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SOME NORMALITY CRITERIA AND A COUNTEREXAMPLE TO THE CONVERSE OF BLOCH'S PRINCIPLE

KULDEEP SINGH CHARAK✉ and SHITTAL SHARMA

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Abstract

In this paper, we prove some value distribution results which lead to normality criteria for a family of meromorphic functions involving the sharing of a holomorphic function by more general differential polynomials generated by members of the family, and improve some recent results. In particular, the main result of this paper leads to a counterexample to the converse of Bloch's principle.

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1. Introduction and main results

A family $\mathcal{F}$ of meromorphic functions in a complex domain D is said to be *normal* in D if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of D with respect to the spherical metric. The concept of normality was introduced in 1907 by Montel [13]. Normal families play a central role in complex dynamics, and are of great interest in their own right. For normal families of meromorphic functions, we refer to Schiff's book [15], Zalcman's survey article [19] and Drasin's paper [7], out of a huge literature on the subject. Drasin [7] brought Nevanlinna value distribution theory [9] into the study of normal families of meromorphic functions and Schwick [16] introduced the concept of sharing of values. In this paper, which continues our earlier work [4], we prove a value distribution result leading to some interesting normality criteria, one of which leads to a counterexample to the converse of Bloch's principle. These normality criteria involve the sharing of holomorphic functions by a more general class of differential polynomials and generalise and improve recent results.

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Let $f \in \mathcal{F}$ and let $h(z)$ be a holomorphic function on D. Let $k \geq 1, l_0, l_1, l_2, \ldots, l_k$, $m_1, m_2, \ldots, m_k$ be nonnegative integers with $l' = \sum_{i=1}^{k} l_i$ and $m' = \sum_{i=1}^{k} m_i$ and let

$$P[f] = f^{l_0}(f^{l_1})^{(m_1)}(f^{l_2})^{(m_2)} \cdots (f^{l_k})^{(m_k)}$$

be a differential polynomial of f with degree $\gamma_P = l_0 + l'$. We assume that $l_0 > 0$ and $l_i \geq m_i$ for $1 \leq i \leq k$ with $l' > m' > 0$. Further, we can see that

$$(f^{l_i})^{(m_i)} = \sum C_{n_0 n_1 n_2 \ldots n_{m_i}} f^{n_0}(f')^{n_1}(f'')^{n_2} \cdots (f^{(m_i)})^{n_{m_i}}$$

is such that $\sum_{j=0}^{m_i} n_j = l_i$ and $\sum_{j=1}^{m_i} j n_j = m_i$. Thus, the weight

$$w((f^{l_i})^{(m_i)}) = \max\left\{\sum_{j=0}^{m_i} (j+1) n_j\right\} = l_i + m_i$$

and so

$$w(P[f]) = l_0 + \sum_{i=1}^{k} (l_i + m_i) = l_0 + l' + m' = \gamma_P + m'.$$

It is assumed that the reader is familiar with the standard notions of Nevanlinna value distribution theory such as $m(r, f), N(r, f), T(r, f), S(r, f)$ and so on (see [9]).

DEFINITION 1.1. Two meromorphic functions f and g in a domain D *share* the function h IM in D if $\overline{E}(h, f) = \overline{E}(h, g)$, where $\overline{E}(h, \phi) = \{z \in D : \phi(z) - h(z) = 0\}$ is the set of zeros of $\phi - h$ in D ignoring multiplicities (IM). If $\overline{E}(h, f) \subseteq \overline{E}(h, g)$, then we say that *$f$ shares h partially* with g on D.

Dethloff *et al.* proved the following Picard-type theorem.

THEOREM 1.2 [6, Corollary 2, page 676]. *Let a be a nonzero complex value, l_0 a nonnegative integer and $l_1, l_2, \ldots, l_k$, $m_1, m_2, \ldots, m_k$ positive integers. Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D such that, for any $f \in \mathcal{F}$, $P[f] - a$ is nowhere vanishing on D. Assume that:*

(a) $l_j \geq m_j$ *for* $1 \leq j \leq k$;
(b) $l_0 + l' \geq 3 + m'$.

Then $\mathcal{F}$ is normal in D.

Dutt and Kumar extended Theorem 1.2 as follows.

THEOREM 1.3 [8, Theorem 1.4, page 2]. *Let a be a nonzero complex value, l_0 a nonnegative integer and $l_1, l_2, \ldots, l_k$, $m_1, m_2, \ldots, m_k$ positive integers such that:*

(a) $l_j \geq m_j$ *for* $1 \leq j \leq k$;
(b) $l_0 + l' \geq 3 + m'$.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that for every pair $f, g \in \mathcal{F}$, $P[f]$ and $P[g]$ share a IM on D. Then $\mathcal{F}$ is normal in D.

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It is natural to consider the following more general question.

Question 1.4. *Is the family $\mathcal{F}$ normal in D if for each pair of functions f and g in $\mathcal{F}$ the differential polynomials $P[f]$ and $P[g]$ share a holomorphic function h IM?*

We answer Question 1.4 as follows.

