h_2| \quad \text{and}$$

$$(h_1 - d_1)/h_1 = (h_2 - d_2)/h_2.$$



Then

$$\left|\frac{f(x+h_1) - f(x)}{h_1} - \frac{f(x+h_2) - f(x)}{h_2}\right|$$

$$\leqq \left|\frac{h_1 - d_1}{h_1} \cdot \frac{f(x+h_1) - f(x+d_1)}{h_1 - d_1}\right.$$

$$\left. - \frac{h_2 - d_2}{h_2} \cdot \frac{f(x+h_2) - f(x+d_2)}{h_2 - d_2}\right|$$

$$+ \left|\frac{f(x+d_1) - f(x)}{h_1} - \frac{f(x+d_2) - f(x)}{h_2}\right|$$

$$\leqq \left|\frac{f(x+d_1) - f(x)}{h_1}\right| + \left|\frac{f(x+d_2) - f(x)}{h_2}\right|$$

$$+ |\Delta f(K) - \Delta f(L)| \cdot |(h_1 - d_1)/h_1| \cdot$$

Now the first two terms tend to 0 as d_1 tends to 0 by the continuity of f. Also $(h_1 - d_1)/h_1$ tends to 1 as d_1 tends to 0 and

$$|\Delta f(K) - \Delta f(L)| \leqq M/m$$

from the definition of the tree $T(M,f)$. So

$$\left|\frac{f(x+h_1)-f(x)}{h_1} - \frac{f(x+h_2)-f(x)}{h_2}\right| \leqq \frac{M}{m}.$$

But x is an interior point of I_M and (*) is true for each m. From this it follows that f is differentiable at x.

"$\Leftarrow$" Suppose now that f is not in ND. Choose x_0 in [0,1] such that f is differentiable at x_0. Then there is a positive integer c such that $f(x)$ always lies between $c.(x-x_0)$ and $-c.(x-x_0)$. Take $M = 2c$. Since f is differentiable at x_0, for all n there exists positive integers $m(n)$ such that for all h with $0 < |h| < 1/m(n)$ and $x_0 + h$ in [0,1] we have

$$|\Delta f(x_0 + h, x_0) - f'(x_0)| \leqq M/2n.$$

We may assume without loss of generality that $m(n)$ is strictly increasing. Let

$$p_n = \max\{0, x_0 - 1/2m(n)\} \quad \text{and} \quad q_n = \min\{1, x_0 + 1/2m(n)\}.$$

Take $I_1 = [0,1]$ and for each $n \geqq 2$ choose I_n in $Q[0,1]$ such that $I_n \subseteq [p_n, q_n]$ and $\bar{I}_n \subseteq I_{n-1}$. Then

$$|I_n| \leqq q_n - p_n \leqq 1/m(n) \leqq 1/n.$$

So for each n, $\langle I_1, \dots, I_n \rangle$ satisfies condition (i) of the definition of $T(M,f)$. We will now show that condition (ii) is also satisfied.

Fix n. Since x_0 is in I_n it will suffice to show that for all K, L in $R[0,1]$ with $x_0 \in K, L \subseteq I_i$ we have

$$|\Delta f(K) - \Delta f(L)| \leqq M/i.$$

For $i = 1$ we have

$$|\Delta f(K) - \Delta f(L)| \leqq |\Delta f(K)| + |\Delta f(L)| \leqq c + c = M/1$$

as required. For $i \geqq 2$ let $\{a,b\}$ and $\{c,d\}$ be the endpoints of K and L respectively. Then

$$\begin{aligned}
|\Delta f(K) - \Delta f(L)| &= \left|\frac{f(b)-f(a)}{b-a} - \frac{f(d)-f(c)}{d-c}\right| \\
&\leqq \left|\frac{f(b)-f(a)}{b-a} - f'(x_0)\right| + \left|\frac{f(d)-f(c)}{d-c} - f'(x_0)\right| \\
&\leqq \left|\frac{f(a)-f(x_0)}{a-x_0} - f'(x_0)\right| + \left|\frac{f(b)-f(x_0)}{b-x_0} - f'(x_0)\right| \\
&\quad + \left|\frac{f(c)-f(x_0)}{c-x_0} - f'(x_0)\right| + \left|\frac{f(d)-f(x_0)}{d-x_0} - f'(x_0)\right| \\
&\leqq M/4i + M/4i + M/4i + M/4i = M/i.
\end{aligned}$$

So $\langle I_1, \ldots, I_n \rangle$ is in $T(M,f)$ for each $n \geqq 1$. Thus $T(M,f)$ is not well founded and consequently $(T)f$ is not well-founded.

Our next aim is to show that the height of $T(f)$ is always a limit ordinal. But first some definitions and two lemmas.

Definition. Let I be in $R[0,1]$ and T be a tree on $R[0,1]$. We define the subtree $T \upharpoonright I$ to be set of all $\langle I_1, I_2, \ldots, I_n \rangle$ in T with $I_2 \subseteq I$.

