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On Super Weakly Compact Convex Sets and Representation of the Dual of the Normed Semigroup They Generate

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Abstract. In this note, we first give a characterization of super weakly compact convex sets of a Banach space X: a closed bounded convex set $K \subset X$ is super weakly compact if and only if there exists a w^* lower semicontinuous seminorm p with $p \geq \sigma_K \equiv \sup_{x \in K}\langle \cdot, x\rangle$ such that p^2 is uniformly Fréchet differentiable on each bounded set of X^*. Then we present a representation theorem for the dual of the semigroup swcc(X) consisting of all the nonempty super weakly compact convex sets of the space X.

1 Introduction

Let X be a Banach space, and let swcc(X) be the normed semigroup of all nonempty super weakly compact convex sets of X. The purpose of this paper is to establish a representation theorem of the dual of swcc(X). This is done by giving a generalized renorming characterization and an approximation property of super weakly compact convex sets.

It is well known that super-reflexive or uniformly convexifiable Banach spaces play an important role in Banach space theory, and they form an extremely useful class of reflexive spaces. The Enflo renorming theorem [9] states that every super-reflexive Banach space is uniformly convexifiable and vice versa (see also [13]). Recently, Cheng, Cheng, Wang, and Zhang [6] introduced a notion of super weakly compact set, and gave the Enflo renorming theorem a localized setting. A closed bounded convex set in a Banach space is uniformly convexifiable if and only if it is super weakly compact. Now, we recall some definitions that will be used in the sequel.

Definition 1.1 Suppose that X is a Banach space, $\varepsilon > 0$. For all $n \in \mathbb{N}$, $A_n \subset X$ are defined by

$$A_n = \{x_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} : \varepsilon_i = 1, 2 \text{ and } i = 1, 2, \ldots, n\}.$$

(i) The subset A_n is called an (n, ε)-*tree* for some $n \in \mathbb{N}$ if it satisfies

$$x_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k} = \tfrac{1}{2}(x_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k,1} + x_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k,2})$$

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and

$$\|x_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k,1} - x_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k,2}\| \geq \varepsilon$$

for $k = 1, 2, \ldots, n-1, \varepsilon_i = 1, 2$ and $i = 1, 2, \ldots, k$.

(ii) A bounded closed convex set $A \subset X$ is said to be *super weakly compact* if for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that A does not contain an (n, ε)-tree.

Definition 1.2 Suppose that $C \subset X$ is a nonempty convex set.

(i) A real-valued convex function f defined on C is said to be *uniformly convex* provided for every $\varepsilon > 0$ there is $\delta > 0$ such that $f(x) + f(y) - 2f(\frac{x+y}{2}) \geq \delta$ whenever $x, y \in C$ with $\|x - y\| \geq \varepsilon$.

(ii) The set C is called uniformly convex provided for every $x_0 \in C$ the function $f := \|\cdot - x_0\|^2$ is uniformly convex on C.

(iii) We say the set C is *uniformly convexifiable* if there is an equivalent norm $|\cdot|$ on X such that C is uniformly convex with respect to $|\cdot|$.

Let $\mathrm{swcc}(X) = \{K \subset X : K$ is nonempty super weakly compact and convex$\}$. Among many other things, the authors, Cheng, et al [6] showed the following property.

Proposition 1.3 *For any Banach space X, the set* swcc(X) *is closed under the two operations of addition and scalar multiplication.*

Definition 1.4 Let G be an Abelian semigroup and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

(i) G is said to be a *module* if there are two operations $(x, y) \in G \times G \to x + y \in G$, and $(\alpha, x) \in (\mathbb{F} \times G) \to \alpha x \in G$ satisfying

$$(\lambda\mu)g = \lambda(\mu g), \quad \forall \lambda, \mu \in \mathbb{F} \quad \text{and} \quad g \in G;$$
$$\lambda(g_1 + g_2) = \lambda g_1 + \lambda g_2, \quad \forall \lambda \in \mathbb{F} \quad \text{and} \quad g_1, g_2 \in G;$$

and

$$1g = g \quad \text{and} \quad 0g = 0 \ \ \forall g \in G.$$

(ii) A module G endowed with a norm is called a *normed semigroup*.

(iii) A function ϕ on a normed semigroup G is called a *linear functional* if it satisfies

$$\phi(\alpha g_1 + \beta g_2) = \alpha\phi(g_1) + \beta\phi(g_2), \ \forall \alpha, \beta \in \mathbb{R}^+ \text{ and } g_1, g_2 \in G.$$

It is said to be *bounded* provided $\|\phi\| = \sup\{|\phi(g)| : g \in G, \|g\| \leq 1\} < \infty$. We denote by G^* the Banach space of all bounded functionals on G and call it the dual of G.

