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ON FUNCTIONS DERIVED FROM REGULARLY VARYING FUNCTIONS

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Introduction and notation

A generalization of Karamata's theorem on integrals of regularly varying functions is proved. Using Laplace–Stieltjes transforms it is shown that any regularly varying function with exponent α $(\alpha+1 \not\in \mathbf{N})$ is asymptotic to another regularly varying function all of whose derivations are regularly varying.

Suppose U is a positive function on $\mathbf{R}^+$. U is *regularly varying* at ∞ (or $0+$) with exponent α, in short α-varying, notation $U \in RV_\alpha^{(\infty)}$ (or $RV_\alpha^{(0)}$ respectively), if for all $x>0$

$$\frac{U(tx)}{U(t)} \to x^\alpha$$

as $t \to \infty$ (or $t \downarrow 0$ respectively); cf. Karamata (1930) and (1933), Feller (1971) chapter VIII, 8 and XIII, 5.

If U is non-decreasing and if for suitable functions $a(t)>0$ and $b(t)$ and all $x>0$

$$\frac{U(tx)-b(t)}{a(t)} \to \log x$$

as $t \to \infty$ we say $U \in \Pi^{(\infty)}$. If U is non-increasing and if for suitable functions $a(t)>0$ and $b(t)$ and all $x>0$

$$\frac{U(tx)-b(t)}{a(t)} \to \log x$$

as $t \downarrow 0$ we say $U \in \Pi^{(0)}$; cf. de Haan (1970), section I, 4.

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1. Integrals of regularly varying functions

We start from a well known result. Suppose F is a probability distribution function and $F(0+)=0$. Then (Feller (1971), VIII, 9 th. 2. cf. Pitman (1968), lemma 3) for $\alpha>0$, $\beta<0$, $\alpha+\beta>0$

(P1)
$$\int_0^x t^\alpha dF(t)\in RV^{(\infty)}_{\alpha+\beta} \Leftrightarrow \int_0^x t^{\alpha-1}(1-F(t))dt \in RV^{(\infty)}_{\alpha+\beta}$$

$$\Leftrightarrow 1-F(x)\in RV^{(\infty)}_\beta \Leftrightarrow \lim_{x\to\infty}\frac{\int_0^x t^{\alpha-1}(1-F(t))dt}{x^\alpha(1-F(x))}=\frac{1}{\alpha+\beta}$$

$$\Leftrightarrow \lim_{x\to\infty}\frac{\int_0^x t^\alpha dF(t)}{x^\alpha(1-F(x))}=\frac{-\beta}{\alpha+\beta}.$$

A variant is the following. Suppose U is non-decreasing, $U(0+)=0$, then for $\alpha>0$, $\beta>0$

(P2)
$$\int_0^x t^\alpha dU(t)\in RV^{(\infty)}_{\alpha+\beta} \Leftrightarrow \int_0^x t^{\alpha-1}U(t)dt \in RV^{(\infty)}_{\alpha+\beta}$$

$$\Leftrightarrow U(x)\in RV^{(\infty)}_\beta \quad \Leftrightarrow \lim_{x\to\infty}\frac{\int_0^x t^{\alpha-1}U(t)dt}{x^\alpha U(x)}=\frac{1}{\alpha+\beta}$$

$$\Leftrightarrow \lim_{x\to\infty}\frac{\int_0^x t^\alpha dU(t)}{x^\alpha U(x)}=\frac{\beta}{\alpha+\beta}.$$

We want to present some analogous statements. Proofs are given in the next section. Firstly for probability distributions F and $\alpha>0$

(P3)
$$\int_0^x t^\alpha dF(t)\in RV^{(\infty)}_0 \Leftrightarrow \int_0^x t^{\alpha-1}(1-F(t))dt \in RV^{(\infty)}_0$$

$$\Leftrightarrow \lim_{x\to\infty}\frac{\int_0^x t^{\alpha-1}(1-F(t))dt}{x^\alpha(1-F(x))}=\infty \Leftrightarrow \lim_{x\to\infty}\frac{\int_0^x t^\alpha dF(t)}{x^\alpha(1-F(x))}=\infty.$$

This is the case $\alpha+\beta=0$ of **(P1)**. A sufficient (but not necessary) condition is $1-F(x)\in RV^{(\infty)}_{-\alpha}$.

