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GOLDIE DIMENSION AND CHAIN CONDITIONS FOR MODULAR LATTICES WITH FINITE GROUP ACTIONS

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ABSTRACT. A relation between Goldie dimensions of a modular lattice L and its sublattice L^G of fixed points under a finite group G of automorphisms of L is obtained. The method used also gives a relation between ACC (DCC) for L and for L^G. The results obtained are applied to rings and modules.

Introduction. In [2] Fisher initiated studies of relations between finiteness conditions on a modular lattice L and its sublattice L^G consisting of fixed points of an action of a finite group G of automorphisms of L. He proved that if L^G satisfies any of a large class of chain conditions then L satisfies the same condition. The aim of this paper is to investigate relations between Goldie dimensions dL and dL^G of L and L^G respectively. In section 3 we prove that $dL^G \leqslant dL \leqslant |G|\, dL^G$. Our methods, which are quite different from those used by Fisher, can also be applied to some chain conditions. In section 2 we show how they work for ACC and DCC. The results obtained are applied in section 4 to rings and modules. In particular we give a new proof of Kharchenko's theorem ([6]) which says that if G is a finite group of automorphisms of a ring R then R contains no infinite direct sum of non-zero right ideals if and only if R contains no infinite direct sum of non-zero G-invariant right ideals.

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1. **Preliminaries**. Thoughout the paper G is a finite group of automorphisms of a lattice L and $L^G = \{x \in L \mid x^g = x \text{ for all } g \in G\}$. We always assume that L contains 0 and 1. By the standard procedure of adjoining 0 and 1 we can omit this assumption in many places. We used the terminology of [1].

LEMMA 1. *If the lattice L is complete then for every $S \subseteq L$ and $g \in G$, $(\bigvee S)^g = \bigvee_{s\in S} s^g$.*

The proof is straightforward.

The lattice L is said to be upper continuous ([1]) if L is complete and for every element $a \in L$ and every chain C in L, $a \wedge \bigvee C = \bigvee_{x\in C} a \wedge x$

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LEMMA 2. *If L is upper continuous then the set $P = \{a \in L \mid \bigwedge_{g \in G} a^g = 0\}$ is closed under taking joins of chains, and so in particular it contains a maximal element.*

PROOF. If $x \in L$ and C is a chain in L such that $x \wedge \bigvee C \neq 0$ then, since $x \wedge \bigvee C = \bigvee_{c \in C} x \wedge c$, $x \wedge c \neq 0$ for some $c \in C$. By induction, if $C_1, \ldots, C_n$ are chains in L such that $\bigvee C_1 \wedge \ldots \wedge \bigvee C_n \neq 0$ then for some $c_1 \in C_1, \ldots, c_n \in C_n$, $c_1 \wedge \ldots \wedge c_n \neq 0$. Now let C be a chain in P. If $\bigwedge_{g \in G} (\bigvee C)^g \neq 0$ then by Lemma 1 and the foregoing, for every $g \in G$ there exists $c_g \in C$ such that $\bigwedge_{g \in G} c_g^g \neq 0$. Since C is a chain, there exists $a \in C$ such that $c_g \leqslant a$ for all $g \in G$. Thus for $g \in G$, $c_g^g \leqslant a^g$ and $\bigwedge_{g \in G} a^g \neq 0$.

For $x \leqslant y$ in L, let $[x, y]$ denote $\{z \in L \mid x \leqslant z \leqslant y\}$. A non-empty subset I of L is called an ideal of L if $x, y \in I$ implies $[0, x \vee y] \subseteq I$. The set $I(L)$ of all ideals of L is an upper continuous lattice with respect to operations: $\bigwedge S_\alpha = \bigcap S_\alpha$ and $\bigvee S_\alpha =$ the ideal generated by $\bigcup S_\alpha$. It is clear that the map $p: L \to I(L)$ given by $p(a) = [0, a]$ is an embedding of lattices, so L may be treated as a sublattice of $I(L)$. It is also obvious that the action of G can be extended to $I(L)$. Elements of $I(L)^G$ are called G-invariant ideals of L.