Let f be in ND and $M > 0$. For each x in $[0,1]$ we define

$$|T(M,f) : x| = \min\{|T(M,f)\upharpoonright I| : x \in I \in R[0,1]\}.$$

LEMMA 3. *If* $|T(M,f)| \geqq \omega.\alpha$ *then there is an* x *in* $[0,1]$ *such that*

$$|T(M,f) : x| \geqq \omega.\alpha.$$

Proof. We will find a sequence of nested closed intervals $\langle L_n \rangle$ with $1/n \leqq |L_n| \leqq 2/n$ such that

$$|T(M,f)|L_n| \geqq \omega.\alpha \quad \text{for each } n \geqq 1.$$

Choosing x in $\cap\{L_n : n \geqq 1\}$ will then give

$$|T(M,f) : x| \geqq \omega.\alpha.$$

We construct L_n by induction on n. Take $L_1 = [0,1]$. Given L_n choose closed intervals L' and L'' such that

$$L' \cup L'' = L_n; \quad |L'|, |L''| < 2/n+1 \quad \text{and} \quad |L' \cap L''| \geqq 1/n+1.$$

Now observe that if $\langle I_1, \ldots, I_{n+1}, \ldots, I_k \rangle$ is in $T(M,f)$ then $I_{n+1} \subseteq L'$ or $I_{n+1} \subseteq L''$. So $\langle I_1, I_{n+1}, \ldots, I_k \rangle$ is in $T(M,f)\upharpoonright L'$ or $T(M,f)\upharpoonright L''$. We claim that at least one of the latter two trees have height $\geqq \omega.\alpha$. Suppose that $|T(M,f)\upharpoonright L'|$ and $|T(M,f)\upharpoonright L''|$ are both $\leqq \beta < \omega.\alpha$. Then

$$\begin{aligned}
|T(M,f)\upharpoonright L_n| &\leqq \sup\{|\mathbf{u}; T(M,f)\upharpoonright L_n| + 1 : |\mathbf{u} = n+1\} + n \\
&\leqq \max\{\sup\{|\mathbf{v}; T(M,f)\upharpoonright L'| + 1 : |\mathbf{v}| = 2\} + n, \\
&\qquad \sup\{|\mathbf{v}; T(M,f)\upharpoonright L''| + 1 : |\mathbf{v}| = 2\} + n\} \\
&\leqq \max\{|T(M,f)\upharpoonright L'|, |T(M,f)\upharpoonright L''|\} + n \\
&\leqq \beta + n < \omega.\alpha,
\end{aligned}$$

which contradicts the induction hypothesis. Take L_{n+1} to be L' if $|T(M,f)L'| \geqq \omega.\alpha$ and to be L'' otherwise. This completes the induction step.

Definition. Let T be a tree on $Q[0,1]$ and $k \geqq 1$. We define the subtree $[T]_k$ to be the set of all $\langle I_1, I_2, \dots I_n \rangle$ in T such that there exists $J_2, \dots, J_k$ in $Q[0,1]$ such that the sequence $\langle I_1, J_2, \dots, J_k, I_2, \dots, I_n \rangle$ is also in T.

LEMMA 4. *Suppose* $|T(M,f)\restriction I| \geqq \omega . \alpha$. *Then for each* $k \geqq 1$,

$$|[T(M,f)\restriction I]_k| \geqq \omega . \alpha .$$

Proof. Observe from the definition of $T(M,f)$ that $\langle I_1, I_2, \dots, I_n \rangle \in T(M,f)$ implies $\langle I_1, J_2, \dots, J_m \rangle$ is in $T(M,f)$ for any subsequence $\langle J_2, \dots, J_m \rangle$ of $\langle I_2, \dots, I_n \rangle$. Now

$$\sup\{|\mathbf{v}; [T(M,f)\restriction I]_k| : |\mathbf{v}| = 2\} + k + 1$$
$$\geqq |\langle I_1 \rangle ; T(M,f)\restriction I| + 1 = |T(M,f)\restriction I| \geqq \omega . \alpha .$$

So

$$\sup\{|\mathbf{v}; [T(M,f)\restriction I]_k| : |\mathbf{v}| = 2\} \geqq \omega . \alpha$$

and hence

$$|[T(M,f)\restriction I]_k| \geqq \omega . \alpha .$$

PROPOSITION 5. *For each* f *in* ND *the height of* $T(f)$ *is a limit ordinal.*

Proof. It will suffice to show that $|T(f)| \geqq \omega . \alpha + 1$ implies

$$|T(f)| \geqq \omega . (\alpha + 1).$$

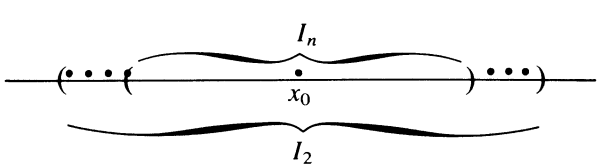
Suppose $|T(f)| \geqq \omega . \alpha + 1$ then for some integer M we must have

$$|(T(M,f)| \geqq \omega . \alpha .$$

By Lemma 3 take x_0 in [0,1] such that

$$|T(M,f) : x_0| \geqq \omega . \alpha .$$

Fix $N \geqq 1$ and choose $I_2, \dots, I_n$ in $Q[0,1]$ such that $|I_i| \leqq 1/i, \bar{I}_i \subseteq I_{i-1}$ and x_0 is in I_n. (Here as always $I_1 = [0,1]$.)



Now define the tree T_n as follows. $\langle I_1, \dots I_i \rangle$ is in T_n for each $i = 1, \dots, N-1$ and for each $\langle I_1, J_n, \dots, J_k \rangle$ in $[T(M,f)\restriction I_n]_n$ we let $\langle I_1, \dots, I_{n-1}, J_n, \dots, K_k \rangle$ be in T_n. Then it is easy to see that T_n is a subtree of $T(MN,f)$. But

$$|\langle I_1 \rangle ; T_n| \geqq |\langle I_1 \rangle : [T(M,f)\restriction I_n]_n| + (N-2)$$
$$\geqq \omega . \alpha + (N-2)$$

by Lemma 4. Thus

$$|T(MN,f)| \geqq \omega.\alpha + (N-2)$$

and so

$$\begin{aligned}|T(f)| &= \sup\{|T(MN,f)| + 1 : N \geqq 1\}\\ &\geqq \sup\{\omega.\alpha + (N-1) : N \geqq 1\} = \omega.(\alpha+1).\end{aligned}$$

Definition . For each f in ND we define the rank $|f|$ of f to be the unique ordinal α such that $|T(f)| = \omega.\alpha$.