We endow the Hausdorff metric d_H on swcc(X), *i.e.,*

$$d_H(K_1, K_2) = \max\Big\{\sup_{x \in K_1} d(x, K_2), \sup_{y \in K_2} d(K_1, y)\Big\}, \text{ for } K_1, K_2 \in \mathrm{swcc}(X),$$

where $d(K,x) = d(x,K) = \inf_{k\in K} \|k-x\|$. This metric induces further a norm $\|\cdot\|_H$ for $K \in \operatorname{swcc}(X)$

$$\|K\|_H = d_H(0,K) = \sup\{\|k\| : k \in K\}.$$

Therefore, combining this with Proposition 1.3, we obtain the following proposition.

Proposition 1.5 swcc(X) *is, endowed with the norm, a normed semigroup.*

In this paper, the letter X will always be a real Banach space and X^* its dual. B_X (B_{X^*}, resp.) stands for the closed ball of X (X^*, resp.); if there is no possible confusion, we simply write by B (B^*, resp.) for B_X (B_{X^*}, resp.). S_X (S_{X^*}, resp.) represents the unit sphere of X (X^*, resp.). We denote by Ω a compact Hausdorff space, and by $C(\Omega)$ the Banach space of all real-valued continuous functions defined on Ω endowed with the sup-norm. For a subset $A \subset X$, σ_A stands for the support function of A, *i.e.*, $\sigma_A(x^*) = \sup_{x\in A}\langle x^*, x\rangle$, and A^0 for the polar of A, *i.e.*,

$$A^0 = \{x^* \in X^* : \langle x^*, x\rangle \le 1 \text{ for all } x \in A\}.$$

We say that a function f defined on a subset A of a Banach space X is a Δ-support function if there are two closed convex sets $C, D \subset X^*$ such that $f = \sigma_C - \sigma_D$ on A.

This paper is organized as follows. In the next section, we show that a sufficient and necessary condition for a nonempty closed convex set $K \subset X$ to be super weakly compact is that there exists a w^* lower semicontinuous seminorm p on X^* with $p \ge \sigma_K$ such that p^2 is uniformly Fréchet differentiable on B^*. In Section 3 we establish the representation theorem of the dual $\operatorname{swcc}(X)^*$ of the normed semigroup swcc(X), and this is done by showing that a nonempty closed convex set $K \subset X$ containing the origin is super weakly compact if and only if there exists a sequence $\{q_n\}$ of w^* lower semicontinuous Minkowski functionals whose squares are uniformly Fréchet differentiable on B^*, such that $q_n \to \sigma_K$ uniformly on B^*.

2 A Characterization of Super Weakly Compact Sets

In this section, we show that a sufficient and necessary condition for a nonempty closed convex set $K \subset X$ to be super weakly compact is that there exists a w^* lower semicontinuous seminorm p on X^* with $p \ge \sigma_K$ such that p^2 is uniformly Fréchet differentiable on B^*. To begin, we recall some more notions.

Given $\varepsilon \ge 0$, for a convex function f defined on a Banach space X, its ε- subdifferential mapping $\partial_\varepsilon f : X \to 2^{X^*}$ is defined by

$$\partial_\varepsilon f(x) = \{x^* \in X^* : f(x+y) - f(x) + \varepsilon \ge \langle x^*, y\rangle,\ \forall y \in X\}.$$

If $\varepsilon = 0$, then $\partial_\varepsilon f$ is called the subdifferential mapping of f, and in this case, we denote it by ∂f instead of $\partial_0 f$. The conjugate function of f, denoted f^*, is defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x), x \in X\},\ x^* \in X^*.$$

Definition 2.1 Suppose that f is a convex function defined on a Banach space X.

(i) We say that f is Gâteaux differentiable at x if there is $x^* \in X^*$ such that

$$\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t} - \langle x^*, y\rangle = 0, \ \ \forall\, y \in X.$$

(ii) f is said to be Fréchet differentiable at $x \in X$ provided

$$\lim_{t\to 0^+} \sup_{y\in B_X} \Big[\frac{f(x+ty)-f(x)}{t} - \langle x^*, y\rangle \Big] = 0.$$

In this case, we denote by $x^* = df(x)$ the Fréchet derivative of f at x.

(iii) f is called uniformly Fréchet differentiable on a subset $A \subset X$ if

$$\lim_{t\to 0^+} \sup_{y\in B_X, x\in A} \Big[\frac{f(x+ty)-f(x)}{t} - \langle df(x), y\rangle \Big] = 0.$$

The following is the Brøndsted–Rockafellar theorem [3](see, also [2, 12]).

