Proceedings of the Edinburgh Mathematical Society (1990) 33, 143–158 ©

A FURTHER RESULT ON THE COMPLEX OSCILLATION THEORY OF PERIODIC SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS*

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(Received 8th August 1988)

We prove the following: Assume that $B(\zeta)=g(\zeta^{\mp 1})+\sum_{i=1}^{p} b_{\pm i}\zeta^{\pm i}$, where p is an odd positive integer, $g(\zeta)$ is a transcendental entire function with order of growth less than 1, and set $A(z)=B(e^{\alpha z})$. Then for every solution $f\not\equiv 0$ of $f''+A(z)f=0$, the exponent of convergence of the zero-sequence is infinite, and, in fact, the stronger conclusion $\log^+ N(r,1/f)\neq o(r)$ holds. We also give an example to show that if the order of growth of $g(\zeta)$ equals 1 (or, in fact, equals an arbitrary positive integer), this conclusion doesn't hold.

1980 Mathematics subject classification (1985 Revision): 30D35

1. Introduction

S. Bank. and I. Laine proved in [1]: Let $A(z)=B(e^{\alpha z})$ be a periodic entire function with period $\omega=2\pi i/\alpha$ and rational in $e^{\alpha z}$. If $B(\zeta)$ has poles of odd order at both $\zeta=\infty$ and $\zeta=0$, then for every solution $f\not\equiv 0$ of equation (1)

$$f''+A(z)f=0, \tag{1}$$

the exponent of convergence of the zero-sequence is infinite.

In [2], S. Bank generalized this result: The above conclusion still holds if we just suppose that both $\zeta=\infty$ and $\zeta=0$ are poles of $B(\zeta)$, and at least one is of odd order. Gao Shian also obtained the same generalization in [4] ([4] was written before seeing the paper of S. Bank), but S. Bank replaced the above conclusion with the stronger conclusion

$$\log^+ N(r,1/f)\neq o(r) \quad \text{as}\, r\to+\infty. \tag{2}$$

In the case where $B(\zeta)$ has a pole at one of $\zeta=\infty$ and $\zeta=0$, and at the other point $B(\zeta)$ is analytic, Gao Shian also proved in [4]:

Let $A(z)=B(e^{\alpha z})$ be a polynomial of odd degree in $e^{\alpha z}$ (including those which can be

*Project supported by the State Natural Science Fund of China, also a part of works as a senior research scholar under SBFSS to Britain.

changed into this case by varying the period of $A(z)$), i.e. $B(\zeta)=\sum_{i=0}^{k} b_i\zeta^i$, where k is an odd positive integer, $b_k\neq 0$. If

$$b_0 \neq -\frac{\alpha^2 s^2}{16},$$

where $s \geqq k$ is an odd positive integer, then for every solution $f\not\equiv 0$ of equation (1), the exponent of convergence of the zero-sequence is infinite. Conversely, if s is an odd positive integer of the form $k(2n+1), n\geqq 0$, then equation (1) may possibly have two linearly independent solutions $f_1\not\equiv 0$, $f_2\not\equiv 0$ whose zero-sequences have exponents not bigger than 1.

It is easy to prove that we can also replace this conclusion about the infinite exponent of convergence of the zero-sequence with the stronger conclusion (2) of S. Bank.

The above conclusions can be summarized as follows:

Assume

$$B(\zeta)=b_p\zeta^p+b_{p-1}\zeta^{p-1}+\cdots+b_0+b_{-1}\zeta^{-1}+\cdots+b_{-q}\zeta^{-q},$$

where b_j are constants, p and q are nonnegative integers, $b_p\neq 0$ if $p\geqq 1, b_{-q}\neq 0$ if $q\geqq 1$. Then,

(i) If $\min(p,q)\geqq 1$, and at least one of p and q is an odd positive integer, then for every solution $f\not\equiv 0$ of equation (1), the exponent of convergence of the zero-sequence is infinite, and, in fact, the stronger conclusion (2) holds, where $A(z)=B(e^{\alpha z})$.

(ii) If $\min(p,q)=0$, and $\max(p,q)=k$ is an odd positive integer, and

$$b_0 \neq -\frac{\alpha^2 s^2}{16},$$

where $s\geqq k$ is an odd positive integer, then for every solution $f\not\equiv 0$ of equation (1), the exponent of convergence of the zero-sequence is infinite, and, in fact, the stronger conclusion (2) holds, where $A(z)=B(e^{\alpha z})$. Conversely, if

$$b_0 = -\frac{\alpha^2 s^2}{16}$$

with s as above, then this conclusion may not hold.