Theorem 1.5. *Let $\mathcal{F}$ be a family of nonconstant meromorphic functions on a domain D such that each $f \in \mathcal{F}$ has poles, if any, of multiplicity at least l_0. Let $h \not\equiv 0$ be a holomorphic function on D having only zeros of multiplicity at most $l_0 - 1$. If $P[f]$ and $P[g]$ share h IM on D for each pair $f, g \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Example 1.6. We show that the condition $h \not\equiv 0$ in Theorem 1.5 is essential. Let $D = \mathbb{D}$, the open unit disc. Consider the family of meromorphic functions on $\mathbb{D}$:

$$\mathcal{F} = \{f_n : f_n(z) = e^{nz^2}, z \in \mathbb{D}\}.$$

Let $P[f] = f(f^2)' = 2f^2 f'$. Then $P[f_n](z) = 2f_n^2(z) f_n'(z) = 4nze^{3nz^2}$. Therefore, for distinct m, n, we see that $P[f_m]$ and $P[f_n]$ share $h \equiv 0$ IM. But the family $\mathcal{F}$ fails to be normal at $z = 0$ in $\mathbb{D}$, since $f_n(0) = 1$ for all n and $f_n(z) \to \infty$ for all $z \neq 0$ in $\mathbb{D}$.

A direct consequence of Theorem 1.5 is the following important result, which, as we will see, leads to a counterexample to the converse of Bloch's principle.

Corollary 1.7. *Let $\mathcal{F}$ be a family of nonconstant meromorphic functions on a domain D. Let $h \not\equiv 0$ be a holomorphic function such that $h(z) \neq 0$ in D. If $P[f] - h$ has no zero in D for any $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Bloch's principle (see [1]) states that a family of holomorphic (meromorphic) functions satisfying a property **P** in a domain D is likely to be normal if the property **P** reduces every holomorphic (meromorphic) function on $\mathbb{C}$ to a constant. Bloch's principle is not universally true (see, for example, [14]).

The converse of Bloch's principle states that if a family of meromorphic functions satisfying a property **P** on an arbitrary domain D is normal, then every meromorphic function on $\mathbb{C}$ with property **P** reduces to a constant. Like Bloch's principle, the converse is not true. For counterexamples, see [2, 5, 11, 12, 15, 18] and [10].

Counterexample 1.8. Suppose $P[f] = f(f^3)'' = f(3f^2 f')' = 3f^3 f'' + 6f^2 f'^2$ and let $f(z) = e^{-z}$ be defined on $\mathbb{C}$. Then

$$P[f](z) = 3e^{-3z}e^{-z} + 6e^{-2z}e^{-2z} = 9e^{-4z}.$$

Take $h(z) = e^{-4z}$, so that $h \not\equiv 0$ and h is holomorphic in $\mathbb{C}$ and hence in every domain $D \subseteq \mathbb{C}$, and also $h(z) \neq 0$ for $z \in D$. Then $(P[f] - h)(z) = 8e^{-4z}$ has no zeros in $\mathbb{C}$. Note that f is nonconstant, which violates the statement of the converse of Bloch's principle in view of Corollary 1.7.

Next we discuss normality of $\mathcal{F}$ when $P[f] - h$ has zeros under different scenarios.

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THEOREM 1.9. *Let $\mathcal{F}$ be a family of nonconstant meromorphic functions on a domain D. Let h be a holomorphic function on D such that $h(z) \neq 0$ in D. If, for each $f \in \mathcal{F}$, any one of the following three conditions holds:*

(i) $(P[f] - h)(z)$ *has at most one zero;*
(ii) $(P[f] - h)(z) = 0$ *implies that* $|f(z)| \geq M$ *for some* $M > 0$;
(iii) $(P[f] - h)(z) = 0$ *implies that* $|(f^{l_i})^{(m_i)}(z)| \leq M$ *for some positive* M, l_i *and* m_i,

then $\mathcal{F}$ is normal in D.

Further, under the weaker hypothesis of partial sharing (see [3, 4]) of holomorphic functions, we can prove the following result.

THEOREM 1.10. *Let $\mathcal{F}$ be a family of nonconstant meromorphic functions on a domain D. Let h be a holomorphic function on D such that $h(z) \neq 0$ in D. If, for every $f \in \mathcal{F}$, there exists $\tilde{f} \in \mathcal{F}$ such that $P[f]$ shares h partially with $P[\tilde{f}]$, then $\mathcal{F}$ is normal in D, provided $h \not\equiv P[\tilde{f}]$ in D.*

REMARK 1.11. Theorem 1.5 improves and generalises Theorems 1.2 and 1.3. Theorem 1.10 is a direct generalisation of [4, Theorem 1.3].

2. Some value distribution results

To facilitate the proofs of our theorems, we prove some value distribution results.