LEMMA 3. $I(L^G) \approx I(L)^G$.

PROOF. Define for $I \in I(L^G)$ and $J \in I(L)^G$, $f(I) =$ the ideal of L generated by I and $g(J) = J \cap L^G$. It is easy to check that $f: I(L^G) \to I(L)^G$ and $g: I(L)^G \to I(L^G)$ are lattice homomorphisms satisfying $f \circ g = id_{I(L)^G}$, $g \circ f = id_{I(L^G)}$.

For the remainder of this paper the lattice L will be modular. This assumption implies that

A. The lattice L^0 dual to L is modular;

B. The lattice $I(L)$ is modular;

C. The following Isomorphism Theorem holds: If $a, b \in L$ then the mapping $f_b: [a, a \vee b] \to [a \wedge b, b]$ defined by $f_b(x) = x \wedge b$, is a lattice isomorphism with inverse given by $g_a(y) = y \vee a$.

2. **Chain conditions**. Let us recall that L is said to satisfy ACC (DCC) if for every chain $a_1 \leqslant a_2 \leqslant \ldots$ ($a_1 \geqslant a_2 \geqslant \ldots$) of elements of L there exists n such that $a_n = a_{n+1} = \ldots$. It is clear that L satisfies ACC if and only if the lattice L^0 dual to L satisfies DCC.

As a consequence of the Isomorphism Theorem one obtains the following well known

LEMMA 4. *If $a_1, \ldots, a_n \in L$ then the lattices $[a_1, 1], \ldots, [a_n, 1]$ satisfy ACC (DCC) if and only if the lattice $[a_1 \wedge \ldots \wedge a_n, 1]$ satisfies ACC (DCC).*

Now we prove

LEMMA 5. *L satisfies ACC if and only if $I(L)$ satisfies ACC.*

PROOF. L is a sublattice of $I(L)$, so if $I(L)$ satisfies ACC then the same is true for L.

Now let $I_1 < I_2 < \ldots$ be a strictly ascending chain of ideals of L. Let us take for every i, $a_i \in I_{i+1} \setminus I_i$ and put $b_i = \bigvee_{k=1}^{i} a_k$. Obviously $b_1 \leq b_2 \leq \ldots$ and $b_i \in I_{i+1}$. Since $a_i \leq b_i$, $a_i \notin I_i$ and I_i is an ideal, we have $b_i \notin I_i$. Thus $b_1 < b_2 < \ldots$. In consequence if L satisfies ACC then the same is true for $I(L)$.

Now we present a new proof of the following

THEOREM 1 (Fisher [2]). *L satisfies ACC (DCC) if and only if L^G satisfies ACC (DCC).*

PROOF. It is clear that if L satisfies ACC then the same is true for L^G. Conversely, let us assume that L^G satisfies ACC. By Lemmas 3 and 5 we can assume (passing if necessary to $I(L)$) that L is upper continuous. If L does not satisfy ACC then there exists $a \in L^G$ maximal with respect to the property that the lattice $[a, 1] = \{x \in L | a \leq x\}$ does not satisfy ACC. Obviously the lattice $[a, 1]$ is upper continuous and G acts on it. Hence by Lemma 2, the set $P = \{x \in [a, 1] | \bigwedge_{g \in G} x^g = a\}$ contains a maximal element m. Now Lemma 4 implies that for some $g \in G$ the lattice $[m^g, 1]$ does not satisfy ACC. But for each $g \in G$ the lattices $[m^g, 1]$ and $[m, 1]$ are isomorphic, so the lattice $[m, 1]$ does not satisfy ACC. Thus we can find in L a strictly ascending chain $m < m_1 < m_2 < \ldots$. By choice of m, $a < \bigwedge_{g \in G} m_1^g$. Hence by choice of a, $[\bigwedge_{g \in G} m_1^g, 1]$ has ACC. But $\bigwedge_{g \in G} m_1^g < m_1 < m_2 < \ldots$, giving a contradiction.

Applying the foregoing to the lattice L^0 dual to L we obtain the result for DCC.