These results are only in the case where $B(\zeta)$ is rational and analytic on $0<|\zeta|<+\infty$. If $B(\zeta)$ is transcendental and analytic on $0<|\zeta|<+\infty$, what can we say? We will try to answer this question in part. In this paper, we first generalize Theorem 4 in [1], and add a new property to it; second, using this generalization and our new property we get a relation between the solutions $f(z)$ and $f(z+\omega)$ of equation (1); finally, by proving another contrary relation between $f(z)$ and $f(z+\omega)$ we obtain our main result: Let $g(\zeta)$ be a transcendental entire function with order of growth less than 1, and

$$B(\zeta)=g(1/\zeta)+\sum_{i=1}^{p} b_i\zeta^i$$

or

$$B(\zeta)=g(\zeta)+\sum_{i=1}^{p} b_{-i}\zeta^{-i},$$

where p is an odd positive integer, then for every solution $f\not\equiv 0$ of equation (1), the exponent of convergence of the zero-sequence is infinite, and, in fact, the stronger conclusion (2) holds, where $A(z)=B(e^{\alpha z})$ in (1). We also give an example to show that if the order of growth of $g(\zeta)$ equals 1 (or, in fact, equals an arbitrary positive integer), this conclusion doesn't hold.

We will use the standard notations of Nevanlinna theory, see [5]. In addition, we will denote the exponent of convergence of the zero-sequence of $f(z)$ by $\lambda(f)$, and the order of growth of $f(z)$ by $\sigma(f)$. The other notations will be shown when we need to use them.

2. Main theorem and corollary

Theorem. *Let $A(z)=B(e^{\alpha z})$ be a periodic entire function with period $\omega=2\pi i/\alpha$ and transcendental in $e^{\alpha z}$, i.e. $B(\zeta)$ is transcendental and analytic on $0<|\zeta|<+\infty$. If there exists a constant δ with $0<\delta<1$ such that*

$$\log T(r,A)<\delta|\alpha|r \quad \text{for } r \text{ near } +\infty, \tag{3}$$

and if $B(\zeta)$ has a pole of odd order at $\zeta=\infty$ or $\zeta=0$ (including those which can be changed into this case by varying the period of $A(z)$), then for every solution $f\not\equiv 0$ of equation (1), $\lambda(f)=+\infty$, and, in fact, the stronger conclusion (2) holds.

Corollary. *Let $g(\zeta)$ be a transcendental entire function with $\sigma(g)<1$, and*

$$B(\zeta)=g(1/\zeta)+\sum_{i=1}^{p} b_i\zeta^i$$

or

$$B(\zeta)=g(\zeta)+\sum_{i=1}^{p} b_{-i}\zeta^{-i},$$

where $b_{\pm i}$ are constants, p is an odd positive integer, $b_{\pm p}\neq 0$, then for every solution $f\not\equiv 0$ of equation (1), $\lambda(f)=+\infty$, and, in fact, the stronger conclusion (2) holds.

In Section 5, we give an example to show that the corollary doesn't hold if p is even. We also give another example to show that the corollary doesn't hold if $\sigma(g)$ is an arbitrary positive integer and p is odd. If p is odd and $\sigma(g)$ isn't a positive integer but is bigger than 1, could the corollary be true or not? This is still an open problem.

Remark. The condition (3) is equivalent to the following condition: There exists a constant δ_0 with $0<\delta_0<1$ such that

$$\log\log M(r,A)<\delta_0|\alpha|r \quad \text{for } r \text{ near} +\infty, \tag{4}$$

where $M(r,A)=\max_{|z|\leq r}|A(z)|$. From [5, Theorem 1.6], we have

$$T(r,A)\leqq \log^+ M(r,A)\leqq \frac{2+\varepsilon}{\varepsilon}T((1+\varepsilon)r,A),$$

where ε is an arbitrary positive constant. It is easy to check that this is true by choosing ε such that $0<\delta(1+\varepsilon)<1$. Hence, we will regard the conditions (4) and (3) as the same from now on.

3. Proof of theorem

The proof of the theorem will be completed by a series of lemmas.

Lemma 1. *Let $V(\zeta)$ be analytic on $0<|\zeta|<+\infty$, and set $w(z)=V(e^{\alpha z})$. If $\log^+ N(r,1/w)=o(r)$ as $r\to+\infty$, then $\lambda_\infty(V)=0$, $\lambda_0(V)=0$, where we denote the exponent of convergence of the zero-sequence of $V(\zeta)$ on $1\leqq|\zeta|<+\infty$ by $\lambda_\infty(V)$, and $\lambda_0(V)=\lambda_\infty(V^*)$, $V^*(\zeta)=V(1/\zeta)$ (see [1]).*

Proof. Denote the counting function of the zeros of $w(z)$ with $|e^{\alpha z}|\geqq 1$ by $N_1(r,1/w)$. It is clear that

$$\log^+ N_1(r,1/w)=o(r) \quad \text{as } r\to+\infty. \tag{5}$$

If we denote the counting function of the zeros of $V(\zeta)$ on $1\leqq|\zeta|<+\infty$ by $N_\infty(\rho,1/V)$, then

$$\lambda_\infty(V)=\limsup_{\rho\to+\infty}\frac{\log N_\infty(\rho,1/V)}{\log\rho}.$$