THEOREM 2.1. *Let f be a transcendental meromorphic function. Then $P[f](z) - \omega(z)$ has infinitely many zeros for any small function $\omega(\not\equiv 0, \infty)$ of f.*

PROOF. Suppose on the contrary that $P[f](z) - \omega(z)$ has only finitely many zeros. Then, by the second fundamental theorem of Nevanlinna for three small functions [9, Theorem 2.5, page 47],

$$\begin{aligned}[1 + o(1)]T(r, P) &\leq \overline{N}(r, P) + \overline{N}\Big(r, \frac{1}{P}\Big) + \overline{N}\Big(r, \frac{1}{P - \omega}\Big) + S(r, P)\\ &= \overline{N}(r, P) + \overline{N}\Big(r, \frac{1}{P}\Big) + S(r, P). \qquad (2.1)\end{aligned}$$

Since the homogeneous differential polynomial

$$P[f] = f^{l_0}(f^{l_1})^{(m_1)}(f^{l_2})^{(m_2)} \cdots (f^{l_k})^{(m_k)} \quad (k \geq 1)$$

is a product of monomials $f^{l_0}, (f^{l_1})^{(m_1)}, (f^{l_2})^{(m_2)}, \ldots, (f^{l_k})^{(m_k)}$, where the exponents $l_0, l_1, \ldots, l_k$ of f are positive integers (since $l_0 > 0, l_i \geq m_i > 0$, for $1 \leq i \leq k$), by [17, Theorem 1, page 792], f and $P[f]$ have the same order of growth and hence $T(r, \omega) = S(r, P)$ as $r \to \infty$. That is, ω is a small function of f if and only if ω is a small function of $P[f]$. Next,

$$\begin{aligned}\overline{N}\Big(r, \frac{1}{P}\Big) &= \overline{N}\Big(r, \frac{1}{f^{l_0}(f^{l_1})^{(m_1)} \cdots (f^{l_k})^{(m_k)}}\Big)\\ &\leq \overline{N}\Big(r, \frac{1}{f}\Big) + \sum_{i=1}^{k} \overline{N}_0\Big(r, \frac{1}{(f^{l_i})^{(m_i)}}\Big) \leq \overline{N}\Big(r, \frac{1}{f}\Big) + \sum_{i=1}^{k} N_0\Big(r, \frac{1}{(f^{l_i})^{(m_i)}}\Big),\end{aligned}$$

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where $N_0(r, 1/(f^{l_i})^{(m_i)})$ represents the count of those zeros of $(f^{l_i})^{(m_i)}$ which are not the zeros of f^{l_i} and hence not of f. Denote by $\overline{N}_{p)}(r, 1/f)$ and $\overline{N}_{(p+1}(r, 1/f)$ the counting functions ignoring multiplicities of those zeros of f whose multiplicity is at most p and at least $p+1$, respectively. Therefore,

$$\begin{aligned}\overline{N}\Big(r, \frac{1}{P}\Big) &\le \overline{N}\Big(r, \frac{1}{f}\Big) + \sum_{i=1}^{k}\Big[m_i\overline{N}(r, f) + N_{m_i)}\Big(r, \frac{1}{f}\Big) + m_i\overline{N}_{(m_i+1}\Big(r, \frac{1}{f}\Big)\Big] + S(r, f)\\ &\le \overline{N}\Big(r, \frac{1}{f}\Big) + \sum_{i=1}^{k} m_i\Big[\overline{N}(r, f) + \overline{N}_{m_i)}\Big(r, \frac{1}{f}\Big) + \overline{N}_{(m_i+1}\Big(r, \frac{1}{f}\Big)\Big] + S(r, f)\\ &= \overline{N}\Big(r, \frac{1}{f}\Big) + \sum_{i=1}^{k} m_i\Big[\overline{N}(r, f) + \overline{N}\Big(r, \frac{1}{f}\Big)\Big] + S(r, f)\\ &= \overline{N}\Big(r, \frac{1}{f}\Big) + m'\Big[\overline{N}(r, f) + \overline{N}\Big(r, \frac{1}{f}\Big)\Big] + S(r, f).\end{aligned}$$

That is,

$$\overline{N}\Big(r, \frac{1}{P}\Big) \le m'\overline{N}(r, f) + (1+m')\overline{N}\Big(r, \frac{1}{f}\Big) + S(r, f). \tag{2.2}$$

Next, if z_0 is a zero of f of order p with $2 \le p \le k$, then z_0 is a zero of $P[f]$ of order $pl_0 + pl' - m' \ge 2l_0 + 2l' - m' \ge 2l_0 + m' \ge 2 + m'$. Similarly, for $p \ge k+1$, z_0 is a zero $P[f]$ of order $\ge (k+1)(l_0 + l') - m' \ge (k+1) + km' = k(1+m') + 1$. Thus, we see that

$$N\Big(r, \frac{1}{P}\Big) - \overline{N}\Big(r, \frac{1}{P}\Big) \ge (m'+1)\overline{N}_{k)}\Big(r, \frac{1}{f}\Big) + k(m'+1)\overline{N}_{(k+1}\Big(r, \frac{1}{f}\Big).$$