Theorem 2.2 (Brøndsted–Rockafellar) *Suppose that $f \neq -\infty$ is an extended real-valued lower semicontinuous convex function defined on a Banach space X and $x_0 \in$ $\mathrm{dom}(f) \equiv \{x \in X : f(x) < \infty\}$. Suppose that $x_0^* \in \partial_\varepsilon f(x_0)$. Then there exist $x_\varepsilon \in \mathrm{dom}\, f$, $x_\varepsilon^* \in X^*$ such that*

$$\text{(i) } x_\varepsilon^* \in \partial f(x_\varepsilon), \quad \text{(ii) } \|x_0 - x_\varepsilon\| \le \sqrt{\varepsilon}, \quad \textit{and} \quad \text{(iii) } \|x_0^* - x_\varepsilon^*\| \le \sqrt{\varepsilon}.$$

The following properties are either classical or easily obtained (see, for instance, [8, 10, 12] for the non-uniform case).

Proposition 2.3 *Suppose that p is an extended real-valued lower semicontinuous Minkowski functional defined on a Banach space X,* i.e., *there exists a closed convex set $C \subset X$ with $0 \in C$ such that $p(x) = \inf\{\alpha > 0 : x \in \lambda C\}$ for all $x \in X$. Let $C^* = \{x^* \in X^* : \langle x^*, x\rangle \le p(x), \forall x \in X\}$. Then*

(i) $C^* = \partial p(0) = \partial p(X) = C^0$, the polar of C;

(ii) $x^* \in \partial p(x)$ if and only if $x^* \in C^*$ with $\langle x^*, x\rangle = p(x)$.

Proposition 2.4 *Suppose that f is a continuous convex function defined on a Banach space X. Then*

(i) *the subdifferential mapping $\partial f : X \to 2^{X^*}$ is always nonempty w^* compact convex valued and norm-to-w^* upper semicontinuous at each point of X;*

(ii) *f is Gâteaux differentiable at $x \in X$ if and only if $\partial f(x)$ is a singleton;*

(iii) *f is Fréchet differentiable at $x \in X$ if and only if ∂f is single-valued and norm-to-norm upper semicontinuous at x;*

(iv) *f is uniformly Fréchet differentiable on a subset $A \subset X$ if and only if ∂f is single-valued and uniformly norm-to-norm continuous on A.*

Proposition 2.5 *Let p be a continuous seminorm on a Banach space X, $S_p = \{x \in X : p(x) \le 1\}$ and let $C^* = \{x^* \in X^* : \langle x^*, x\rangle \le p(x), \ \forall x \in X\}$. Then p is uniformly Fréchet differentiable on S_p if and only if for every sequence $\{x_n\} \subset X$ with $p(x_n) = 1$ and all sequences $\{x_n^*\}$, $\{y_n^*\} \subset C^*$ with $x_n^* \in \partial\|x_n\|$ for all $n \in \mathbb{N}$, we have $\|x_n^* - y_n^*\| \to 0$ whenever $\langle y_n^*, x_n\rangle \to 1$.*

Proof *Sufficiency.* We want to show that ∂p is norm-to-norm uniformly continuous on S_p. Let $\{x_n\}, \{y_n\} \subset S_p$ be two sequences with $\|x_n - y_n\| \to 0$. Since p is continuous, C^* is bounded. For any selection ϕ of the subdifferential mapping $\partial\|\cdot\|$ of the norm $\|\cdot\|$, $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are bounded, and they satisfy that $\langle\phi(x_n), y_n\rangle \to 1$ and $\langle\phi(y_n), x_n\rangle \to 1$. Therefore, $\|\phi(x_n) - \phi(y_n)\| \to 0$.

Necessity. Since p is a continuous seminorm and uniformly Fréchet differentiable on S_p, ∂p is single-valued and uniformly norm-to-norm continuous on S_p. Let $\{x_n\} \subset X$ with $p(x_n) = 1$, and let $\{x_n^*\}, \{y_n^*\} \subset C^* \equiv \partial p(0)$ with $x_n^* \in \partial p(x_n)$ for all $n \in \mathbb{N}$, and with $\langle y_n^*, x_n\rangle \to 1$. Therefore, $y_n^* \in \partial_\varepsilon(x_n)$ for all sufficiently large $n \in \mathbb{N}$. By the Brøndsted-Rockafellar theorem, for every $\varepsilon > 0$ we obtain that two sequences $\{x_{\varepsilon,n}\} \subset X$, $\{x_{\varepsilon,n}^*\} \subset X^*$ such that