Assuming that $\lambda_\infty(V)>0$, then there must exist a constant $\bar{\delta}>0$ and a sequence $\{\rho_j\}\to+\infty$ such that

$$\log N_\infty(\rho_j,1/V)>\bar{\delta}\log\rho_j.$$

Denote the zeros of $V(\zeta)$ on $1\leqq|\zeta|<\rho_j$ by $\zeta_1,\zeta_2,\ldots,\zeta_{p_j}$. Let $e^{\alpha z_k}=\zeta_k, k=1,2,\ldots,p_j$, then

$z_1, z_2, \ldots, z_{p_j}$ are zeros of $w(z)$ satisfying $|e^{\alpha z}| \geqq 1$. The set $\{z; |e^{\alpha z}| = \rho_j\}$ is clearly unbounded, so there is a point $z_j^* \in \{z; |e^{\alpha z}| = \rho_j\}$ such that $|z_k| < |z_j^*|, k = 1, 2, \ldots, p_j$. Let $e^{\alpha z_j^*} = \rho_j e^{i\theta_j}$, $|\theta_j| \leqq \pi$. From $\alpha z_j^* = \log \rho_j + i\theta_j$, it follows that $|\alpha z_j^*| \leqq 2 \log \rho_j$ if j is large enough, and $z_j^* \to \infty$ as $\rho_j \to +\infty$. Hence,

$$\log N_\infty(\rho_j, 1/V) > \delta \frac{|\alpha|}{2} |z_j^*|.$$

But it is clear (because if ζ_0 is a zero of $V(\zeta)$ and $e^{\alpha z_0} = \zeta_0$, then

$$z_0 + \frac{2k\pi}{\alpha} i$$

are zeros of $w(z)$) that

$$N_1(|z_j^*|, 1/w) \geqq N_\infty(\rho_j, 1/V).$$

Thus

$$\log N_1(|z_j^*|, 1/w) > \delta \frac{|\alpha|}{2} |z_j^*|,$$

and this contradicts (5). Hence, $\lambda_\infty(V) = 0$. We can prove $\lambda_0(V) = 0$ by the same reasoning.

The following Lemma 2 is Lemma C in [**2**].

Lemma 2. *Let $A(z)$ be a nonconstant periodic entire function with period ω, and $f \not\equiv 0$ be a solution of equation* (1) *such that*

$$\log^+ N(r, 1/f) = o(r) \quad \text{as } r \to +\infty, \tag{6}$$

then $f(z)$ and $f(z + 2\omega)$ are linearly dependent solutions of equation (1).

Lemma 3. *Let $A(z)$ be a nonconstant periodic entire function with period ω, i.e. $A(z) = B(e^{\alpha z})$, where $B(\zeta)$ is analytic on $0 < |\zeta| < +\infty$,*

$$\alpha = \frac{2\pi i}{\omega},$$

and let $A(z)$ satisfy the condition (3). *Assume $f \not\equiv 0$ is a solution of equation* (1) *which satisfies condition* (6), *and $f(z)$ and $f(z + \omega)$ are linearly independent. Set $E(z) = f(z) f(z + \omega)$. Then:*

(*a*) *there exists a constant δ_1 with $0 < \delta_1 < 1$ such that*

$$\log T(r,E) < \delta_1 |\alpha| r \quad \textit{for } r \textit{ near } +\infty;$$

(b) $E(z)^2$ *is a periodic function with period* ω, *so we can write* $E(z)^2 = \Phi(e^{\alpha z})$, *where* $\Phi(\zeta)$ *is analytic on* $0 < |\zeta| < +\infty$.

(c) if $B(\zeta)$ *has an essential singularity at* $\zeta = \infty$ *(resp.* $\zeta = 0$*), then* $\Phi(\zeta)$ *has also an essential singularity at* $\zeta = \infty$ *(resp.* $\zeta = 0$*).*

Proof. (a) Since $f(z)$ satisfies (6), it is easy to check that $f(z+\omega)$ satisfies (6) also, and so does $E(z)$. From [**1**, Section 5(b) and Section 4(a)] and (3), we obtain

$$\log T(r,E) < \delta |\alpha| \beta r \quad \text{for } r \text{ near } +\infty,$$

where β is an arbitrary constant with $\beta > 1$. We can choose $\beta > 1$ such that $0 < \delta\beta < 1$, and then set $\delta_1 = \delta\beta$. So part (a) is true.

(b) By Lemma 2, we have $E(z+\omega) \equiv cE(z)$, where c is a nonzero constant. Thus, E'/E and E''/E have period ω, and so does $E(z)^2$ from [**1**, Section 5(a)].