That is,

$$\overline{N}_{k)}\Big(r, \frac{1}{f}\Big) \le \frac{1}{m'+1}\Big[N\Big(r, \frac{1}{P}\Big) - \overline{N}\Big(r, \frac{1}{P}\Big)\Big] - k\overline{N}_{(k+1}\Big(r, \frac{1}{f}\Big).$$

Since $(1-k)(1+m') \le 0$ for $k \ge 1$, (2.2) with the help of the last inequality gives

$$\begin{aligned}\overline{N}\Big(r, \frac{1}{P}\Big) &\le m'\overline{N}(r, f) + (1+m')\overline{N}_{k)}\Big(r, \frac{1}{f}\Big) + (1+m')\overline{N}_{(k+1}\Big(r, \frac{1}{f}\Big) + S(r, f)\\ &\le m'\overline{N}(r, f) + N\Big(r, \frac{1}{P}\Big) - \overline{N}\Big(r, \frac{1}{P}\Big) + (1-k)(1+m')\overline{N}_{(k+1}\Big(r, \frac{1}{f}\Big) + S(r, f)\\ &\le m'\overline{N}(r, f) + N\Big(r, \frac{1}{P}\Big) - \overline{N}\Big(r, \frac{1}{P}\Big) + S(r, f),\end{aligned}$$

which implies that

$$\overline{N}\Big(r, \frac{1}{P}\Big) \le \frac{m'}{2}\overline{N}(r, f) + \frac{1}{2}N\Big(r, \frac{1}{P}\Big) + S(r, f). \tag{2.3}$$

Putting (2.3) into (2.1) and noting that $\overline{N}(r, f) = \overline{N}(r, P)$ and $S(r, f) = S(r, P)$ gives

$$[1 + o(1)]T(r, P) \le \Big[1 + \frac{m'}{2}\Big]\overline{N}(r, f) + \frac{1}{2}N\Big(r, \frac{1}{P}\Big) + S(r, P). \tag{2.4}$$

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Also, a pole of f of order $p \geq 1$ is a pole of $P[f]$ of order

$$pl_0 + pl' + m' \geq l_0 + l' + m' \geq 1 + m' + 1 + m' = 2 + 2m'.$$

Therefore, $N(r, P) \geq (2 + 2m')\overline{N}(r, f)$, which implies that

$$\overline{N}(r, f) \leq \frac{1}{2 + 2m'} N(r, P).$$

Hence, (2.4) yields

$$[1 + o(1)]T(r, P) \leq \left[\frac{1}{2} - \frac{m'}{4(1 + m')}\right] N(r, P) + \frac{1}{2} N\left(r, \frac{1}{P}\right) + S(r, P),$$

which implies that

$$\left[\frac{m'}{4(1 + m')} + o(1)\right] T(r, P) \leq S(r, P).$$

But this gives $T(r, P) \leq S(r, P)$, which is a contradiction. □

Theorem 2.2. *Let $\omega(z) \not\equiv 0$ be a polynomial of degree $m < l_0$. Let f be a nonconstant rational function having poles, if any, of multiplicity at least l_0. Then $P[f] - \omega$ has at least two distinct zeros.*

Remark 2.3. For $m = 0$, Theorem 2.2 holds without any restriction on the multiplicity of poles of f.

Proof. The proof of Theorem 2.2 is based on ideas from [4] but with a number of modifications. Since the computations are a little involved, we give the proof in full.

Suppose on the contrary that $P[f] - \omega$ has at most one zero. We consider the following cases.

Case 1. If f is a nonconstant polynomial, then $P[f]$ is also a polynomial of degree at least $l_0 + l' - m' \geq l_0 + 1$. Since $\omega(z)$ is a polynomial of degree $m < l_0$, $P[f](z) - \omega(z)$ is a polynomial of degree ≥ 1. By the fundamental theorem of algebra, $P[f] - \omega$ has exactly one zero. We can set

$$P[f](z) - \omega(z) = A(z - z_0)^n, \tag{2.5}$$

where A is a nonzero constant and $n > m + 1$. Then

$$\frac{d^{m+1} P[f]}{dz^{m+1}}(z) = P^{(m+1)}[f](z) = An(n - 1)(n - 2) \cdots (n - m)(z - z_0)^{n-m-1},$$

which implies that z_0 is the only zero of $P^{(m+1)}[f](z)$. Since each zero of f is a zero of $P[f]$ of order at least $l_0 + l' - m' > m + 1$, it follows that z_0 is a zero of $P[f]$ also. Thus, $P^{(m)}[f](z_0) = 0$. But (2.5) gives $P^{(m)}[f](z_0) = \omega^{(m)}(z_0) \neq 0$, which is a contradiction.