Since $Q[0,1]$ is countable, $T(f)$ is a countable tree, so $|f| < \omega_1$.

PROPOSITION 6. $|\cdot| : ND \longrightarrow \omega_1$ *is a coanalytic norm.*

Proof. It will suffice to show that the function h defined by $h(f) = |T(f)|$ is a coanalytic norm. Let $g : Q[0,1] \longrightarrow P$ (where $P = \{1,2,3,\dots\}$) be a Borel measurable bijection. Let $s(f)$ be the tree consisting of all $\langle N, g(I_1), \dots, g(I_n)\rangle$ such that $\langle N, I_1, \dots, I_n\rangle$ is in $T(f)$. Then $s : C \longrightarrow F(P^*,2)$ is a Borel measurable map (the tree $s(f)$ is viewed here as an element of the Polish space $F(P^*,2)$). Also

$$\mathrm{ND} = s^{-1}[\mathrm{WF}] \quad \text{and} \quad h(f) = |s(f)|.$$

So it follows from Propositions C and D that h is a coanalytic norm.

Definition . Let f be in C. We define the *amplitude* of the difference quotient of f at x by

$$A(f;x) = \limsup_{h_1,h_2 \to 0} \left| \frac{f(x+h_1) - f(x)}{h_1} - \frac{f(x+h_2) - f(x)}{h_2} \right|.$$

So f is in ND if and only if $A(f;x) > 0$ for each x in $[0,1]$. Moreover $A(f;x)$ is finite if and only if all the Dini derivates of f at x are finite. So f is a Banach function if and only if $A(f;x) = +\infty$ for each x in $[0,1]$.

PROPOSITION 7. $|f| = 1$ *if and only if* f *is a Banach function.*

Proof. "$\Rightarrow$" Suppose f is not a Banach function. Then there is an x_0 in $[0,1]$ such that $A(f;x_0)$ is finite. We can thus find a positive integer M such that for all K, L in $R[0,1]$ with x_0 in K, L we have

$$|\Delta f(K) - \Delta f(L)| \leqq M.$$

We claim that $|T(M,f)| \geqq \omega$. Indeed let us fix n. Choose I_2 in $Q[0,1]$ such that $|I_2| \leqq 1/n$ and x_o is in I_2. By the continuity of f, we can find an I_n in $Q[0,1]$ with $\bar{I}_n \subseteq I_2$ and x_0 in I_n such that

$$|\Delta f(K) - \Delta f(L)| \leqq M/n$$

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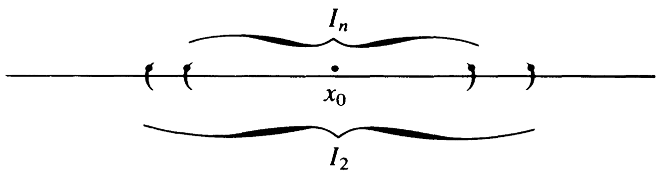
for all K, L in $R[0, 1]$ with $I_n \subseteq K, L \subseteq I_2$. (It suffices to choose I_n with endpoints close enough to I_2.)



Now take $I_1 = [0, 1]$ and $I_3, \ldots, I_{n-1}$ in $Q[0, 1]$ such that $\bar{I}_{i+1} \subseteq I_i$ $(i = 2, \ldots, n-1)$. Then it is easy to see that $\langle I_1, \ldots, I_n \rangle$ is in $T(M,f)$. (Note $\langle I_1, \ldots, I_n \rangle$ depends on n.) So

$$|T(M,f)| \geqq n + 1.$$

But this is true for each $n \geqq 1$, so

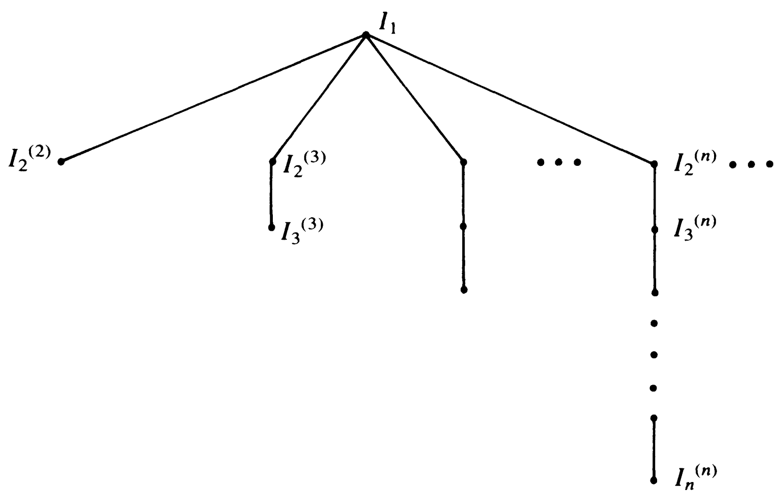
$$|T(M,f)| \geqq \omega.$$

Thus $|T(f)| > \omega$ and so $|f| > 1$.

"$\Leftarrow$" Suppose now that $|f| > 1$. Then $|f| \geqq 1$ and so by Proposition 5 $|T(f)| \geqq \omega \cdot 2$. Hence there is a positive integer M such that

$$|T(M,f)| \geqq \omega + 1.$$

So $T(M,f)$ must have a subtree as shown below.