(c) $\Phi(\zeta)$ satisfies (see [**1**, p. 8])

$$\alpha^2(\zeta^2 \Phi\Phi'' - \tfrac{3}{4}\zeta^2(\Phi')^2 + \zeta\Phi\Phi') + 4B(\zeta)\Phi^2 + c\Phi = 0, \tag{7}$$

or

$$\alpha^2\left(\zeta^2 \frac{\Phi''}{\Phi} - \tfrac{3}{4}\zeta^2\left(\frac{\Phi'}{\Phi}\right)^2 + \zeta\frac{\Phi'}{\Phi}\right) + \frac{c^2}{\Phi} = -4B(\zeta).$$

From this, it is easy to see that part (c) is true.

The following Lemma 4 generalizes Theorem 4 in [1], and includes a new property (vii).

Lemma 4. *Let* $A(z) = B(e^{\alpha z})$ *be a periodic entire function with period* $\omega = 2\pi i/\alpha$, *and be transcendental in* $e^{\alpha z}$, *i.e.* $B(\zeta)$ *is transcendental and analytic on* $0 < |\zeta| < +\infty$. *Also let* $f \not\equiv 0$ *be a solution of equation* (1) *which satisfies condition* (6). *Then, the following are true:*

(A) *if* $f(z)$ *and* $f(z+\omega)$ *are linearly dependent, then* $f(z)$ *can be represented in the form*

$$f(z) = e^{dz} H(e^{\alpha z}) \exp(g(e^{\alpha z})), \tag{8}$$

where

(i) *d is a constant,*

(ii) $H(\zeta)$ *and* $g(\zeta)$ *are analytic on* $0 < |\zeta| < +\infty$,

(iii) $\sigma_0(H) = \sigma_\infty(H) = 0$,

(iv) $g(\zeta)$ *has at most a pole at* $\zeta = \infty$ *(resp.* $\zeta = 0$*) if and only if* $B(\zeta)$ *has at most a pole at* $\zeta = \infty$ *(resp.* $\zeta = 0$*),*

(v) $\sigma_\infty(g)=\sigma_\infty(B)$,

(vi) $\sigma_0(g)=\sigma_0(B)$,

(vii) *if* $B(\zeta)$ *has at most a pole at* $\zeta=\infty$ (*resp.* $\zeta=0$), *then* $H(\zeta)$ *has at most a pole at* $\zeta=\infty$ (*resp.* $\zeta=0$).

(*For the notations* $\sigma_0(H), \sigma_\infty(H), \ldots$, *the reader is referred to* [**1**, *p.* 4–5].)

(B) *If* $f(z)$ *and* $f(z+\omega)$ *are linearly independent, then* $f(z)$ *can be represented in the form*

$$f(z)=e^{dz}H(e^{(\alpha/2)z})\exp(g(e^{(\alpha/2)z})), \tag{9}$$

where d, H *and* g *satisfy the conditions* (i)–(vii) *listed in part* (A).

Proof. Part (A). Assume $f(z)$ and $f(z+\omega)$ are linearly dependent. By [**1**, p. 14], we have $f(z)=e^{\beta z}U(z)$, where β is a constant and $U(z)$ is a periodic entire function with period ω. Thus we can write $U(z)=G(e^{\alpha z})$, where $G(\zeta)$ is analytic on $0<|\zeta|<+\infty$. Since $N(r,1/U)=N(r,1/f)$, from (6) we have

$$\log^+ N(r,1/U)=o(r) \quad \text{as } r\to+\infty.$$

Then, from Lemma 1 we have $\lambda_0(G)=\lambda_\infty(G)=0$. Let $H_1(\zeta)$ (resp. $H_2(t)$) be the canonical product formed with the zeros of $G(\zeta)$ in $|\zeta|\geqq 1$ (resp. $G^*(t)=G(1/t)$ in $|t|>1$), and denote $H(\zeta)=H_1(\zeta)H_2(\zeta^{-1})$. Since $\sigma(H_1)=\lambda_\infty(G)=0$, $\sigma(H_2)=\lambda_0(G)=0$, $|H(\zeta)|=0(H_1(\zeta))$ as $\zeta\to\infty$ and $H(\zeta)=0(H_2(\zeta^{-1}))$ as $\zeta\to 0$, we get $\sigma_\infty(H)=0$ and $\sigma_0(H)=0$. It is clear that $G_1(\zeta)=G(\zeta)/H(\zeta)$ is analytic and has no zeros on $0<|\zeta|<+\infty$. Thus $G_1(e^{\alpha z})$ is entire and has no zeros. Hence $G_1(e^{\alpha z})=e^{v(z)}$, where $v(z)$ is entire. Since $v'(z)=\alpha G_1' e^{\alpha z}/G_1$ is periodic with period ω, we have $v(z+\omega)-v(z)\equiv K$, where K is a constant. Choose $d_1=-K/\omega$, we see that $v(z)+d_1 z$ is periodic with period ω. Hence $v(z)+d_1 z=g(e^{\alpha z})$, where $g(\zeta)$ is analytic on $0<|\zeta|<+\infty$. Setting $d=\beta-d_1$, we finally get the representation (8) and (i), (ii), (iii) have been verified also.