Case 2. Suppose that f is a rational function but not a polynomial, say

$$f(z) = A \frac{\prod_{j=1}^{s} (z - \alpha_j)^{n_j}}{\prod_{j=1}^{t} (z - \beta_j)^{p_j}}, \tag{2.6}$$

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where A is a nonzero constant, $n_j \geq 1 (j = 1, 2, \ldots, s)$ and $p_j \geq l_0 (j = 1, 2, \ldots, t)$. Put

$$\sum_{j=1}^{s} n_j = S \quad \text{and} \quad \sum_{j=1}^{t} p_j = T. \tag{2.7}$$

Thus, $S \geq s$ and $T \geq l_0 t \geq t$. We see from (2.6) that

$$P = P[f](z) = \frac{\prod_{j=1}^{s} (z - \alpha_j)^{n_j(l_0+l')-m'}}{\prod_{j=1}^{t} (z - \beta_j)^{p_j(l_0+l')+m'}} g_P(z) = \frac{p(z)}{q(z)}, \quad \text{say}, \tag{2.8}$$

where $g_P(z)$ is a polynomial of degree at most $m'(s + t - 1)$. On differentiating (2.8),

$$P^{(m)} = \frac{\prod_{j=1}^{s} (z - \alpha_j)^{n_j(l_0+l')-(m'+m)}}{\prod_{j=1}^{t} (z - \beta_j)^{p_j(l_0+l')+(m'+m)}} \tilde{g}(z), \tag{2.9}$$

where $\tilde{g}$ is a polynomial of degree at most $(m' + m)(s + t - 1)$, and

$$P^{(m+1)} = \frac{\prod_{j=1}^{s} (z - \alpha_j)^{n_j(l_0+l')-(m'+m+1)}}{\prod_{j=1}^{t} (z - \beta_j)^{p_j(l_0+l')+(m'+m+1)}} \tilde{\tilde{g}}(z), \tag{2.10}$$

where $\tilde{\tilde{g}}$ is a polynomial of degree at most $(m' + m + 1)(s + t - 1)$.

Case 2.1. First assume that $P[f] - \omega$ has exactly one zero, say z_0. In view of (2.8),

$$P[f](z) = \omega(z) + \frac{B(z - z_0)^l}{\prod_{j=1}^{t} (z - \beta_j)^{p_j(l_0+l')+m'}}, \tag{2.11}$$

where l is a positive integer and B is a nonzero constant. On differentiating (2.11),

$$P^{(m)} = C + \frac{(z - z_0)^{l-m} \hat{g}(z)}{\prod_{j=1}^{t} (z - \beta_j)^{p_j(l_0+l')+(m'+m)}}, \tag{2.12}$$

where $\hat{g}$ is a polynomial with degree at most mt and $C \neq 0$ is a constant, and

$$P^{(m+1)} = \frac{(z - z_0)^{l-(m+1)} \hat{\hat{g}}(z)}{\prod_{j=1}^{t} (z - \beta_j)^{p_j(l_0+l')+(m'+m+1)}}, \tag{2.13}$$

where $\hat{\hat{g}}$ is a polynomial of degree at most $(m + 1)t \leq l_0 t$. On comparing (2.9) and (2.12), we see that $z_0 \neq \alpha_j$ $(j = 1, 2, \ldots, s)$ (otherwise, for some j, z_0 is a zero of $P^{(m)}[f]$ from (2.9) and then from (2.12), $P^{(m)}[f](z_0) = 0$, which implies that $C = 0$, which is a contradiction).

Case 2.1.1. Suppose that $l \neq T(l_0 + l') + tm' + m$. Then from (2.11) and using (2.8), we see that $\deg(p) \geq \deg(q)$ and $T(l_0 + l') + tm' \leq S(l_0 + l') - m's + \deg(g_P)$. This implies that

$$T(l_0 + l') \leq S(l_0 + l') - m' < S(l_0 + l'),$$

whence $T < S$. Also, from (2.10) and (2.13),

$$S(l_0 + l') - (m' + m + 1)s \leq \deg(\hat{\hat{g}}) \leq l_0 t \leq T,$$

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which gives

$$S(l_0 + l') \le (m' + m + 1)s + T \le (m' + l_0)S + T < (m' + 1 + l_0)S \le (l' + l_0)S,$$

that is, $S < S$, which is absurd.

Case 2.1.2. Suppose that $l = T(l_0 + l') + tm' + m$. Then we have two possibilities: either $S > T$ or $S \le T$. For the case $S > T$, we can proceed exactly as in Case 2.1.1. Therefore, we need only consider the case $S \le T$. From (2.10) and (2.13), $(z - z_0)^{l-m-1}$ divides $\tilde{\tilde{g}}(z)$ and so $l - m - 1 \le \deg(\tilde{\tilde{g}}) \le (m' + m + 1)(s + t - 1)$. This implies that

$$\begin{aligned} T(l_0 + l') + tm' + m - m - 1 &\le (m' + m + 1)(s + t - 1) \\ &= m'(s - 1) + (m + 1)(s + t - 1) + tm' \end{aligned}$$

and so

$$\begin{aligned} T(l_0 + l') &\le m'(s - 1) + (m + 1)(s + t) - m \\ &\le m'(s - 1) + (m + 1)(s + t) \le m'(s - 1) + l_0(s + t) \\ &< (m' + l_0)S + T \le (m' + l_0 + 1)T \le (l' + l_0)T, \end{aligned}$$

which is again absurd.