3. **Goldie dimension**. A subset $\{x_1, \ldots, x_n\}$ of L is said to be join-independent if for each $1 \leq i \leq n$, $x_i \wedge (x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n) = 0$.

The Goldie dimension of L is defined as $dL = \sup\{k | L$ contains a join-independent subset of k elements$\}$. The Goldie dimension of the lattice L^0 dual to L is called the hollow dimension and denoted by hL.

An element $a \in L$ is said to be essential in L if for every $0 \neq x \in L$, $a \wedge x \neq 0$. We say that a non-zero element $u \in L$ is uniform if every non-zero element $x \leq u$ is essential in $[0, u]$.

The following characterization of the Goldie dimension was given in [4].

THEOREM 2. *$dL = n < \infty$ if and only if L contains a join-independent subset $\{a_1, \ldots, a_n\}$ of uniform elements such that the element $a_1 \vee \ldots \vee a_n$ is essential in L.*

LEMMA 6. *For each $a \in L$, $dL \leq d[0, a] + d[a, 1]$.*

PROOF. First, we claim that if $\{a, x_1, \ldots, x_r\}$ is a join-independent set in L, then $\{a \vee x_1, \ldots, a \vee x_r\}$ is a join-independent set in $[a, 1]$. Indeed, using modularity we see that for every $1 \leq i \leq r$,

$$\begin{aligned}(a \vee x_i) \wedge (a \vee x_1 \vee \ldots a \vee x_{i-1} \vee a \vee x_{i+1} \vee \ldots \vee a \vee x_r) \\ = a \vee ((a \vee x_i) \wedge (x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_r) \\ = a.\end{aligned}$$

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Now the inequality in the statement of the lemma is trivial if either term on the right hand side of that inequality is infinite, so assume both are finite. Then the preceding observation gives a finite bound on the cardinalities r of families $x_1, \ldots, x_r$ such that $\{a, x_1, \ldots, x_r\}$ is join-independent. Let us choose $x_1, \ldots, x_r$ with this property so as to maximize r. Then clearly the x_i are uniform, $r \leqslant d[a, 1]$ and $a \vee x_1 \vee \ldots \vee x_r$ is essential in $[a, 1]$. Since $d = d[0, a]$ is finite, there exists a join-independent family $\{y_1, \ldots, y_d\}$ of uniform elements of $[0, a]$. Clearly $y_1 \vee \ldots \vee y_d$ is essential in $[0, a]$. Now by Lemmas 2 and 3 of [4], $y_1 \vee \ldots \vee y_d \vee x_1 \vee \ldots \vee x_r$ is essential in L. Hence Theorem 2 gives $dL = d + r \leqslant d[0, a] + d[a, 1]$, as required.

COROLLARY 1. *If* $a_1, \ldots, a_n \in L$ *and* $a_1 \wedge \ldots \wedge a_n = 0$ *then* $dL \leqslant \Sigma_{i=1}^{n} d[a_i, 1]$.

PROOF. We proceed by induction on n. For $n = 1$ the result is clear. Thus let $n \geqslant 2$ and $\bar{a}_1 = a_2 \wedge \ldots \wedge a_n$. By the induction assumption $d[\bar{a}_1, 1] \leqslant \Sigma_{i=2}^{n} d[a_i, 1]$. Since $a_1 \wedge \bar{a}_1 = 0$, $[0, a_1] = [a_1 \wedge \bar{a}_1, a_1] \approx [\bar{a}_1, a_1 \vee \bar{a}_1] \subseteq [\bar{a}_1, 1]$. Hence $d[0, a_1] \leqslant \Sigma_{i=2}^{n} d[a_i, 1]$. Now by Lemma 6, $dL \leqslant \Sigma_{i=1}^{n} d[a_i, 1]$.

LEMMA 7. $dL = dI(L)$.