Let x_0 be a limit point of the set of all midpoints of the intervals $I_n^{(n)}$ $(n = 2, 3, \ldots)$. Then we can find intervals $I_n^{(n)}$ arbitrarily close to x_0, for large enough

n. Let $h_1, h_2 \neq 0$ be such that $x_0 + h_1, x_0 + h_2$ are in $[0, 1]$. Then as in the "$\Rightarrow$" part of the proof of Proposition 2 we get that

$$\left| \frac{f(x_0 + h_1) - f(x_0)}{h_1} - \frac{f(x_0 + h_2) - f(x_0)}{h_2} \right| \leqq M.$$

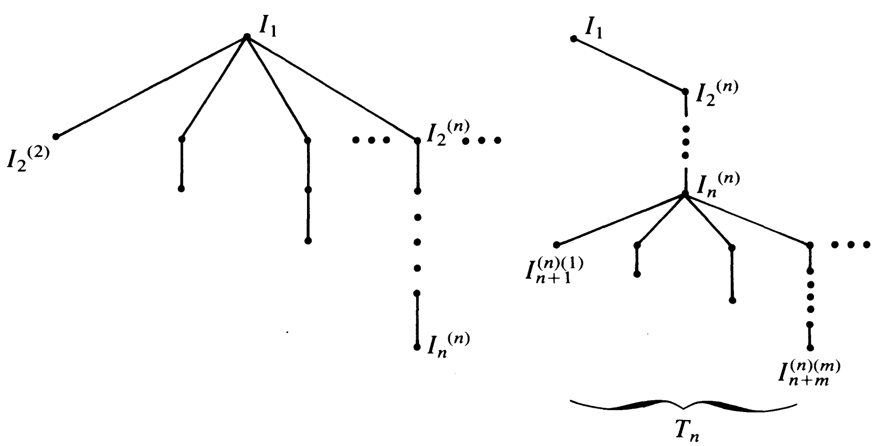
Since this is true for all h_1, h_2 we get $A(f; x) \leqq M$. So f is not a Banach function.

PROPOSITION 8. *If there is a constant $c > 0$ such that $A(f; x) \geqq c$ for all x in* $[0, 1]$, *then* $|f| \leqq 2$.

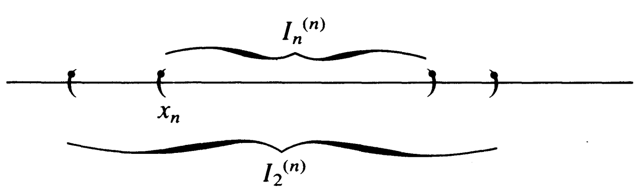
Proof. Suppose $|f| > 2$. Then $|T(f)| \geqq \omega.3$. Hence there is an integer M such that

$$|T(M,f)| \geqq \omega.2 + 1.$$

So $T(M,f)$ must have a subtree as shown below, where each of the nodes $\langle I_1, I_2^{(n)}, \ldots, I_n^{(n)} \rangle$ is of rank at least ω in $T(M,f)$.



Fix $n \geqq 2$ and consider the subtree T_n of $T(M,f)$ shown above. As in the "$\Leftarrow$" part of the proof of Proposition 7 we see that there is a point x_n in $\bar{I}_n^{(n)} \subseteq i_{n-1}^{(n)}$ such that $A(f; x) \leqq M/(n-1)$.



But this is true for each $n \geqq 2$. So for large enough n we will get $A(f;x_n) < c$, which is a contradiction. Hence $|f| \leqq 2$.

Remark. The converse of Proposition 8 is false. By using the techniques we will develop in the next section it will be easy to construct a function f with $|f| = 2$ and $A(f;x_n) \longrightarrow 0$ for any given sequence $\langle x_n \rangle$ of distinct points in $[0, 1]$ tending to zero. Using 8 it is not hard to check that all of the natural examples of functions in ND given in [**4**], [**5**], [**6**], [**11**], [**12**], [**21**], [**23**], [**24**], [**26**] have rank $\leqq 2$. A little more effort along with Proposition 7 shows that most of these functions in fact have rank 1 (see [**20**] for details).

3. Unboundedness of the rank function. In this section we will show that for each countable ordinal there is a Besicovitch function with rank $\geqq \alpha$. By refining this construction we also show that for each countable ordinal $\alpha \geqq 1$, there is a Besicovitch function f with $|f| = \alpha$. To this end we introduce the following definitions.

Definition . For each I in $Q[0, 1]$ we define the subtree $T(M,f;I)$ of $T(M,f)$ as the set of all $\langle I_1, \ldots, I_n \rangle$ in $T(M,f)$ such that there exists $J_1, \ldots J_k$ in $Q[0, 1]$ such that $\langle I_1, \ldots, I_n, J_1, \ldots, J_k \rangle$ is in $T(M,f)$ and $J_k \subseteq I$.

For each x in $[0, 1]$ we also define

$$|T(M,f;x)| = \min\{|T(M,f;I)| : \quad x \in I \in Q[0, 1]\}.$$

The ordinal $|T(M,f;x)|$ can be thought of as the *local height* of the tree $T(M,f)$ at x. The following result is therefore natural.

LEMMA 9. *Suppose* $|T(M,f)| \geqq \omega$. *Then there is an* x *in* $[0, 1]$ *such that*

$$|T(M,f;x)| = |T(M,f)|.$$

Proof. The proof is very similar to that of Lemma 3. We shall find a nested sequence of closed intervals $\langle L_2 \rangle$ with $1/n < |L_n| < 2/n$ such that

$$|T(M,f;L_n)| = |T(M,f)| \quad \text{for each } n \geqq 1.$$

Taking x in $\cap\{L_n : n \geqq 1\}$ gives the result. Let $L_1 = [0, 1]$ and given L_n choose L' and L'' as in Lemma 3. Now observe that if $\langle I_1, \ldots, I_{n+1} \rangle$ is a node of length $n+1$ in $T(M,f;L_n)$ then $I_{n+1} \subseteq L'$ or $I_{n+1} \subseteq L''$. So all the nodes in $T(M,f;L_n)$ of length at least $n+1$ must lie in one of the trees $T(M,f;L'), T(M,f;L'')$. Since

$$|T(M,f;L_n)| = |T(M,f)| \geqq \omega$$

by the induction hypothesis, we must have

$$\max\{|T(M,f;L')|, |T(M,f;L'')|\} = |T(M,f)|.$$

We take L_{n+1} to be L' if

$$|T(M,f;L')| = |T(M,f)|,$$

and L''_n otherwise. This completes the proof.