Case 2.2. Finally, we suppose that $P[f] - \omega$ has no zeros. Then $l = 0$ in (2.11), giving

$$P[f](z) = \omega(z) + \frac{B}{\prod_{j=1}^{t} (z - \beta_j)^{p_j(l_0+l')+m'}},$$

where $B \ne 0$ is a constant, and so

$$P^{(m+1)} = B \frac{h(z)}{\prod_{j=1}^{t} (z - \beta_j)^{p_j(l_0+l')+m'+m+1}},$$

where $\deg(h) \le (m + 1)t - 1 < (m + 1)t \le l_0 t$. Proceeding as in Case 2.1 leads to a contradiction. □

3. Proofs of the main results

Since normality is a local property, we can assume that D is the open unit disc $\mathbb{D}$.

Proof of Theorem 1.5. Suppose on the contrary that $\mathcal{F}$ is not normal at $z = 0$. We consider the following cases.

Case 1. Let $h(0) \ne 0$. Then, by Zalcman's lemma [19, page 216], there are a sequence $\{f_j\}$ in $\mathcal{F}$, a sequence $\{z_j\}$ of complex numbers in $\mathbb{D}$ with $z_j \to 0$ as $j \to \infty$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \to 0$ as $j \to \infty$ such that the sequence $g_j(z) := \rho_j^{-\alpha} f_j(z_j + \rho_j z)$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $g(z)$ having bounded spherical derivative on $\mathbb{C}$. Clearly, $(g_j^{l_i})^{(m_i)} \to (g^{l_i})^{(m_i)}$ and so $P[g_j] \to P[g]$ locally uniformly on $\mathbb{C}$.

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Since g is nonconstant and $l_i \geq m_i$ for all $i = 1, 2, \ldots, k$, it follows that $P[g] \not\equiv 0$. We claim that $P[g]$ is nonconstant. For, suppose that

$$P[g] \equiv a, \quad a \in \mathbb{C} \setminus \{0\}. \tag{3.1}$$

Then, by definition of $P[g]$ with $l_0 > 0$ and $l_i \geq m_i$ for all i, we see that g is entire and nonvanishing. So, for some $c \neq 0$, $g(z) = e^{cz+d}$, whence

$$P[g](z) = \prod_{i=1}^{k} (l_i c)^{m_i} e^{(l_0+l')(cz+d)},$$

which is nonconstant, in contradiction to (3.1). Hence, the claim follows.

Taking $\alpha = m'/(l_0 + l')$, we find that $P[g_j](z) = P[f_j](z_j + \rho_j z)$. Thus, on every compact subset of $\mathbb{C}$ not containing poles of g,

$$P[f_j](z_j + \rho_j z) - h(z_j + \rho_j z) = P[g_j](z) - h(z_j + \rho_j z) \longrightarrow P[g](z) - h(0) = P[g](z) - h_0,$$

spherically uniformly, where $h_0 = h(0) \neq 0$. In view of Theorems 2.1 and 2.2, let u_0 and v_0 be two distinct zeros of $P[g] - h_0$ in $\mathbb{C}$. Since zeros are isolated, we consider two nonintersecting neighbourhoods, $N(u_0)$ and $N(v_0)$, such that $N(u_0) \cup N(v_0)$ does not contain any other zero of $P[g] - h_0$. By Hurwitz's theorem, we find that for sufficiently large values of j, there exist points $u_j \in N(u_0)$ and $v_j \in N(v_0)$ such that

$$P[f_j](z_j + \rho_j u_j) - h(z_j + \rho_j u_j) = 0 \quad \text{and} \quad P[f_j](z_j + \rho_j v_j) - h(z_j + \rho_j v_j) = 0.$$

Since $P[f]$ and $P[g]$ share h IM in $\mathbb{D}$, for each pair f, g of members of $\mathcal{F}$, for a fixed n and for all j,

$$P[f_n](z_j + \rho_j u_j) - h(z_j + \rho_j u_j) = 0 \quad \text{and} \quad P[f_n](z_j + \rho_j v_j) - h(z_j + \rho_j v_j) = 0.$$

Taking $j \to \infty$ and noting that $z_j + \rho_j u_j \to 0$ and $z_j + \rho_j v_j \to 0$ as $j \to \infty$, we find that $P[f_n](0) - h(0) = 0$, that is, $P[f_n](0) = h(0) = h_0 \neq 0$. Since the zeros of $P[f_n] - h$ have no accumulation point, for sufficiently large j, $z_j + \rho_j u_j = 0 = z_j + \rho_j v_j$. But this means that $u_j = -z_j/\rho_j = v_j$ is a point in both of the neighbourhoods $N(u_0)$ and $N(v_0)$, which is a contradiction.

Case 2. Suppose that $h(0) = 0$. Then we can write $h(z) = z^m h_1(z)$, where $m \in \mathbb{N}$ and $h_1(z)$ is a holomorphic function in $\mathbb{D}$ such that $h_1(0) \neq 0$. We may take $h_1(0) = 1$. Since $0 < (m + m')/(l_0 + l') < 1$, as in Case 1, by Zalcman's lemma [19, page 216], we obtain a sequence of rescaled functions $g_j(z) = \rho_j^{-(m+m')/(l_0+l')} f_j(z_j + \rho_j z)$ which converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $g(z)$ on $\mathbb{C}$ having bounded spherical derivatives.