PROOF. Obviously, $dL \leqslant dI(L)$. Conversely, let $I_1, \ldots, I_n$ be a join-independent subset of $I(L)$ and let $0 \neq x_i \in I_i$ for $1 \leqslant i \leqslant n$. For every $1 \leqslant j \leqslant n$, $x_j \wedge (x_1 \vee \ldots \vee x_{j-1} \vee x_{j+1} \vee \ldots \vee x_n) \in I_j \wedge (I_1 \vee \ldots \vee I_{j-1} \vee I_{j+1} \vee \ldots \vee I_n) = 0$. Hence $\{x_1, \ldots, x_n\}$ is a join-independent subset of L. Thus $dI(L) \leqslant dL$.

Now we can prove the main result of the paper.

THEOREM 3. $dL^G \leqslant dL \leqslant |G| dL^G$.

PROOF. By Lemmas 3 and 7 we can assume that L is upper continuous. Thus using Lemma 2 we can find a maximal element l in the set $\{x \in L | \bigwedge_{g \in G} x^g = 0\}$. Now by Corollary 1 it is enough to show that $d[l, 1] \leqslant dL^G$, which will clearly follow if we show that for every family of join-independent elements $x_1, \ldots, x_m \in [l, 1]$, the elements $\bar{x}_i = \bigwedge_{g \in G} x_i^g$ are join-independent in L^G. By the choice of l, $\bar{x}_i \neq 0$ for $1 \leqslant i \leqslant m$. Thus we have to show that for $1 \leqslant i \leqslant m$, $y_i = \bar{x}_i \wedge (\bar{x}_1 \vee \ldots \vee \bar{x}_{i-1} \vee \bar{x}_{i+1} \vee \ldots \vee \bar{x}_m) = 0$. Since for every $1 \leqslant i \leqslant m$, $l < l \vee \bar{x}_i \leqslant x_i$ and the set $\{x_1, \ldots, x_m\}$ is join-independent in $[l, 1]$, the set $\{l \vee \bar{x}_1, \ldots, l \vee \bar{x}_m\}$ is join-independent in $[l, 1]$. In consequence for all i, $l = (l \vee \bar{x}_i) \wedge (\bigvee_{1 \leqslant j \neq i \leqslant m} l \vee \bar{x}_j) \geqslant y_i$. This, the choice of l and the fact that $y_i \in L^G$ imply, $0 = \bigwedge_{g \in G} l^g \geqslant \bigwedge_{g \in G} y_i^g = y_i$.

Applying Theorem 3 to the lattice L^0 dual to L we obtain

COROLLARY 2. $hL^G \leqslant hL \leqslant |G| hL^G$.

REMARK. In [5] the notion of Goldie dimension was extended to the infinite case. By similar arguments to those used in the proof of Theorem 3 one can obtain that for an infinite cardinal α, $dL = \alpha$ if and only if $dL^G = \alpha$.

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4. **Applications**. Let G be a finite group and S a Clifford system for G i.e. ([9]) S is a ring such that

$$S = \sum_{g \in G} S_g$$

for additive subgroups S_g, $g \in G$, satisfying $S_g S_h = S_{gh}$ for all $g, h \in G$. Obviously S_e is a subring of S.

Let M_s be a right S-module and $L(M_{S_e})$ the lattice of S_e-submodules of M. Define for $g \in G, f_g : L(M_{S_e}) \to L(M_{S_e})$ by $f_g N = NS_g$. Since $f_g \circ f_{g^{-1}} = f_{g^{-1}} \circ f_g = id$ and for all $N, K \in L(M_{S_e})$ if $N \subseteq K$ the $f_g(N) \subseteq f_g(K)$, all f_g are automorphisms of $L(M_{S_e})$. Thus G acts on $L(M_{S_e})$ by automorphisms and $L(M_{S_e}) = \{N \in L(M_{S_e}) | NS_g = N$ for all $g \in G\} = L(M_S)$ is the lattice of S-submodules of M.

For every right R-module M, $dL(M_R)$ is equal to the Goldie rank, rank M_R, of M_R and $hL(M_R)$ is equal to corank M_R, defined in [10]. Hence Theorem 3 yields

COROLLARY 3. rank $M_S \leqslant$ rank $M_{S_e} \leqslant |G|$ rank M_S
corank $M_S \leqslant$ corank $M_{S_e} \leqslant |G|$ corank M_S.