The proofs of (iv), (v) and (vi) are the same as [**1**, p. 15].

To prove (vii) in the case $\zeta=\infty$, we first show that $\zeta=\infty$ is not a cluster point of zeros of $G(\zeta)$. If we assume the contrary, then $G(\zeta)$ has an essential singularity at $\zeta=\infty$. It is easy to see that $G(\zeta)$ satisfies the linear equation

$$\alpha^2\zeta^2 G''+\zeta(2\alpha\beta+\alpha^2)G'+(B(\zeta)+\beta^2)G=0 \tag{10}$$

whose coefficients each have at most a pole at $\zeta=\infty$. From the Wiman–Valiron theory summarized in [**1**, p. 4–6], we can write $G(\zeta)=\zeta^m\Psi(\zeta)u(\zeta)$, where m is an integer, $\Psi(\zeta)$ is analytic and nonvanishing at $\zeta=\infty$, and $u(\zeta)$ is a transcendental entire function of finite order of growth. Clearly, $\zeta=\infty$ is also the cluster point of zeros of $u(\zeta)$, hence $u(\zeta)$ has infinitely many zeros. We have the representation $u(\zeta)=H_0(\zeta)\exp(Q(\zeta))$, where $H_0(\zeta)$ is the canonical product formed with the zeros of $u(\zeta)$, and $Q(\zeta)$ is a polynomial. From

(10), $G_1=Ge^{-Q}$ satisfies also a linear equation whose coefficients each have at most a pole at $\zeta=\infty$. But since $G_1(\zeta)=\zeta^m\Psi(\zeta)H_0(\zeta)$, again using the Wiman–Valiron theory summarized in [1, p. 4–6], we have $\sigma(H_0)=\delta>0$. Hence $\lambda(u)=\lambda(H_0)=\sigma(H_0)=\delta>0$. It is easy to see that $\lambda_\infty(G)=\lambda(u)=\delta>0$. So by Lemma 1, $\log^+N(r,1/U)\neq o(r)$ as $r\to+\infty$. But $\log^+N(r,1/f)=\log^+N(r,1/U)$, therefore $\log^+N(r,1/f)\neq o(r)$ and this contradicts assumption (6). Thus $\zeta=\infty$ is not a cluster point of zeros of $G(\zeta)$, and $G(\zeta)$ has only finitely many zeros in $|\zeta|\geqq 1$. Then, the canonical product $H_1(\zeta)$ can be replaced with a polynomial with these zeros. Since $H_2(\zeta^{-1})$ is analytic at $\zeta=\infty$, $H(\zeta)=H_1(\zeta)H_2(\zeta^{-1})$ has at most a pole at $\zeta=\infty$. Setting $G^*(t)=G(t^{-1})$, we can prove (vii) in the case $\zeta=0$ by the same reasoning.

Part (B). In this case, $f(z)$ and $f(z+\omega)$ are linearly independent, but $f(z)$ and $f(z+2\omega)$ are still linearly dependent by Lemma 2. Considering that $A(z)$ has period 2ω and using Part (A), we obtain the representation (9) with the asserted properties.

Before proving the following Lemma 5, we define an R-set to be a countable union of discs in the plane the sum of whose radii is finite, and remark that the set of θ for which the ray $re^{i\theta}$ meets infinitely many discs of a given R-set has measure zero (see [3, p. 11–12]).

Lemma 5. *Let $A(z)=B(e^{\alpha z})$ be a periodic entire function with period $\omega=2\pi i/\alpha$, and be transcendental in $e^{\alpha z}$, i.e. $B(\zeta)$ is transcendental and analytic on $0<|\zeta|<+\infty$. Assume also that $A(z)$ satisfies condition* (3). *If $f\not\equiv 0$ is a solution of equation* (1) *and satisfies condition* (6), *then $f(z)$ and $f(z+\omega)$ are linearly dependent.*

Proof. Suppose that $f(z)$ and $f(z+\omega)$ are linearly independent, and set

$$E(z)=f(z)f(z+\omega).$$

We first assume that $B(\zeta)$ has an essential singularity at $\zeta=\infty$. From Lemma 3, $F(z)=E(z)^2=\Phi(e^{\alpha z})$ is a periodic entire function with period ω, and $\Phi(\zeta)$ has an essential singularity at $\zeta=\infty$, and $\log T(r,F)=\log T(r,E)+\log 2<\delta_1|\alpha|r$ for r near $+\infty$, where δ_1 is a constant with $0<\delta_1<1$.