Let S be the square in the plane with vertices at (0,0), (1/2,1/2), (1,0) and $(1/2, -1/2)$. We define SF to be the collection of all Besicovitch functions whose graphs lie inside the square S. It is not difficult to see that SF is non-empty. Indeed all we need to do is to take a Morse-Besicovitch function f with $f(0) = f(1) = 0$ and $|f(x)| \leqq 1/2$ for all x in [0,1] (see [**17**] and continuously squeeze it into a new function g whose graph lies in S. The squeezing transformation is given by

$$g(x) = \begin{cases} x . \tan[\frac{1}{2}\tan^{-1}\{f(x)/x\}] & \text{for } 0 < x \leqq 1/2 \\ (1-x). \tan[\frac{1}{2}\tan^{-1}\{f(x)/(1-x)\}] & \text{for } 1/2 < x < 1. \end{cases}$$

If f is in C we define a *scaled copy* of f onto the interval $[a, b]$ to be the function g given by

$$g(x) = f(x-a)/(b-a)) \quad \text{for } x \text{ in } (a, b], \text{and}$$

$$g(a) = f(0).$$

Let $I \subseteq [0, 1]$ be a closed interval with rational endpoints and with $|I| > 0$. Then there is an obvious way to define the tree $T(M,f\restriction I)$. Let $R(I)$ be the set of all intervals which are open in I, and $Q(I)$ be the set of all intervals in $R(I)$ with rational endpoints.

Definition Let f be in C and $M > 0$. We define the tree $T(M,f\restriction I)$ as follows. $\langle I_1, \dots, I_n \rangle$ is in $T(M,f\restriction I)$ if and only if

(i) $I_1 = I, I_i \in Q(I), \bar{I}_{i+1} \subseteq I_i, |I_i| \leqq |I|/i$ and

(ii) for all K, L in $R(I)$ with $I_n \subseteq K, L \subseteq I_i$ we have

$$|\Delta f(K) - \Delta f(L)| \leqq M/i.$$

Observe that if f and g are in C and $g\restriction I$ is a scaled copy of f onto I, then $T(M,f)$ and $T(M, g\restriction I)$ are isomorphic trees.

PROPOSITION 10. *For each countable ordinal α there is an f in SF with $|f| \geqq \alpha$.*

Proof. It will suffice to show that for each $\alpha < \omega_1$ there is an f in SF with $|T(f)| \geqq \omega . \alpha$. We prove this by induction. By Proposition 7 we know that if f is in SF then $|T(f)| \geqq \omega . 2$, so the result is clear for $\alpha = 0, 1$ and 2. Suppose the result is true for $\alpha (\alpha \geqq 2)$. We have to prove it for $\alpha + 1$. Let K_n be the interval $[2^{-n}, 2^{1-n}], n \geqq 1$. Choose f in SF with $|T(f)| \geqq \omega . \alpha$. The basic idea is to put scaled copies of f/n onto K_n and so obtain a function g in C. Since

$$T(1, g\restriction K_n) \simeq T(1, f/n) = T(n,f)$$

we would expect to get

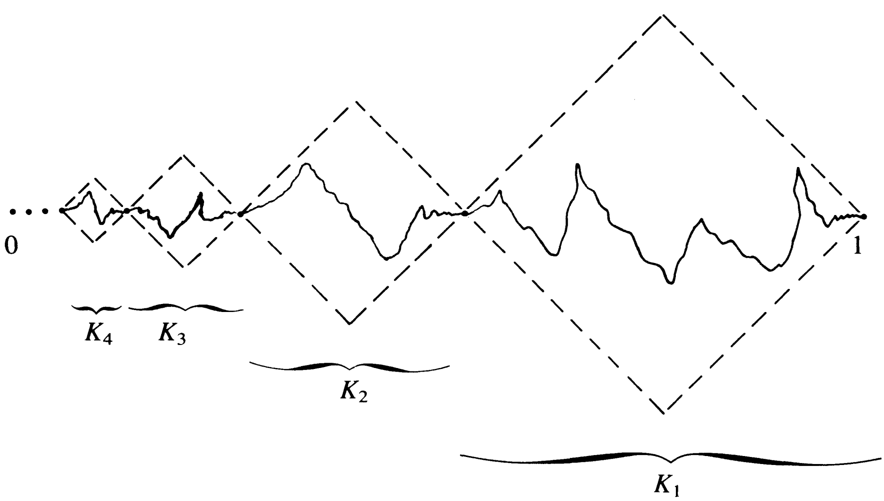
$$(*)\ \ |T(1,g)| \geqq \sup\{|T(1,g\restriction K_n)| : n \geqq 1\}$$
$$= \sup\{|T(n,f)| : n \geqq 1\} \geqq \omega.\alpha.$$

So $|T(g)| \geqq \omega.(\alpha+1)$ as required. But two things can go wrong here. Firstly the g obtained might not be in SF (in fact if we use the process above the g obtained would be differentiable at 0). Moreover if the rank of f is concentrated at the endpoints (i.e. if $T(M,f;x)| < |T(M,f)|$ for all x except 0 and 1) the inequality (*) might not hold. So we need to modify our procedure accordingly.

Let H be the middle open third of [0,1] (i.e., $H = (1/3, 2/3)$) and H_n be the middle open third of K_n. Choose f in SF with $|T(f)| \geqq \omega.\alpha$ such that for each $M > 0$ there is an x_M in H with

$$|T(M,f;x_M)| = |T(M,f)|.$$

(This latter condition can be easily obtained by replacing f, if necessary, by five scaled copies of f, each onto a fifth of [0,1].) Let g be the function defined by $g(0) = 0$, $g\restriction K_n$ is a scaled copy of f onto K_n for n even, and $g\restriction K_n$ is a scaled copy of $f/2n$ onto K_n for n odd.