If G is a finite group of R-automorphisms of an R-module M then M is a module over the group ring $R^1[G]$, where R^1 denotes the natural extension of R to a ring with unity. Moreover G-invariant submodules of M are exactly those submodules which are $R^1[G]$-submodules of M. Obviously $S = R^1[G]$ is a Clifford system for G with $S_e = R^1$. Thus if irank M (icorank M) denotes rank (corank) of M taken with respect to G-invariant submodules of M, we have the following

COROLLARY 4. irank $M \leqslant$ rank $M \leqslant |G|$ irank M
icorank $M \leqslant$ corank $M \leqslant |G|$ icorank M.

Let G be a finite group of automorphisms of a ring R and let $R^1 * G$ be the skew group ring. The ring R has a natural structure of a right $R^1 * G$-module (cf. [7]). Right $R^1 * G$-submodules of R are precisely G-invariant right ideals of R. This observation and Corollary 3 give a relation between the Goldie rank (corank) and the G-invariant rank (corank) of R. Symmetric arguments give the same relations for left ranks (coranks).

COROLLARY 5. irank $R \leqslant$ rank $R \leqslant |G|$ irank R
icorank $R \leqslant$ corank $R \leqslant |G|$ icorank R.

In particular Corollary 5 gives a quite different proof of Kharchenko's theorem presented in [6, 7] which says that R contains no infinite direct sum of non-zero right ideals if and only if R contains no infinite direct sum of non-zero right G-invariant ideals.

It is known (cf. [7]) that if R is a semiprime ring and multiplication by $|G|$ is a bijection on R then the right G-invariant Goldie rank of R is equal to the right Goldie rank of the fixed ring R^G. Thus (cf. [7])

$$\text{rank } R^G \leqslant \text{rank } R \leqslant |G| \text{ rank } R^G.$$

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LEMMA 8. *If G is a finite group of automorphisms of a ring R with unity and $|G|^{-1} \in R$ then* icorank R = corank R^G.

PROOF. Let $t:R \to R^G$ be the trace map i.e. $t(x) = |G|^{-1} \Sigma_{g \in G} x^g$. Obviously $t(R) = R^G$ and if I is a proper G-invariant right ideal of R then $t(I) = I \cap R^G$ is a proper right ideal of R^G. For every right ideal J of R^G, JR is a G-invariant right ideal of R. If $a \in JR \cap R^G$ then for some $j_1, \ldots, j_n \in J$ and $r_1, \ldots, r_n \in R$, $a = j_1r_1 + \ldots + j_nr_n$. Now $a = t(a) = j_1t(r_1) + \ldots + j_nt(r_n) \in J$. Hence $JR \cap R^G = J$. In particular if J is a proper right ideal of R^G then JR is a G-invariant proper right ideal of R.

Now if $J_1, \ldots, J_n$ are proper right ideals of R^G such that for every $1 \leq j \leq n$, $J_j + \bigcap_{i \neq j} J_i = R^G$ then $R = R^G R = (J_j + \bigcap_{i \neq j} J_i)R \subseteq J_jR + \bigcap_{i \neq j} J_iR$. Thus for every $1 \leq j \leq n$, $R = J_jR + \bigcap_{i \neq j} J_iR$ and J_kR, $1 \leq k \leq n$, are proper G-invariant right ideals of R. This proves that corank $R^G \leq$ icorank R. Conversely, let $I_1, \ldots, I_n$ be proper G-invariant right ideals of R such that for every $1 \leq j \leq n$, $I_j + \bigcap_{i \neq j} I_i = R$. Then $R^G = t(R) = t(I_j) + t(\bigcap_{i \neq j} I_i) \subseteq t(I_j) + \bigcap_{i \neq j} t(I_i)$. Thus $R^G = t(I_j) + \bigcap_{i \neq j} t(I_i)$ and $t(I_k)$, $1 \leq k \leq n$, are proper right ideals of R^G. In consequence icorank $R \leq$ corank R^G.