$$\text{(i) } x_{\varepsilon,n}^* \in p(x_{\varepsilon,n}), \quad \text{(ii) } \|y_n - x_{\varepsilon,n}\| \le \sqrt{\varepsilon} \quad \text{and} \quad \text{(iii) } \|y_n^* - x_{\varepsilon,n}^*\| \le \sqrt{\varepsilon}$$

for all sufficiently large $n \in \mathbb{N}$. Note that the continuity of p, $\|x_n - x_{\varepsilon,n}\| \le \sqrt{\varepsilon}$, and $p(x_n) = 1$ imply that there exists a constant $a > 0$ such that $\|x_n - y_{\varepsilon,n}\| \le a\sqrt{\varepsilon}$, where $y_{\varepsilon,n} \equiv x_{\varepsilon,n}/p(x_{\varepsilon,n}) \in S_p$. The arbitrariness of ε, the homogeneity of p, and the uniform continuity of ∂p on S_p entail that $x_{\varepsilon,n}^* \in \partial p(y_{\varepsilon,n})$ and

$$\|x_n^* - y_n^*\| \le \|x_n^* - x_{\varepsilon,n}^*\| + \|x_{\varepsilon,n}^* - y_n^*\| \to 0. \qquad \blacksquare$$

We also need the following notion.

Definition 2.6 Suppose that X is a linear space and that $|\cdot|$ and $\|\cdot\|$ are two norms on X.

(i) We say that the normed space $(X, |\cdot|)$ is relatively uniformly convex with respect to $\|\cdot\|$ provided that for any two sequences $\{x_n\}, \{y_n\} \subset (X, |\cdot|)$, we have $\|x_n - y_n\| \to 0$ whenever $2(|x_n|^2 + |y_n|^2) - |x_n + y_n|^2 \to 0$; equivalently, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x|^2 + |y|^2 - \frac{1}{2}|x+y|^2 > \delta$ whenever $\|x - y\| \ge \varepsilon$.

(ii) The normed space $(X, |\cdot|)$ is called uniformly convex if it is relatively uniformly convex with respect to $|\cdot|$.

The following lemma is due to Cheng et al. [6, Theorem 4.8 and Corollary 3.11].

Lemma 2.7 *Suppose that K is a super weakly compact convex set of a Banach space $(X, \|\cdot\|)$. Then there exists a reflexive Banach space $(E, |\cdot|)$ such that*

(i) *$K \subset B_E \subset X$;*
(ii) *$\|\cdot\| \le \lambda|\cdot|$ on E for some $\lambda > 0$;*
(iii) *$|\cdot|^2$ is uniformly convex and $\|\cdot\|$-uniformly continuous on K;*
(iv) *$(E, |\cdot|)$ is relatively uniformly convex with respect to $\|\cdot\|$.*

Lemma 2.8 *Suppose that K is a bounded closed convex set of a Banach space $(X, \|\cdot\|)$. Suppose that there is a Banach space $(E, |\cdot|)$ satisfying*

(i) *$K \subset \lambda B_E \subset X$ for some $\lambda > 0$;*
(ii) *$|\cdot|$ is relatively uniformly convex with respect to $\|\cdot\|$ on K.*

Then K is super weakly compact in X.

Proof Assume that K is not super weakly compact. Then there exists $\varepsilon > 0$ such that, for each $n \in \mathbb{N}$, there is an (n, ε)-*tree* $A_n \subset K$,

$$A_n = \{x^n_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k} : k = 1, 2, \dots, n,\ \varepsilon_i = 1, 2 \quad \text{and} \quad i = 1, 2, \dots, k\},$$

where

$$x^n_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k} = \frac{1}{2}(x^n_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k,1} + x^n_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k,2})$$

and

$$\|x^n_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k,1} - x^n_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k,2}\| \geq \varepsilon$$

for $k = 1, 2, \dots, n-1, \varepsilon_i = 1, 2,$ and $i = 1, 2, \dots, k$. Let $f = |\cdot|^2$. Note that f is bounded by λ^2 on K. By Definition 2.6, there exists $\delta > 0$ such that

$$\begin{aligned}
0 &\leq \inf_{x\in K} f(x) \leq f(x^n_{\varepsilon_1}) < \frac{1}{2}\big(f(x^n_{\varepsilon_1,1}) + f(x^n_{\varepsilon_1,2})\big) - \frac{1}{2}\delta \\
&< \frac{1}{2^2}\big(f(x^n_{\varepsilon_1,1,1}) + f(x^n_{\varepsilon_1,1,2}) + f(x^n_{\varepsilon_1,2,1}) + f(x^n_{\varepsilon_1,2,2})\big) - \delta \\
&< \frac{1}{2^n}\big(f(x^n_{\varepsilon_1,1,\dots,1}) + f(x^n_{\varepsilon_1,1,\dots,2}) + \cdots + f(x^n_{\varepsilon_1,2,\dots,1}) + f(x^n_{\varepsilon_1,2,\cdots,2})\big) - 2^{n-1}\delta \\
&\leq \lambda^2 - 2^{n-1}\delta \longrightarrow -\infty, \quad \text{as } n \to \infty.
\end{aligned}$$

This is a contradiction. ■

Now, we are ready to prove the main result of this section. We restate it as follows.