From (7), $\Phi(\zeta)$ and $B(\zeta)$ satisfy

$$\alpha^2\left(\zeta^2\frac{\Phi''}{\Phi}-\tfrac{3}{4}\zeta^2\left(\frac{\Phi'}{\Phi}\right)^2+\zeta\frac{\Phi'}{\Phi}\right)+\frac{c^2}{\Phi}=-4B(\zeta). \tag{11}$$

Since $\Phi(\zeta)$ has an essential singularity at $\zeta=\infty$, we can write $\Phi(\zeta)=\zeta^m\Psi(\zeta)u_1(\zeta)$, where m is an integer, $\Psi(\zeta)$ is analytic and nonvanishing at $\zeta=\infty$, and $u_1(\zeta)$ is a transcendental entire function. We assert that $\sigma(u_1)<1$. If we assume the contrary, i.e.

$$\limsup_{\rho\to+\infty}\frac{\log\log \mathrm{M}(\rho,u_1)}{\log\rho}\geqq 1,$$

then for then for an arbitrary $\varepsilon_1>0$, there exists a sequence $\{\rho_j\}\to+\infty$ such that

$\log\log M(\rho_j, u_1) > (1-\varepsilon_1)\log\rho_j$. Let ζ_{ρ_j} be the points with $|\zeta_{\rho_j}| = \rho_j$ at which $|u_1(\zeta_{\rho_j})| = M(\rho_j, u_1)$, then $\log\log|u_1(\zeta_{\rho_j})| > (1-\varepsilon_1)\log\rho_j$. From this, it is easy to see that $\log\log|\Phi(\zeta_{\rho_j})| > (1-\varepsilon_1)\log\rho_j - \log 2$ for sufficiently large j. Let z_j be points with $e^{\alpha z_j} = \zeta_{\rho_j}$. Setting $\zeta_{\rho_j} = \rho_j e^{i\theta_j}, |\theta_j| \leqq \pi$, since $\alpha z_j = \log\rho_j + i\theta_j$, we have $|\alpha||z_j| \leqq (1+\varepsilon_1)\log\rho_j$ for sufficiently large j, and $z_j \to \infty$ as $\rho_j \to +\infty$. Therefore,

$$\log\log|F(z_j)| = \log\log|\Phi(\zeta_{\rho_j})| > \frac{1-\varepsilon_1}{1+\varepsilon_1}\cdot|\alpha|\,|z_j| - \log 2.$$

Since we can choose $\varepsilon_1 > 0$ such that

$$\frac{1-\varepsilon_1}{1+\varepsilon_1} > \delta_1,$$

this contradicts the condition above which is satisfied by $F(z)$. Hence we must have $\sigma(u_1) < 1$. In addition, it is easy to see that

$$\frac{\Psi^{(k)}(\zeta)}{\Psi(\zeta)} = 0(1) \quad \text{as } \zeta \to \infty.$$

Thus, if $|\zeta| \geqq 1$ and $\zeta \notin V$, where V is an R-set, standard estimates (see [7, p. 74]) yield an $M > 0$ such that

$$\left|\zeta^2\frac{\Phi''}{\Phi} - \tfrac{3}{4}\zeta^2\left(\frac{\Phi'}{\Phi}\right)^2 + \zeta\frac{\Phi'}{\Phi}\right| \leqq |\zeta|^M.$$

So by (11), if $|\zeta| \geqq 1, \zeta \notin V$ and $|u_1(\zeta)| > 1$, we have

$$|B(\zeta)| \leqq |\zeta|^N \tag{12}$$

for some positive integer N.

On the other hand, $B(\zeta)$ has the expansion

$$B(\zeta) = \sum_{k=-\infty}^{+\infty} b_k\zeta^k, \quad 0 < |\zeta| < +\infty.$$

Denote $h(\zeta) = \sum_{k=0}^{+\infty} b_k\zeta^k$. Clearly, $h(\zeta)$ is a transcendetal entire function. We assert that $\sigma(h) < 1$. If we assume the contrary, i.e.

$$\limsup_{\rho\to+\infty}\frac{\log\log M(\rho, h)}{\log\rho} \geqq 1,$$

then for an arbitrary $\varepsilon_1 > 0$, there exists a sequence $\{\rho_j\} \to +\infty$ such that $\log\log M(\rho_j, h) > (1-\varepsilon_1)\log\rho_j$. Let ζ_{ρ_j} be the points with $|\zeta_{\rho_j}| = \rho_j$ at which $|h(\zeta_{\rho_j})| = M(\rho_j, h)$, then $\log\log|h(\zeta_{\rho_j})| > (1-\varepsilon_1)\log\rho_j$. From this, it is easy to see that

$$\log\log|B(\zeta_{\rho_j})|=\log\log\left|\sum_{k=-\infty}^{-1} b_k\zeta_{\rho_j}^k+h(\zeta_{\rho_j})\right|>\log\log\tfrac{1}{2}|h(\zeta_{\rho_j})|$$