We now consider the following two subcases of Case 2.

Case 2.1. Suppose that there exists a subsequence of z_j/ρ_j, which for convenience we take to be z_j/ρ_j itself, such that $z_j/\rho_j \to \infty$ as $j \to \infty$. Consider the family

$$\mathcal{G} := \{G_j(z) = z_j^{-(m+m')/(l_0+l')} f_j(z_j + z_j z) : f_j \in \mathcal{F}\}$$

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defined on $\mathbb{D}$, for which

$$\begin{aligned}P[G_j](z) &= G_j^{l_0}(G_j^{l_1})^{(m_1)}\cdots(G_j^{l_k})^{(m_k)}(z)\\ &= z_j^{-(m+m'/l_0+l')(l_0+l')+m'}P[f_j](z_j+z_jz) = z_j^{-m}P[f_j](z_j+z_jz),\end{aligned}$$

that is, $P[f_j](z_j+z_jz) = z_j^m P[G_j](z)$. Now, by hypothesis, for $f_a, f_b \in \mathcal{F}$,

$$\begin{aligned}&(P[f_a]-h)(z_j+z_jz) = 0 \Leftrightarrow (P[f_b]-h)(z_j+z_jz) = 0\\ &\quad\Rightarrow z_j^m P[G_a](z) = z_j^m(1+z)^m h_1(z_j+z_jz) \Leftrightarrow z_j^m P[G_b](z) = z_j^m(1+z)^m h_1(z_j+z_jz)\\ &\quad\Rightarrow P[G_a](z) = (1+z)^m h_1(z_j+z_jz) \Leftrightarrow P[G_b](z) = (1+z)^m h_1(z_j+z_jz).\end{aligned}$$

Since $(1+z)^m h_1(z_j+z_jz) \neq 0$ at the origin, it follows from Case 1 that $\mathcal{G}$ is normal in $\mathbb{D}$ and hence there exists a subsequence of $\{G_j\}$ in $\mathcal{G}$, which we may take to be $\{G_j\}$ itself, such that $G_j \to G$, locally uniformly on $\mathbb{D}$ with respect to the spherical metric.

If $G(0) \neq 0$, then we see that

$$\begin{aligned}g_j(z) &= \rho_j^{-(m+m')/(l_0+l')}f_j(z_j+\rho_jz) = \left(\frac{z_j}{\rho_j}\right)^{(m+m')/(l_0+l')} z_j^{-(m+m')/(l_0+l')}f_j(z_j+\rho_jz)\\ &= \left(\frac{z_j}{\rho_j}\right)^{(m+m')/(l_0+l')} G_j\left(\frac{\rho_j}{z_j}z\right),\end{aligned}$$

which converges locally uniformly with respect to the spherical metric to ∞ on $\mathbb{C}$. This implies that $g(z) \equiv \infty$, which is a contradiction. Thus, we must have $G(0) = 0$, which implies that $G'(0) \neq \infty$. Next, for each $z \in \mathbb{C}$,

$$g_j'(z) = \rho_j^{-(m+m')/(l_0+l')+1}f_j'(z_j+\rho_jz) = \left(\frac{\rho_j}{z_j}\right)^{-(m+m')/(l_0+l')+1} G_j'\left(\frac{\rho_j}{z_j}z\right)$$

and $(m+m')/(l_0+l') < 1$. Therefore, $g_j'(z) \to 0$ spherically uniformly as $j \to \infty$. But this implies that g is constant, which is a contradiction.

Case 2.2. Suppose that there exists a subsequence of z_j/ρ_j, which, for simplicity, we take to be z_j/ρ_j itself, such that $z_j/\rho_j \to c$ as $j \to \infty$, where c is a finite number. Then

$$H_j(z) = \rho_j^{-(m+m')/(l_0+l')}f_j(\rho_jz) = g_j\left(z - \frac{z_j}{\rho_j}\right) \stackrel{\chi}{\to} g(z-c) := H(z)$$

on $\mathbb{C}$. Note that $P[H_j](z) = \rho_j^{-m}P[f_j](\rho_jz)$. For each f_a and f_b in $\mathcal{F}$, $P[f_a]$ and $P[f_b]$ share h IM, so

$$P[f_a](\rho_jz) = h(\rho_jz) \Leftrightarrow P[f_b](\rho_jz) = h(\rho_jz). \tag{3.2}$$

That is,

$$P[H_a](z) = z^m h_1(\rho_jz) \Leftrightarrow P[H_b](z) = z^m h_1(\rho_jz). \tag{3.3}$$

We claim that $P[H](z) \not\equiv z^m$. If, on the contrary, $P[H] \equiv z^m$, then $z = 0$ is the only possible zero of H. If H is transcendental, then $H(z) = z^\alpha e^{Q(z)}$ for some nonnegative integer α and a polynomial Q. Thus, $(H^{l_i})^{(m_i)}(z) = p(z)e^{l_iQ(z)}$, where $p(z)$ $(\not\equiv 0)$ is a rational function. It follows that $P[H]$ is also transcendental, which is not the case.