Theorem 2.9 *Suppose that K is a closed bounded convex set of a Banach space $(X, \|\cdot\|)$. Then K is super weakly compact if and only if there exists a w^* lower semicontinuous seminorm p on X^* with $p \geq \sigma_K$ such that p^2 is uniformly Fréchet differentiable on B_{X^*}.*

Proof *Sufficiency.* Since p is a w^* lower semicontinuous seminorm on X^*, it is necessarily continuous. Let $C^* = \{x^* \in X^* : p(x^*) \leq 1\}$, and let $C \subset X$ be a closed convex set such that $C^0 = C^*$. Then C^* is nonempty, convex, and w^* compact. Since $p \geq \sigma_K$ entails that $K \subset C$, it suffices to show that C is super weakly compact. Put $S_p = \{x^* \in X^* : p(x^*) = 1\}$. The uniform Fréchet differentiability of p^2 is equivalent to that p is uniformly Fréchet differentiable on S_p. By [5], C is weakly compact. Since p is w^* lower semicontinuous on X^*, the Fréchet derivative $dp(x^*) \in C$ for every $x^* \in S_p$[5]. Let q be the Minkowski functional generated by C, and let $X_q = \cup_{n=1}^{\infty} nC$. Then q is lower semicontinous on X, and (X_q, q) is a Banach space (see, for instance, the proof of in [15]).

By Lemma 2.8, we need only show that (X_q, q) is relatively uniformly convex with respect to $\|\cdot\|$. Note that C is just the closed unit ball of (X_q, q). We are done if we

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can prove that for any two sequences $\{x_n\}, \{y_n\} \subset C$ with $q(x_n) = q(y_n) = 1$ such that $q(x_n + y_n) \to 2$, we have $\|x_n - y_n\| \to 0$.

Let x_n^*, y_n^* and $z_n^* \in S_p$ such that

$$p(x_n^*) = \langle x_n^*, x_n\rangle = q(x_n) = 1, \quad p(y_n^*) = \langle y_n^*, y_n\rangle = q(y_n) = 1$$

and $p(z_n^*) = \langle z_n^*, z_n\rangle = 1$ for all $n \in \mathbb{N}$, where $z_n = (x_n + y_n)/q(x_n + y_n)$. By Proposition 2.3,

$$dp(x_n^*) = x_n, \; dp(y_n^*) = y_n, \text{and } dp(z_n^*) = z_n.$$

We have that $q(x_n + y_n) \to 2$ implies that

$$\langle z_n^*, x_n\rangle \to 1 = \langle x_n^*, x_n\rangle \text{ and } \langle z_n^*, y_n\rangle \to 1 = \langle y_n^*, y_n\rangle.$$

Uniform Fréchet differentiability of p on S_p and Proposition 2.4 entail that

$$\|z_n - x_n\| = \|dp(z_n^*) - x_n\| \to 0 \text{ and } \|z_n - y_n\| = \|dp(z_n^*) - y_n\| \to 0,$$

and which further imply that $\|x_n - y_n\| \to 0$.

Necessity. Let X_K be the closure of span$K$ in X. Since K is also super weakly compact in X_K, by Lemma 2.7, there is a reflexive Banach space $(E, |\cdot|)$ such that $K \subset B_E \subset \lambda B_{X_K}$ for some $\lambda > 0$, and $(E, |\cdot|)$ is relatively uniformly convex with respect to $\|\cdot\|$. Next, we extend $|\cdot|$ from E to X by $|x|_X = |x|$ if $x \in E$ and $|x|_x = +\infty$ otherwise. Then $|\cdot|_X$ is extended real-valued and lower semicontinuous on X, since B_E is closed in X. Let

$$p \equiv \sigma_{B_E} = |\cdot|_X^* = \sqrt{2(\tfrac{1}{2}|\cdot|_X^2)^*}, \quad S_p = \{x^* \in X^* : p(x^*) = 1\},$$