Since the corank of a ring with unity is finite if and only if the ring is semilocal (cf. [8]), Lemma 8 and Corollary 5 imply

COROLLARY 6 (cf. [7]). *If G is a finite group of automorphisms of a ring R and multiplication by $|G|$ is a bijection on R then R is semilocal if and only if the fixed ring R^G is semilocal.*

We close the paper with some remarks concerning radicals. Let the lattice L be complete. We define the radical $r(L)$ of L as the intersection of all maximal elements of L (if L contains no maximal element then $r(L) = 1$). Obviously for a module M, $r(L(M))$ is equal to the Jacobson radical $J(M)$ of M.

PROPOSITION 1. $r(L) \leq r(L^G)$.

PROOF. Let a be a maximal element of L^G. By Theorem 1 the lattice $[a, 1]$ contains a maximal element b. It is clear that the element b is maximal in L and $\bigwedge_{g \in G} b^g = a$. In consequence $r(L) \leq r(L^G)$.

The following example shows that $r(L)$ is not always equal to $r(L^G)$.

EXAMPLE. *Let K be a field of characteristic $p > 0$ and G a finite group with $p \| |G|$. If L is the lattice of all K-subspaces of the group algebra $K[G]$ then $r(L) = 0$. Defining for $g \in G$, $f_g:L \to L$ by $f_g(N) = gN$ we obtain an action of G on L. Obviously $r(L^G) = J(K[G]) \neq 0$.*

Proposition 1, the foregoing remarks on modules over Clifford systems and Maschke's theorem for Clifford systems ([9]) imply

COROLLARY 7. *If S is a Clifford system for a finite group G and M_S is a right module over S then*

a) $J(M_{S_c}) \subseteq J(M_S)$;
b) *if multiplication by* $|G|$ *is a bijection on S then* $J(M_{S_c}) = J(M_S)$.

As a special case of Corollary 7 we obtain

COROLLARY 8. *If G is a finite group of automorphisms of a ring R then* $J(R)$ *is contained in the intersection* $J^G(R)$ *of all maximal G-invariant right ideals of R. If in addition multiplication by* $|G|$ *is a bijection on R then* $J(R) = J^G(R)$.

Using the trace map one can easily check that if R has unity and $|G|^{-1} \in R$ then $J(R^G) = J^G(R) \cap R^G$. As an effect of this and Corollary 8 we obtain

COROLLARY 9 (cf. [7]). *If G is a finite group of automorphisms of a ring with unity and* $|G|^{-1} \in R$ *then* $J(R) \cap R^G = J(R^G)$.

REFERENCES

1. P. Grawley and R. P. Dilworth, *Algebraic Theorey of Lattices*, Prentice-Hall, Englewood Cliffs, N.J., 1973.
2. J. Fisher, *Chain conditions for modular lattices with finite group actions*, Canad. J. of Math., **31** (1979), pp. 558–564.
3. G. Grätzer, *General Lattice Theorey*, Birkhauser-Verlag, Basel, 1978.
4. P. Grzeszczuk and E. R. Puczyłowski, *On Goldie and dual Goldie dimensions*, J. Pure Appl. Algebra, **31** (1984), pp. 47–54.
5. P. Grzeszczuk and E. R. Puczyłowski, *On infinite Goldie dimension of modular lattices and modules*, J. Pure Appl. Algebra, **35** (1985), pp. 151–155.
6. V. K. Kharchenko, *Galois extensions and quotient rings*, Algebra i Logika, **13** (1974), pp. 460–484 (Russian).
7. S. Montgomery, *Fixed rings of finite automorphism groups of associative rings*, LNM 818, Springer Verlag, 1980.
8. B. Sarath and K. Varadarajan, *Dual Goldie dimension — II*, Comm. in Algebra, **17** (1979), pp. 1885–1899.
9. F. Van Oystaeyen, *On Clifford systems and generalized crossed products*, J. Algebra, **87** (1984), pp. 396–415.
10. K. Varadarajan, *Dual Goldie dimension*, Comm. in Algebra, **7** (1979), pp. 565–610.

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