It is easy to see that g is in SF. (The only difficulty lies in showing that g has no right derivative at 0, but this was ensured by choosing $g\restriction K_n$ as scaled copies of f onto K_n for n even.)

Let n be odd and consider the function $g\restriction K_n$. Let T_n be the tree $[T(1, g\restriction K_n; H_n)]_3$ (recall that H_n is the middle open third of K_n). Observe that each interval in T_n is in $Q[0,1]$ and

$$|T_n| + 3 \geqq |T(1, g\restriction K_n)|.$$

Note also that

$$T_n \simeq [T(1,f/2n;H)]_3 = [T(2n,f;H)]_3.$$

So

$$\sup\{|T_n| + 4 : n \geqq 1\} \geqq |T(f)|.$$

Since $|T(f)|$ is a limit ordinal it follows that

$$\sup\{|T_n| : n \geqq 1\} \geqq |T(f)|.$$

Now let T'_n be the tree defined by $\langle [0,1], I_2, \ldots, I_n \rangle$ is in T'_n if and only if $\langle K_n, I_2, \ldots, I_n \rangle$ is in T_n. We claim that T'_n is a subtree of $T(4,g)$. It will suffice to show that for any interval I_m in T'_n and for all K, L in $R[0,1]$ with $I_m \subseteq K, L \subseteq [0,1]$ we have

$$|\Delta\, g(K) - \Delta\, g(L)| \leqq 4.$$

Now if K contains an endpoint of some K_n then $|\Delta\, g(K)| \leqq 2$ by our construction; and if $K \subseteq K_n$ for some n then

$$|\Delta\, g(K)| = |\Delta\, g(K) - \Delta\, g(K_n)| \leqq 1$$

since T_n was a subtree $T(1, g{\restriction}K_n)$ and g is zero at the endpoints of K_n. The same holds for L. So we always have

$$|\Delta\, g(K) - \Delta\, g(L)| \leqq |\Delta\, g(K)| + |\Delta\, g(L)| \leqq 4.$$

Thus T'_n is a subtree of $T(4,g)$. Hence

$$\begin{aligned} |T(4,g)| &\geqq \sup\{|T'_n| : n \geqq 1, n \text{ odd}\} \\ &= \sup\{|T_n| : n \geqq 1, n \text{ odd}\} \\ &\geqq |T(f)|. \end{aligned}$$

So

$$|T(g)| = \sup\{|T(N,g)| + 1 : N \geqq 1\} > |T(f)| \geqq \omega.\alpha.$$

Since $|T(g)|$ is a limit ordinal it follows that $|T(g)| \geqq \omega.(\alpha + 1)$ and this part of the induction is complete.

Suppose now that the result is true for all $\alpha < \lambda$, where λ is a limit ordinal. We have to show that the result is true for λ. Let $\langle \alpha_n \rangle$ be an increasing sequence of ordinals with $\lim \alpha_n = \lambda$. For each $n \geqq 1$ choose h_n in SF with

$$|T(h_n)| \geqq \omega.\alpha_n + 1$$

such that for each $M > 0$ there is an $x_M^{(n)}$ in H with

$$|T(M, h_n : x_M^{(n)})| = |T(M, h_n)|.$$

Then for each $n \geqq 1$ there is a positive integer $M(n)$ such that

$$|T(M(n), h_n)| \geqq \omega . \alpha_n.$$

Let $f_n = h_n/M(n)$. Then for each $n \geqq 1$ there exists an x_n in H such that

$$|T(1,f_n; x_n)| = |T(1,f_n)| \geqq \omega . \alpha_n.$$

We now define g as in the case for successor ordinals. Let $g(0) = 0, g \restriction K_n$ be a scaled copy of f_2 onto K_n for n even, and $g \restriction K_n$ be a scaled copy of f_n for n odd. As before the scaled copies of f_2 onto K_n for n even, ensure that g has no derivative at $x = 0$. It is thus clear that g is in SF. Finally by the same argument as in the successor ordinal case we get

$$\begin{aligned} |T(4, g)| &\geqq \sup\{|T(1,f_n)| : n \geqq 1\} \\ &\geqq \sup\{\omega . \alpha_n : n \geqq 1\} = \omega . \lambda. \end{aligned}$$

So

$$|T(g)| = \sup\{|T(n, g)| + 1 : N \geqq 1\} \geqq \omega . \lambda + 1 \geqq \omega . \lambda$$

and hence the result is true for λ. This completes the proof.

PROPOSITION 11. *For each countable ordinal $\alpha \geqq 1$ there is an f in BF with $|f| = \alpha$.*

Proof. For $\alpha = 1$ we simply take an f in BC $\cap$ BF, a Morse-Besicovitch function (see **[17]**) would do nicely. We will now show by induction that for each $\alpha \geqq 2$ there is an f in SF with $|f| = \alpha$. For $\alpha = 2$ take an f in SF with $A(f; x) = +\infty$ for each x in (0,1). Since $A(f; 0)$ and $A(f; 1)$ are finite we see that $|f| = 2$.