Next suppose U is as above. For $\alpha>0$

(P4)
$$U\in\Pi^{(\infty)} \Leftrightarrow \int_0^x t^\alpha dU(t)\in RV^{(\infty)}_\alpha \Leftrightarrow \int_x^\infty t^{-\alpha}dU(t)\in RV^{(\infty)}_{-\alpha}.$$

This is the case $\beta=0$ of **(P2)**.

Finally suppose U_1 is non-decreasing, U_2 is continuous and strictly increasing, $U_2(0+)=0$. Suppose $\alpha>0$, $\beta>0$.

(P5) Any two of the following statements imply the others.

a. $U_1 \in RV_\alpha^{(\infty)}$

b. $U_2 \in RV_\beta^{(\infty)}$

c. $\int_0^x U_1(t)dU_2(t) \in RV_{\alpha+\beta}^{(\infty)}$

d. $\lim_{x\to\infty} \frac{\int_0^x U_1(t)dU_2(t)}{U_1(x)U_2(x)} = \frac{\beta}{\alpha+\beta}$

This generalizes **(P2)**. Similarly for functions in $\Pi^{(\infty)}$ we have the following.

(P6) Suppose $U_1 \in RV_\alpha^{(\infty)}$ $(\alpha>0)$, U_1 is continuous and strictly increasing, $U_1(0+)=0$.

$$U_2(x)\in\Pi^{(\infty)} \Leftrightarrow \int_0^x U_1(t)dU_2(t)\in RV_\alpha^{(\infty)} \Leftrightarrow \int_x^\infty \frac{dU_2(t)}{U_1(t)}\in RV_{-\alpha}^{(\infty)}.$$

This generalizes **(P4)**.

Remark. Property **P5** may be used to generalize a result on convergence of moments for sample extremes, see Pickands (1968).

2. Proofs and remarks

Proof of **(P3)**.

$$\int_0^x t^\alpha dF(t)\in RV_0^{(\infty)}$$

if and only if

$$\lim_{x\to\infty}\int_0^x t^\alpha dF(t)/\{x^\alpha(1-F(x))\}=\infty$$

by Feller (1971), VIII, 9 th. 2 (part iii). Now

$$\lim_{x\to\infty}\int_0^x t^\alpha dF(t)/\{x^\alpha(1-F(x))\}=\infty$$

$$\Leftrightarrow \lim_{x\to\infty}\int_0^x t^{\alpha-1}(1-F(t))dt/\{x^\alpha(1-F(x))\}=\infty$$

is a matter of partial integration. If

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$$\alpha(x) = \int_0^x t^{\alpha-1}(1-F(t))dt/\{x^\alpha(1-F(x))\} \to \infty \qquad (x \to \infty)$$

then

$$\int_0^x t^{\alpha-1}(1-F(t))dt = \left\{\int_0^1 t^{\alpha-1}(1-F(t))dt\right\} \exp \int_1^x \{t\alpha(t)\}^{-1}dt$$

and the latter is in $RV_0^{(\infty)}$ by the representation theorem for regularly varying functions. If $\int_0^x t^{\alpha-1}(1-F(t))dt \in RV_0^{(\infty)}$ then by an obvious extension of the argument in Feller (1971, prop. 8 p. 22)

$$\lim_{x\to\infty} \int_0^x t^{\alpha-1}(1-F(t))dt/\{x^\alpha(1-F(x))\} = \infty.$$

Remark. The statements of **(P3)** with $\alpha = 1$ are the necessary and sufficient conditions for a weak law of large numbers for positive random variables (Feller (1971), VII, 7 th. 2).