and note that $\text{co}\{(S_p \cup \ker p\} \supset \lambda^{-1}B_{X^*}$. We need only show that p is uniformly Fréchet differentiable on S_p. Let $\{x_n^*\}, \{y_n^*\} \subset S_p$ satisfy $p(x_n^* - y_n^*) \to 0$. Since B_E is (super) weakly compact in X, there exist $\{x_n\}, \{y_n\} \subset S_E$ such that $\langle x_n^*, x_n\rangle = 1$ and $\langle y_n^*, y_n\rangle = 1$. Therefore, $\langle x_n^*, y_n\rangle = \langle y_n^*, y_n\rangle - \langle x_n^* - y_n^*, y_n\rangle \to 1$ and $\langle y_n^*, x_n\rangle = \langle x_n^*, x_n\rangle - \langle y_n^* - x_n^*, x_n\rangle \to 1$. These entail $|x_n + y_n| \to 2$. The relative uniform convexity of $|\cdot|$ implies that $\|x_n - y_n\| \to 0$. Therefore, p is uniformly Fréchet differentiable on S_p. ■

3 Representation of swcc$(X)^*$

In this section, we shall give the dual of swcc(X). To begin with, we present some notions. The concept of Δ-convex function is used in Cepedello–Boiso [4] (see, also [1, p. 94]). Analogously, we call that a function f defined on a convex subset A of a Banach space X Δ-support function if there are two nonempty (bounded convex) subsets $C, D \subset X^*$ such that $f = \sigma_C - \sigma_D$ on A. In particular, if $0 \in C \cap D$, we say that the function f a Δ-Minkowski functional.

We would like to mention two remarkable results concerning embedding of cc(X) (the normed semigroup of all compact convex sets of a Banach space X and representation of cc$(\mathbb{R}^n)^*$. Radstrom [14] showed that cc(X) is (additivity and nonnegative scalar multiplication preserved) isometric to cone of a real Banach space.

Keimel and Roth [11] proved that $\mathrm{cc}(\mathbb{R}^{n*})^* \simeq C(S_{X^*})^*$, where S_{X^*} denotes the unit sphere of $(\mathbb{R}^n)^*$, and $C(S_{X^*})$ stands for the space of all continuous functions on S_{X^*} equipped with the sup-norm. In Cheng and Zhou [7], it is shown that $\mathrm{cc}(X)^* = C_{\mathrm{PH}}(B_{X^*})^*$, where $C_{\mathrm{PH}}(B_{X^*})$ denotes the Banach space of all w^* continuous positively homogenous functions on X^* restricted to B_{X^*}, while the dual of $\mathrm{wcc}(X)$ (the normed semigroup of all nonempty weakly compact convex sets of X) is just the dual of $C_{\Delta\mathrm{SSFD}}(B_{X^*})$. (The normed space of all w^* lower semicontinuous positively homogenous functions on X^* restricted to B_{X^*} satisfying that for each element f of the space there exist two weakly compact convex sets C and $D \subset X$ such that $f = \sigma_C - \sigma_D$ and such that σ_C^2 and σ_D^2 are Fréchet differentiable on B_{X^*}.)

Inspired by the preceding results, in this section we show that $\mathrm{swcc}(X)^* = C_{\Delta MSUFD}(B_{X^*})^*$, where $C_{\Delta MSUFD}(B_{X^*})$ denotes the normed space of all w^* lower semicontinuous Δ-Minkowski functionals defined on X^* restricted to B_{X^*} satisfying that for each element f of the space there exist two closed bounded convex sets C and $D \subset X$ with $0 \in C \cap D$ such that $f = \sigma_C - \sigma_D$ and such that σ_C^2 and σ_D^2 are uniformly Fréchet differentiable on B_{X^*}.

For a real Banach space X, let

$$\begin{aligned}
P_{\mathrm{swcc}(X)} &= \{\sigma_K : K \in \mathrm{swcc}(X)\};\\
M_{\mathrm{swcc}(X)} &= \{\sigma_{K_1} - \sigma_{K_2} : K_1, K_2 \in \mathrm{swcc}(X)\};\\
\mathrm{swcc}_0(X) &= \{K \in \mathrm{swcc}(X) \text{ with } 0 \in K\};\\
P_{\mathrm{swcc}_0(X)} &= \{\sigma_K : K \in \mathrm{swcc}_0(X)\};\\
M_{\mathrm{swcc}_0(X)} &= \{\sigma_{K_1} - \sigma_{K_2} : K_1, K_2 \in \mathrm{swcc}_0(X)\}.
\end{aligned}$$

Proposition 3.1 *Suppose that X is a Banach space. Then $M_{\mathrm{swcc}(X)} = M_{\mathrm{swcc}_0(X)}$.*