$$>\log\log|h(\zeta_{\rho_j})|-\log 2>(1-\varepsilon_1)\log\rho_j-\log 2$$

for sufficiently large j. Let z_j be points with $e^{\alpha z_j}=\zeta_{\rho_j}$. Setting $\zeta_{\rho_j}=\rho_j e^{i\theta_j}$, $|\theta_j|\leqq\pi$, since $\alpha z_j=\log\rho_j+i\theta_j$, we have $|\alpha||z_j|\leqq(1+\varepsilon_1)\log\rho_j$ for sufficiently large j, and $z_j\to\infty$ as $\rho_j\to+\infty$. Therefore,

$$\log\log|A(z_j|=\log\log|B(\zeta_{\rho_j})|>\frac{1-\varepsilon_1}{1+\varepsilon_1}|\alpha||z_j|-\log 2.$$

Since we can choose $\varepsilon_1>0$ such that

$$\frac{1-\varepsilon_1}{1+\varepsilon_1}>\delta_0,$$

this contradicts the condition (3) which is satisfied by $A(z)$. So we must have $\sigma(h)<1$. We can also write

$$B(\zeta)=h_1(\zeta)+h_2(\zeta),$$

where

$$h_1(\zeta)=\sum_{k\leqq N} b_k\zeta^k,\quad h_2(\zeta)=\sum_{k<N} b_k\zeta^k.$$

Clearly, $|h_1(\zeta)|=0(|\zeta|^N)$. Since (12) holds if $\zeta\notin V$ and $|u_1(\zeta)|>1$, we have $|h_2(\zeta)|=0(|\zeta|^N)$ if $\zeta\notin V$ and $|u_1(\zeta)|>1$. Thus. we can choose a constant $K>0$ such that

$$\frac{|\zeta^{-N}h_2(\zeta)|}{K}\leqq 1$$

if $\zeta\notin V$ and $|u_1(\zeta)|>1$. Set

$$u_2(\zeta)=\frac{\zeta^{-N}h_2(\zeta)}{K}.$$

It is easy to see that $u_2(\zeta)$ is a transcendental entire function with $\sigma(u_2)<1$. Moreover, $|u_2(\zeta)|\leqq 1$ if $\zeta\notin V$ and $|u_1(\zeta)|>1$, as has been shown above.

Clearly, $D_j^*=\{\zeta;|u_j(\zeta)|>1\}$ are open sets, $j=1,2$. Denote the boundary of D_j^* by Γ_j^*. It is clear that $|u_j(\zeta)|=1$ for $\zeta\in\Gamma_j^*$. Since $u_j(\zeta)$ are transcendental entire functions, there must exist unbounded connected components D_j of D_j^*. Denote the boundary of D_j by

Γ_j. Then set $E_j(\rho)=\{\theta;\rho e^{i\theta}\in D_j\}$, $E(\rho)=\{\theta;\rho e^{i\theta}\in V\}$. It is easy to check that $E_1\cap E_2\subseteq E$. Also set

$$\theta_j(\rho)=\int_{E_j(\rho)} d\theta,\ j=1,2,\quad \theta(\rho)=\int_{E(\rho)} d\theta.$$

Clearly, for an arbitrarly $\varepsilon>0$, there exists a $\rho_0>0$ such that $\theta(\rho)<\varepsilon$ for $\rho\geqq\rho_0$, We also can choose $\rho_0>0$ such that the circle $|\zeta|=\rho$ intersects D_j for $\rho\geqq\rho_0$.

Since $\sigma(u_j)<1$, from [6, Theorem III.68., p. 117] there exists a constant $\beta>0$ and a $\rho_1\geqq\rho_0$ such that (see the following remark)

$$\int_{\rho_0}^{\rho/2}\frac{\pi}{\theta_j(\rho)}\frac{d\rho}{\rho}<(1-\beta)\log\rho$$

for $\rho\geqq\rho_1$ and $j=1,2$. So

$$\int_{\rho_0}^{\rho/2}\pi\,\frac{\theta_1(\rho)+\theta_2(\rho)}{\theta_1(\rho)\theta_2(\rho)}\frac{d\rho}{\rho}<(2-2\beta)\log\rho.$$

Thus, since

$$\sqrt{ab}\leqq\frac{a+b}{2}(a,b\geqq 0),$$

$$\int_{\rho_0}^{\rho/2}\frac{4\pi}{\theta_1(\rho)+\theta_2(\rho)}\frac{d\rho}{\rho}<(2-2\beta)\log\rho.$$

But $\theta_1(\rho)+\theta_2(\rho)\leqq 2\pi+\varepsilon$ for $\rho\geqq\rho_1$. This gives

$$\frac{4\pi}{2\pi+\varepsilon}\log\frac{\rho}{2\rho_0}<(2-2\beta)\log\rho.$$

Since $\beta>0$ is a constant and $\varepsilon>0$ is arbitrary, this is impossible.