Remark. The statements of **(P3)** are implied (Feller (1971), VIII, 9 th. 2) by the set of equivalent statements ($\alpha > 0$, $\beta > \alpha - 1$)

$$1 - F(x) \in RV_{-\alpha}^{(\infty)} \Leftrightarrow \int_0^x t^{\alpha-1}(1-F(t))dt \in \Pi^{(\infty)}$$

$$\Leftrightarrow \int_0^x t^\alpha dF(t) \in \Pi^{(\infty)} \Leftrightarrow \int_0^x t^\beta(1-F(t))dt \in RV_{\beta-\alpha+1}^{(\infty)}$$

(the equivalence of these statements follows from **(P4)**).

Proof of **(P5)**. U_2 has a proper inverse U_2^{-1}. So

$$\int_0^x U_1(t)dU_2(t) = \int_0^{U_2(x)} U_1(U_2^{-1}(s))ds.$$

We shall write $U_1 \circ U_2^{-1}(s)$ for the compound function $U_1(U_2^{-1}(s))$.

Assume a) and b). Then $U_1 \circ U_2^{-1} \in RV_{\alpha/\beta}^{(\infty)}$, and hence

$$\lim_{y\to\infty} \frac{\int_0^y U_1(t)dU_2(t)}{U_1(y)U_2(y)} = \lim_{x\to\infty} \frac{\int_0^{U_2^{-1}(x)} U_1(t)dU_2(t)}{xU_1 \circ U_2^{-1}(x)}$$
$$= \lim_{x\to\infty} \frac{\int_0^x U_1 \circ U_2^{-1}(s)ds}{xU_1 \circ U_2^{-1}(x)} = \frac{\beta}{\alpha+\beta}.$$

Assume b) and c). The compound function

$$\int_0^{U_2^{-1}(x)} U_1(t)dU_2(t) = \int_0^x U_1 \circ U_2^{-1}(s)ds$$

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then belongs to $RV^{(\infty)}_{\beta^{-1}(\alpha+\beta)}$. Since $U_1 \circ U_2^{-1}$ is monotone, it follows $U_1 \circ U_2^{-1} \in RV^{(\infty)}_{\beta^{-1}\alpha}$. Hence

$$U_1 = U_1 \circ U_2^{-1} \circ U_2 \in RV^{(\infty)}_{\alpha}.$$

Assume a) and c). It is well known that there is a continuous and strictly increasing function U_3 such that $U_3(0+) = 0$ and $U_1(x) \sim U_3(x)$ as $x \to \infty$. Then $\int_0^x U_3(t)dU_2(t) \sim \int_0^x U_1(t)dU_2(t)$ as $x \to \infty$. The compound function

$$\int_0^{U_3^{-1}(x)} U_3(t)dU_2(t) = \int_0^x s dU_2 \circ U_3^{-1}(s)$$

is in $RV^{(\infty)}_{\alpha^{-1}(\alpha+\beta)}$. By **(P2)** then $U_2 \circ U_3^{-1} \in RV^{(\infty)}_{\alpha^{-1}\beta}$. Hence $U_2 = U_2 \circ U_3^{-1} \circ U_3 \in RV^{(\infty)}_{\beta}$.

Assume d) then

$$\lim_{x\to\infty} \frac{\int_0^x U_1 \circ U_2^{-1}(s)ds}{xU_1 \circ U_2^{-1}(x)} = \lim_{x\to\infty} \frac{\int_0^{U_2(x)} U_1 \circ U_2^{-1}(s)ds}{U_2(x)U_1 \circ U_2^{-1}(U_2(x))}$$

$$= \lim_{x\to\infty} \frac{\int_0^x U_1(t)dU_2(t)}{U_1(x)U_2(x)} = \frac{\beta}{\alpha+\beta}.$$

Hence $U_1 \circ U_2^{-1} \in RV^{(\infty)}_{\beta^{-1}\alpha}$. Once we know this a) and b) are equivalent. If in addition to d) we assume c) then $U_1(x)U_2(x) = U_2(x)U_1 \circ U_2^{-1}(U_2(x)) = U_4 \circ U_2(x) \in RV^{(\infty)}_{\alpha+\beta}$ where $U_4(x) = xU_1 \circ U_2^{-1}(x)$. Clearly $U_4 \in RV^{(\infty)}_{\beta^{-1}(\alpha+\beta)}$ hence $U_2 \in RV^{(\infty)}_{\beta}$.