Proof The one side inclusion $M_{\mathrm{swcc}(X)} \supset M_{\mathrm{swcc}_0(X)}$ is trivial. To show $M_{\mathrm{swcc}(X)} \subset M_{\mathrm{swcc}_0(X)}$, let $f = \sigma_{K_1} - \sigma_{K_2}$ for some $K_1, K_2 \in \mathrm{swcc}(X)$. Choose any $x_i \in K_i$ for $i = 1, 2$, and let $K = \mathrm{co}\{\pm x_1, \pm x_2\}$. Then K is convex compact (hence, super weakly compact). By Proposition 1.3, $C \equiv K_1 + K$ and $D \equiv K_2 + K$ are super weakly compact and convex. Therefore, $C, D \in P_{\mathrm{swcc}_0}(X)$ and

$$\begin{aligned}
f = \sigma_{K_1} - \sigma_{K_2} &= (\sigma_{K_1} + \sigma_K) - (\sigma_{K_2} + \sigma_K)\\
&= \sigma_{K_1+K} - \sigma_{K_2+K} = \sigma_C - \sigma_D \in M_{\mathrm{swcc}_0}(X).
\end{aligned}$$ ■

Lemma 3.2 *Suppose that X is a Banach space. Then* $\mathrm{swcc}_0(X)$ *is order isometric to* $P_{\mathrm{swcc}_0(X)}$;

Proof For all $\lambda \geq 0$ and for all $K, K_1, K_2 \in \mathrm{swcc}_0(X)$, we have $\sigma_{K_1+K_2} = \sigma_{K_1} + \sigma_{K_2}$, $\sigma_{\lambda K} = \lambda\sigma_K$. Since $d_H(K_1, K_2) = \|\sigma_{K_1} - \sigma_{K_2}\|$, $\mathrm{swcc}(X)$ is order isometric to $P_{\mathrm{swcc}_0(X)}$, and the lemma follows. ■

Recall that an extended real-valued Minkowski functional p on a Banach space X is a nonnegative-valued sublinear function, *i.e.*, $p(x) \in \mathbb{R}^+ \cup \{+\infty\}$ with $p(\lambda x) = \lambda p(x)$ for all $x \in X$, $\lambda \geq 0$ and with $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Lemma 3.3 *Suppose that X is a Banach space and $p\colon X \to \mathbb{R}^{+}\cup\{+\infty\}$ is an extended real-valued lower semicontinuous Minkowski functional with $p \geq \|\cdot\|$ on X. Let $q_n^2 = p^2 + c_n\|\cdot\|^2$ for all $n \in \mathbb{N}$, where $0 < c_n \to 0$. Then $(q_n^2)^* \to (p^2)^*$ uniformly on B^*.*

Proof By definition of conjugate function, it suffices to note that for all $x^* \in X^*$,

$$\begin{aligned}\frac{1}{\sqrt{1+c_n}}(p^2)^*(x^*) &= \{(1+c_n)p^2\}^* \leq (q_n^2)^*(x^*)\\ &= \sup\{\langle x^*, x\rangle - (p^2(x) + c_n\|x\|^2) : x \in \operatorname{dom} p\}\\ &\leq (p^2)^*(x^*) \leq (\|\cdot\|^2)^*(x^*).\end{aligned}$$

■

Theorem 3.4 *$P_{\mathrm{swcc}_0(X)} = \overline{U}$, the closure of*

$$U \equiv \{\sigma_K : K \in \mathrm{swcc}_0(X), {\sigma_K}^2 \text{ is uniformly Frechet differentiable on } B_{X^*}\}.$$