Now suppose the result is true for α $(\alpha \geqq 2)$. We must show that it is true for $\alpha + 1$. Choose f in SF such that $|f| = \alpha$ and for each $M > 0$ there is an x_M in H with

$$|T(M,f; x_M)| = |T(M,f)|.$$

Let g be constructed from f as in Proposition 10. We claim that $|g| = \alpha + 1$. In view of Proposition 10 it will suffice to show that

$$|T(M, g)| \leqq \omega . (\alpha + 1).$$

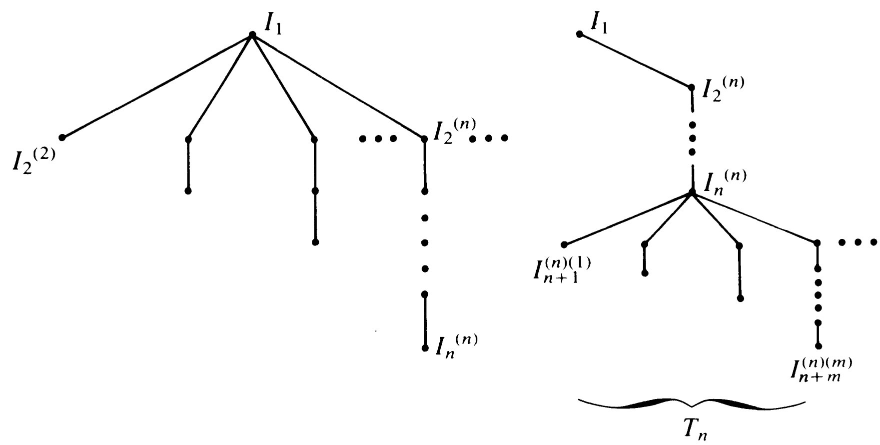
Suppose $|T(g)| > \omega.(\alpha+1)$. Then there is a positive integer M such that

$$|T(M,g)| \geqq \omega.(\alpha+1).$$

So $T(M,g)$ must have a subtree as shown below with each of the nodes

$$\mathbf{v}_n = \langle I_1, I_2^{(n)}, \ldots, I_n^{(n)} \rangle$$

having rank at least $\omega.\alpha$ in $T(M,g)$.



Fix $n \geqq 2$ and consider the subtree T_n through the node $\mathbf{v}_n$ as shown on the right. Since $|\mathbf{v}_n : T_n| \geqq \omega.\alpha$ it follows as in Lemma 3 that there is an x_n in $\bar{I}_n \subseteq I_{n-1}$ with

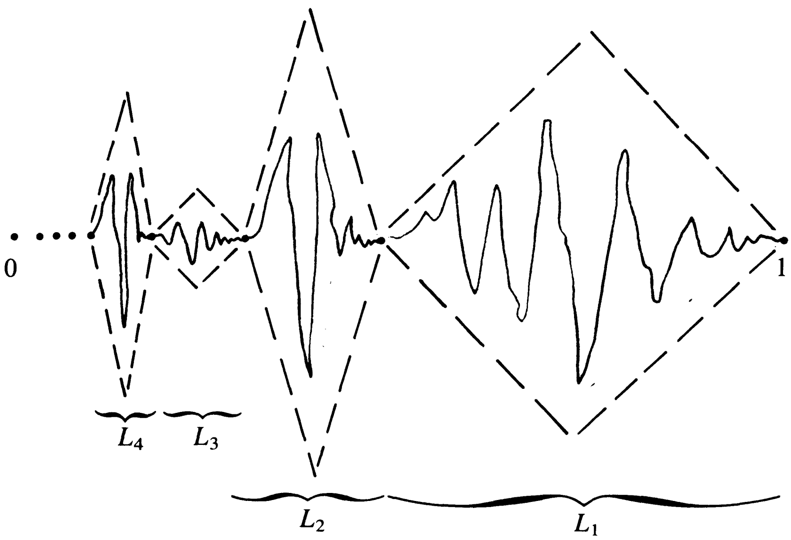
$$|T(M,g);x_n| \geqq \omega.\alpha.$$

But from the way g was constructed we know that x_n must be 0 (because in each K_n we had $|T(M,g\restriction K_n)| < \omega.\alpha$). Now as in Proposition 8 we see that

$$A(g;0) \leqq M/(n-1).$$

Since this is true for each n we get that $A(g;0) = 0$ which contradicts the fact that g is in ND. So we must have $|T(g)| \leqq \omega.(\alpha+1)$.

Now suppose the result is true for all $\alpha < \lambda$, where λ is a limit ordinal. We must show that it is true for λ. In order to do this we need to modify the construction in Proposition 10 (because the g we obtained there always has rank $\geqq \lambda+1$). Let $\langle \alpha_n \rangle$ be a strictly increasing sequence of ordinals with $\lim \alpha_n = \lambda$. Choose f_n in SF as in Proposition 10 and let L_n be the interval $[1/(n+1), 1/n]$ for $n \geqq 1$. Define g by $g(0) = 0, g\restriction L_n$ is a scaled copy of $2nf$ onto L_n for n even, and $g\restriction L_n$ is a scaled copy of f_n onto L_n for n odd.



It is once again easy to see that f is in SF. Also for each odd n we can show exactly as in Proposition 10 that

$$|T(16n, g)| \geqq \omega . \alpha_n .$$

Thus

$$\begin{aligned} |T(g)| &= \sup\{|T(16n, g)| + 1 : n \geqq 1, n \text{ odd} \\ &\geqq \sup\{\omega . \alpha_n : n \geqq 1, n \text{ odd}\} = \omega . \lambda . \end{aligned}$$

It will thus suffice to show that $|T(g)| \leqq \omega . \lambda$ in order to complete the proof.

Fix $M > 0$ and consider the intervals L_n for n even. Let

$$B = \sup\{f_2(x)| : x \in [0, 1]\}$$

and choose x_0 such that $|f(x_0)| = B$. Let x_n be the image of x_0 in L_n when $2nf_2$ is scaled onto L_n. Then

$$g(x_n) = 2nB/(n(n+1)) = 2B/(n+1).$$

So

$$|\Delta g(x_n, 1/(n+2))| = [2B/(n+1)]/[x_n - 1/(n+2)] \geqq nB$$

whenever n is even. Let $N = 2[M/B] + 1$. We claim that for all odd n's with $n \geqq N$ the subintervals of L_n cannot be in the tree $T(M, g)$. Indeed suppose $J \subseteq L_{n+1}$ (where $n+1$ is odd) and J is in $T(M, g)$. Let

$$K = (1/(n+2), 1/(n+1)) \quad \text{and} \quad L = (1/(n+2), x_n).$$

Then $J \subseteq K, L \subseteq [0,1]$ but

$$|\Delta g(K) - \Delta g(L)| = nB > M,$$

which contradicts the definition of the tree $T(M, g)$.