The implications abd $\Rightarrow$ c and bcd $\Rightarrow$ a are trivial. Suppose abc then from $U_1 \circ U_2^{-1} \in RV_{\beta^{-1}\alpha}$ it follows as above

$$\lim_{x\to\infty} \frac{\int_0^x U_1(t)dU_2(t)}{U_1(x)U_2(x)} = \lim_{x\to\infty} \frac{\int_0^x U_1 \circ U_2^{-1}(s)ds}{xU_1 \circ U_2^{-1}(x)} = \frac{\beta}{\alpha+\beta}.$$

PROOF OF **(P6)**.

$$U_2 \in \Pi^{(\infty)} \Leftrightarrow U_2 \circ U_1^{-1} \in \Pi^{(\infty)} \Leftrightarrow \int_0^x t dU_2 \circ U_1^{-1}(t) \in RV^{(\infty)}_1$$

$$\Leftrightarrow \int_0^x U_1(t)dU_2(t) = \int_0^{U_1(x)} t dU_2 \circ U_1^{-1}(t) \in RV^{(\infty)}_{\alpha}$$

and similarly for the third statement of **(P6)**. The second equivalence above follows from de Haan (1970), theorem 1.4.1.b.

As to the first equivalence: suppose $U_2 \in \Pi^{(\infty)}$ and $U_1^{-1} \in RV^{(\infty)}_{\alpha^{-1}}$, then for all $x > 0$

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$$\lim_{t\to\infty} \frac{U_2\left(\dfrac{U_1^{-1}(tx)}{U_1^{-1}(t)}\cdot U_1^{-1}(t)\right) - U_2(U_1^{-1}(t))}{U_2\left(\dfrac{U_1^{-1}(te)}{U_1^{-1}(t)}\cdot U_1^{-1}(t)\right) - U_2(U_1^{-1}(t))} = \frac{\log x^{\alpha^{-1}}}{\log e^{\alpha^{-1}}} = \log x.$$

For the converse implication write $U_2 = U_2 \circ U_1^{-1} \circ U_1$.

3. Derivatives

We prove the following:

THEOREM 1. *Any α-varying function U with $\alpha + 1 \notin \mathbf{N}$ is asymptotic to a function U_1 with the property that the absolute values of all its derivatives are regularly varying.*

PROOF. First let $\alpha < 0$. There is a decreasing function U_2 such that $U(x) \sim U_2(x)$ as $x \to \infty$. Define $U_3(x) = U_2\left(\frac{1}{x}\right)$ then $U_3 \in RV_{-\alpha}^{(0)}$. Denote its Laplace–Stieltjes transform by $\check{U}_3$. Then $U_3(x) \sim \{\Gamma(1-\alpha)\}^{-1}\check{U}_3\left(\frac{1}{x}\right)$ as $x \downarrow 0$. So $U(x) \sim \{\Gamma(1-\alpha)\}^{-1}\check{U}_3(x)$ as $x \to \infty$ and latter function satisfies the requirements (property 8 p. 22 de Haan. (1970)).

Next let $\alpha > 0$ $(\alpha \notin \mathbf{N})$. There is an increasing function U_2 such that $U_2(0+) = 0$ and $U(x) \sim U_2(x)$ as $x \to \infty$. Denote its Laplace–Stieltjes transform by $\check{U}_2$. Then

$$U(x) \sim U_2(x) \sim \frac{1}{\Gamma(1+\alpha)}\check{U}_2\left(\frac{1}{x}\right) \quad \text{as} \quad x \to \infty.$$

We shall prove that $U_1(x) = (\Gamma(1+\alpha))^{-1}\check{U}_2\left(\frac{1}{x}\right)$ satisfies the requirements. We have (Abramowitz and Stegun (1970) Ch. 24, 1.2.I.c.)