Proof We show first that $\overline{U} \subset \mathrm{P}_{\mathrm{swcc}_0(X)}$. Suppose that $K \subset X$ is a closed convex set with $\sigma_K \in \overline{U}$, then K also contains the origin. We claim that K is super weakly compact. Let $\sigma_n \equiv \sigma_{K_n} \in U$ such that $\sigma_n \to \sigma_K$ in $C(B_{X^*})$. Then, by Theorem 2.9, K_n are super weakly compact for all $n \in \mathbb{N}$. This entails that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $K \subset K_n + \varepsilon B_X$. According to [6, Lemma 4.5], K is super weakly compact. Conversely, let $K \in \mathrm{swcc}_0(X)$, q_K be the (extended real-valued and lower semicontinuous) Minkowski functional generated by K, *i.e.*, $q_K(x) = \inf\{\alpha > 0 : x \in \alpha^{-1}K\}$. Next, let $p = \sigma_K$, and let X_K be the closure of $\mathrm{span}K$ in X. Then we obtain that $p = \sqrt{2(q_K^2/2)^*}$. Since K is also super weakly compact in X_K, by Lemma 2.7, there is a reflexive space $(E, |\cdot|)$ such that $K \subset B_E \subset \lambda B_{X_K}$ for some $\lambda > 0$, and $(E, |\cdot|)$ is relatively uniformly convex with respect to $\|\cdot\|$. Therefore, the Minkowski functional q_K satisfies $q_K \geq |\cdot|$ on $(E, |\cdot|)$, and for all $a, b > 0$, $f \equiv aq_K^2 + b|\cdot|^2$ is relatively uniformly convex with respect to $\|\cdot\|$, *i.e.*, for any two bounded sequences $\{x_n\}, \{y_n\} \subset (E, |\cdot|)$, we have $\|x_n - y_n\| \to 0$ whenever $f(x_n) + f(y_n) - 2f((x_n + y_n)/2) \to 0$. Let $f_m = \frac{1}{2}q_K^2 + 2^{-m}|\cdot|^2$ for all $m \in \mathbb{N}$. According to Lemma 3.2, $f_m^* \to (\frac{1}{2}q_K^2)^* = \frac{1}{2}p^2$ uniformly on each bounded subset of $(E, |\cdot|)^*$. Applying relative uniform convexity of f_m and a similar discussion of the proof of the necessity part of Theorem 2.9, we can see that f_m^* is uniformly Fréchet differentiable on each bounded subset of $(E, |\cdot|)^*$. Note that $|\cdot|$ is stronger than $\|\cdot\|$ on E and that E is, with respect to the original norm $\|\cdot\|$, a dense subspace of X_K. Within the natural norm-preserved restriction to E, we obtain $X_K^* \subset E^*$ and $B_{X_K^*} \subset \lambda^{-1}B_{E^*}$. These further imply that f_n^* are w^*-lower semicontinuous and uniformly Fréchet differentiable on each bounded subset of $X_K^* = X^*/X_K^0$. Now, we define Minkowski functionals $\{p_n\}_{n\in\mathbb{N}}$ for $x^* \in X^*$ by $p_n(x^*) = \sqrt{2f_n^*(Q(x^*))}$, where $Q : X^* \to X^*/X_K^0$ denotes the quotient mapping. Then it is easy to see that $p_n \to p$ and p_n^2 are uniformly Fréchet differentiable on each bounded subset of X^*. ■

Corollary 3.5 *$M_{\mathrm{swcc}(X)}$ is a dense subspace of $C_{\Delta\mathrm{MSUFD}}(B^*)$.*

Proof By Proposition 3.1 and Theorem 3.4,

$$M_{\mathrm{swcc}(X)} = M_{\mathrm{swcc}_0(X)} = P_{\mathrm{swcc}_0(X)} - P_{\mathrm{swcc}_0(X)} = \overline{U} - \overline{U} \subset C_{\Delta\mathrm{MSUFD}}(B^*).$$

According to definition of $C_{\Delta\text{MSUFD}}(B^*)$, for every $\varepsilon > 0$ and for every $f \in C_{\Delta\text{MSUFD}}(B^*)$, there exists $f_\varepsilon = \sigma_{K_1} - \sigma_{K_2}$ for some closed bounded convex sets $K_1, K_2 \in X$ with $0 \in K_1 \cap K_2$ such that both $\sigma_{K_1}^2$ and $\sigma_{K_2}^2$ are uniformly Fréchet differentiable on B_{X^*} satisfying

$$|f(x^*) - f_\varepsilon(x^*)| < \varepsilon \text{ uniformly for } x^* \in B_{X^*}.$$

By Theorem 3.4 again, we get $K_1, K_2 \in \text{swcc}_0$ and $f_\varepsilon \in M_{\text{swcc}(X)}$. ∎

The following result is the main theorem of this section.

Theorem 3.6 *Suppose that X is a Banach space. Then*

$$\text{swcc}(X)^* = C_{\Delta\text{MSUFD}}(B^*)^*.$$

Proof Since $M_{\text{swcc}(X)}$ is a dense subspace of $C_{\Delta\text{MSUFD}}(B^*)$(Corollary 3.5), we have $M^*_{\text{swcc}(X)} = C_{\Delta\text{MSUFD}}(B^*)^*$. Since swcc$(X)$ is (ordered isometric to) a reproducing cone of $M_{\text{swcc}(X)}$ with nonempty interior, by definition of the dual of a normed semigroup it is easy to show that $\text{swcc}(X)^* = M^*_{\text{swcc}(X)}$. ∎

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