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On the other hand, if H is rational and $z = 0$ is a zero of H, then H is a polynomial. Clearly, $\deg(P[H]) \geq l_0 + 1 > m$, which is again a contradiction.

On compact subsets of $\mathbb{C}$, not containing poles of H, we see that

$$P[H_j](z) - z^m h_1(\rho_j z) \longrightarrow P[H](z) - z^m,$$

spherically uniformly. Since $P[H](z) \not\equiv z^m$, by Theorems 2.1 and 2.2, $P[H](z) - z^m$ has at least two distinct zeros in $\mathbb{C}$. By proceeding in the same way as in Case 1, we arrive at a contradiction.

Putting all the cases together, it follows that $\mathcal{F}$ must be normal in $\mathbb{D}$. □

Proof of Theorem 1.9. Irrespective of any of the conditions (i), (ii) and (iii), the ideas used in Case 1 of the proof of Theorem 1.5 lead us to the conclusion that $P[g](z) \not\equiv h(0) = h_0$ in $\mathbb{C}$.

If condition (i) holds, then we claim that $P[g](z) - h_0$ has at most one zero in $\mathbb{C}$, in violation of the conclusions of Theorems 2.1 and 2.2, thereby proving the normality of $\mathcal{F}$. Suppose, on the contrary, that $P[g](z) - h_0$ has at least two distinct zeros, say ζ_0 and ζ_0^*. By Hurwitz's theorem, there exist points $\zeta_j \to \zeta_0$ and $\zeta_j^* \to \zeta_0^*$ such that

$$P[f_j](z_j + \rho_j\zeta_j) - h(z_j + \rho_j\zeta_j) = 0 \quad \text{and} \quad P[f_j](z_j + \rho_j\zeta_j^*) - h(z_j + \rho_j\zeta_j^*) = 0$$

for sufficiently large j. Since $P[f_j](z_j + \rho_j z) - h(z_j + \rho_j z)$ has at most one zero, this contradicts the fact that ζ_0 and ζ_0^* are distinct. Hence, the claim follows.

Next we prove the normality of $\mathcal{F}$ when condition (ii) holds. By Theorems 2.1 and 2.2, $P[g](z) - h_0$ must have a zero, say ζ_0, and hence $g(\zeta_0) \neq \infty$. Further, by Hurwitz's theorem, for sufficiently large j, there exists a sequence $\{\zeta_j\}$ converging to ζ_0 such that

$$P[f_j](z_j + \rho_j\zeta_j) - h(z_j + \rho_j\zeta_j) = 0.$$

By hypothesis,

$$|g_j(\zeta_j)| = \rho_j^{-m'/(l_0+l')}|f_j(z_j + \rho_j\zeta_j)| \geq \rho_j^{-m'/(l_0+l')}M.$$

Since $g(\zeta_0) \neq \infty$ in some neighbourhood N of ζ_0, it follows that for sufficiently large j, $g_j(z)$ converges uniformly to $g(z)$ in N. Thus, for given $\epsilon > 0$ and for every $z \in N$,

$$|g_j(z) - g(z)| < \epsilon$$

for sufficiently large j. Therefore, for sufficiently large values of j,

$$|g(\zeta_j)| \geq |g_j(\zeta_j)| - |g(\zeta_j) - g_j(\zeta_j)| > \rho_j^{-m'/(l_0+l')}M - \epsilon,$$

which implies that g has a pole at ζ_0, which is not the case.

Finally, we prove the normality of $\mathcal{F}$ when condition (iii) holds. As in the preceding discussion,

$$P[f_j](z_j + \rho_j\zeta_j) - h(z_j + \rho_j\zeta_j) = 0.$$

Since $\alpha = m'/(l_0 + l')$ for some positive l_i and m_i,

$$|(g_j^{l_i})^{(m_i)}(\zeta_j)| = \rho_j^{m_i - \alpha l_i}|(f_j^{l_i})^{(m_i)}(z_j + \rho_j\zeta_j)| \leq M\rho_j^{m_i - (m' l_i / l_0 + l')} \to 0 \quad \text{as } j \to \infty.$$

Thus, $(g^{l_i})^{(m_i)}(\zeta_0) = \lim_{j\to\infty}(g_j^{l_i})^{(m_i)}(\zeta_j) = 0$, which implies that $P[g](\zeta_0) = 0 \neq h_0$. This is a contradiction. □

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KULDEEP SINGH CHARAK, Department of Mathematics,
University of Jammu, Jammu-180 006, India
e-mail: kscharak7@rediffmail.com

SHITTAL SHARMA, Department of Mathematics,
University of Jammu, Jammu-180 006, India
e-mail: shittalsharma_mat07@rediffmail.com

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