$$\frac{d^n}{dx^n}\check{U}_2\left(\frac{1}{x}\right) = \sum_{m=1}^{n} \frac{n!}{m!}\binom{n-1}{m-1}(-1)^n x^{-n-m}\check{U}_2^{(m)}\left(\frac{1}{x}\right).$$

By property 8 of de Haan (1970), p. 22) for $m = 1, 2, \cdots$

$$x^{-m}\check{U}_2^{(m)}\left(\frac{1}{x}\right) \sim (-\alpha)(-\alpha-1)\cdots(-\alpha-m+1)\check{U}_2\left(\frac{1}{x}\right)$$

as $x \to \infty$. Hence as $x \to \infty$

$$\frac{d^n}{dx^n}\check{U}_2\left(\frac{1}{x}\right) \sim n!(-1)^n x^{-n}\check{U}_2\left(\frac{1}{x}\right)\sum_{m=1}^{n}\binom{n-1}{n-m}\binom{-\alpha}{m}$$

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$$= \binom{-\alpha+n-1}{n} n!(-1)^n x^{-n} \check{U}_2\left(\frac{1}{x}\right)$$

$$= \alpha(\alpha-1)\cdots(\alpha-n+1)x^{-n}\check{U}_2\left(\frac{1}{x}\right).$$

REMARK. A. A. Balkema has given another proof of this result using the convolution of the function with a probability distribution.

The theorem says that within each equivalence class of asymptotically equivalent regularly varying functions there is at least one function satisfying the requirements. A similar statement with a different definition of the equivalence class holds for functions in Π. This is the analogue of the previous theorem for $\alpha = 0$.

THEOREM 2. *Any function $U \in \Pi^{(\infty)}$, i.e. any non-decreasing function satisfying*

$$\lim_{t\to\infty} \frac{U(tx)-b(t)}{a(t)} = \log x$$

for all $x > 0$ and suitably chosen functions $a(t) > 0$ and $b(t)$ has a companion function U_1 such that $(-1)^{n+1}U_1^{(n)}(x) \in RV_{-n}^{(\infty)}$ for $n = 1, 2, \cdots$ and

$$\lim_{t\to\infty} \frac{U(t)-U_1(t)}{a(t)} = 0.$$

PROOF. The Laplace–Stieltjes transform $\check{U}(t)$ of U exists for all $t > 0$. We shall prove that $U_1(x) = \check{U}(x^{-1}e^{-\gamma})$ satisfies the requirements; here γ is Euler's constant. By de Haan (1976) $\check{U} \in \Pi^{(0)}$ and

$$\lim_{t\to\infty} \frac{U(t)-\check{U}\left(\frac{1}{t}\right)}{a(t)} = \gamma.$$

As in the previous proof we have

$$\frac{d^n}{dx^n}\check{U}\left(\frac{1}{x}\right) = \sum_{m=1}^{n} \frac{n!}{m!}\binom{n-1}{m-1}(-1)^n x^{-n-m}\check{U}^{(m)}\left(\frac{1}{x}\right).$$

By the lemma in de Haan (1976) we have for the derivative $-\check{U}^{(1)}(1/x) \in RV_{-1}^{(0)}$ and by property 8, p. 22 of de Haan (1970) for $m = 1, 2, \cdots$ as $x \to \infty$

$$x^{-m}\check{U}^{(m)}\left(\frac{1}{x}\right) \sim (-1)^{m+1}(m-1)!\, x^{-1}\check{U}^{(1)}\left(\frac{1}{x}\right).$$

Hence as $x \to \infty$

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$$\frac{d^n}{dx^n}\check{U}\left(\frac{1}{x}\right) \sim (-1)^n x^{-n-1}\check{U}^{(1)}\left(\frac{1}{x}\right)\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{(m-1)!\,n!}{m!}(-1)^{m+1}$$

$$= (-1)^n x^{-n-1}(n-1)!\,\check{U}^{(1)}\left(\frac{1}{x}\right).$$

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