So for each x in $[0,1]$ we have

$$\begin{aligned}|T(M,g);x| &\leqq \sup\{|T(M,f_n);x| : n \leqq N\}\\ &\leqq \sup\{|T(f_n)| : n \leqq N\}\\ &\leqq |T(f_N)|\end{aligned}$$

since N is odd and $|T(f_n)|$ is increasing. Hence by Lemma 3 we get

$$|T(M,g)| < |T(f_N)| + \omega \leqq \omega.\, \alpha_N + \omega < \omega.\, \lambda.$$

So $|T(M,g)| < \omega.\, \lambda$. Thus

$$|T(g)| = \sup\{T(M,g)| + 1 : M > 0\} \leqq \omega.\, \lambda.$$

This completes the proof.

COROLLARY 12. ND *and* BF *are not Borel Subsets of* C.

Proof. Let φ be our rank function which maps ND onto ω_1. Then φ is a coanalytic norm which is unbounded in ω_1. So by Proposition A, ND is not Borel. Let ψ be the restriction of φ to BF. Then BF and ψ satisfy the conditions of Proposition B. Hence BF is not Borel.

4. An alternative definition. In this section we give an alternative definition of our rank function. Let $J \subseteq [0, 1]$ be any closed interval with rational endpoints and recall the definition of $R(J)$ and $Q(J)$. Let $\bar{Q}(J)$ be the set of all $\bar{I}$ such that $I \in Q(J)$. If $H \subseteq J$ we denote the interior of H with respect to the topology of J by $\mathrm{int}_J(H)$. For each f in $C, M > 0$ and $J \in \bar{Q}[0, 1]$ we will define a sequence $\langle P^\alpha(M, f{\restriction}J)\rangle$ of closed subsets of J and a relation

$$S[x, P^\alpha(M, f{\restriction}J)](W) = S(W)$$

on the closed subsets of J which reflect the properties of $f{\restriction}J$. M is to be thought of as being large and $S(W)$ is to be interpreted as the relation "W *witnesses* that x is in $P^\alpha(M, f{\restriction}J)$".

We define $p^\alpha(M, f{\restriction}J)$ and $S[x, P^\alpha(M, f{\restriction}J)(W)$ by induction. Let $P^1(M, f{\restriction}J)$ be the set of all x in J with

$$\sup\{|\Delta f(K) - \Delta f(L)| : x \in K, L \in R(J)\} \leqq M,$$

and let $S[x, P^1(M,f\restriction J)](W)$ hold if and only if x is in $W \cap P^1(M,f\restriction J)$. Let $P^{\alpha+1}(M,f\restriction J)$ be the set of all x in J such that for all I in $Q(J)$ with $x \in I$ there exist H in $\bar{Q}(I)$, y in $\text{int}_J(H)$ and $V \subseteq P^1(M,f\restriction J)$ such that

$$S[y, P^{\alpha}(M.|I|,f\restriction H)](V).$$

Let $S[x, P^{\alpha+1}(M,f\restriction J)](W)$ hold $\Leftrightarrow$ for all I in $Q(J)$ with $x \in I$ there exist H in $\bar{Q}(I)$, y in $\text{int}_J(H)$ and $V \subseteq W \cap P^1(M,f\restriction J)$ such that

$$S[y, P^{\alpha}(M.|I|,f\restriction H)](V)$$

holds. Finally if λ is a limit ordinal we let

$$P^{\lambda}(M,f\restriction J) = \cap\{P^{\alpha}(M,f\restriction J) : \alpha < \lambda\},$$

and let

$$S[x, P^{\lambda}(M,f\restriction J)](W)$$

hold if and only if

$$(\forall \alpha < \lambda) S[x, P^{\alpha}(M,f\restriction J|)](W)$$

holds.

Observe that our definition is made by use of simultaneous induction on α, M and J. For fixed M and J it is easy to see that $P^{\alpha}(M,f\restriction J)$ decreases as α increases. Also for fixed α and J, $P^{\alpha}(M,f\restriction J)$ increases as M increases. When $J = [0, 1]$ we shall refer to $P^{\alpha}(M,f\restriction J)$ simply as $P^{\alpha}(M,f)$. We have the following result.

PROPOSITION 13. *f is in* ND $\Leftrightarrow$ *for each $M > 0$ there exists $\alpha < \omega_1$ such that $P^{\alpha}(M,f) = \emptyset$.*

The "$\Leftarrow$" part of this result is very easy because if f is differentiable at x_0 then it is easy to verify that for a fixed large enough M, x_0 is in $P^{\alpha}(M,f)$ for all α. The other part is much more complicated and tedious so we omit the proof (see [**20**] for details). Proposition 13 allows us to make the following definition and the proof implicitly gives the next result.

Definition. For each f in ND we define $r(f)$ to be the least ordinal α such that $P^{\alpha}(M,f) = \emptyset$ for all positive integers M.

PROPOSITION 14. *For each f in* ND, $|f| = r(f)$.

Concluding remarks. It is clear that our alternative definition is very complicated as compared with those in [**1**] and [**10**] (where for instance parametric induction is used instead of simultaneous induction). So the question arises as to whether there is a simpler definition. We feel that a simpler definition is not possible because the trees $T(M, f)$ have a nested-like structure which makes the use of simultaneous induction necessary.

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