In the case where $B(\zeta)$ has an essential singularity at $\zeta=0$, we make the change of variable $\zeta=1/t$ and reason as above at $\zeta=\infty$.

Remark. The estimate in [6, Theorem III.68., p. 117] is that

$$\int_{\rho_0}^{\rho/2}\frac{\pi}{\theta_j^*(\rho)}\frac{d\rho}{\rho}<\log\log M(\rho,u_j)+O(1),$$

where

https://doi.org/10.1017/S0013091500028959 Published online by Cambridge University Press

$$\theta_j^*(\rho)=\begin{cases}\theta_j(\rho) & \text{if } E_j(\rho)\neq[0,2\pi]\\ +\infty & \text{if } E_j(\rho)=[0,2\pi].\end{cases}$$

But if $E_1(\rho)=[0,2\pi]$, then $\theta_2(\rho)<\varepsilon$, and so

$$\int\limits_{\substack{\rho_0\\E_1(\rho)=[0,2\pi]}}^{\rho/2}\frac{\pi\,d\rho}{\rho\varepsilon}<\log\log M(\rho,u_2)+0(1).$$

Thus

$$\int\limits_{\substack{\rho_0\\E_1(\rho)=[0,2\pi]}}^{\rho/2}\frac{\pi}{\theta_1(\rho)}\frac{d\rho}{\rho}<\frac{\varepsilon}{2\pi}K_1\log\rho$$

if $K_1>\sigma(u_2)$ and ρ is large enough. So we get

$$\int_{\rho_0}^{\rho/2}\frac{\pi}{\theta_1(\rho)}\frac{d\rho}{\rho}<\log\log M(\rho,u_1)+\frac{\varepsilon}{2\pi}K_1\log\rho+0(1).$$

By the same reasoning, we also get

$$\int_{\rho_0}^{\rho/2}\frac{\pi}{\theta_2(\rho)}\frac{d\rho}{\rho}<\log\log M(\rho,u_2)+\frac{\varepsilon}{2\pi}K_2\log\rho+0(1)$$

for $K_2>\sigma(u_1)$.

Lemma 6. *Let $A(z)=B(e^{\alpha z})$ be a periodic entire function with period $\omega=2\pi i/\alpha$ and be transcendental in $e^{\alpha z}$, i.e. $B(\zeta)$ is transcendental and analytic on $0<|\zeta|<+\infty$. If $B(\zeta)$ has a pole of odd order at $\zeta=\infty$ or at $\zeta=0$ (including those which can be changed into this case by varying the period of $A(z)$), and equation (1) has a solution $f\not\equiv 0$ which satisfies condition (6), then $f(z)$ and $f(z+\omega)$ are linearly independent.*

Proof. If we set $\alpha'=-\alpha$, the pole $\zeta=0$ of $B(\zeta)$ can be changed into the pole $t=\infty$ of $B(t^{-1})$. Thus, noting that $f(z)=kf(z-\omega)$ is equivalent to $f(z+\omega)=kf(z)$, it is enough to only consider the case that $\zeta=\infty$ is the pole of $B(\zeta)$.

Assume equation (1) has a solution $f\not\equiv 0$ which satisfies condition (6), and $f(z), f(z+\omega)$ are linearly dependent. From Part (A) of Lemma 4, we can write $f(z)=e^{dz}G(e^{\alpha z})$, where

$$G(\zeta)=\left(\sum_{j=-\infty}^{q}c_j\zeta^j\right)\exp\left(\sum_{k=-\infty}^{v}d_k\zeta^k\right), 0<|\zeta|<+\infty, \tag{13}$$

q and ν are integers, $c_q d_\nu \neq 0$. Substituting $e^{dz}G(e^{\alpha z})$ for $f(z)$ in (1), we obtain

$$\alpha^2\zeta^2 G''(\zeta)+(2\alpha d+\alpha^2)\zeta G'(\zeta)+[B(\zeta)+d^2]G(\zeta)=0. \tag{14}$$

Since $B(\zeta)$ has a pole of odd order at $\zeta=\infty$, $B(\zeta)$ can be written as

$$B(\zeta)=\sum_{i=-\infty}^{p} b_i\zeta^i, 0<|\zeta|<+\infty,$$

where p is an odd positive integer, $b_p \neq 0$. From (13), it is easy to check that we have for ζ near ∞

$$\frac{G'(\zeta)}{G(\zeta)}=\begin{cases} a\zeta^{-1}+0(|\zeta|^{-2}) & \text{if } \nu<1, \\ b\zeta^{\nu-1}+0(|\zeta|^{\nu-2}) & \text{if } \nu\geqq 1, \end{cases} \tag{15}$$

$$\frac{G''(\zeta)}{G(\zeta)}=\begin{cases} a(a-1)\zeta^{-2}+0(|\zeta|^{-3}) & \text{if } \nu<1, \\ b^2\zeta^{2\nu-2}+0(|\zeta|^{2\nu-3}) & \text{if } \nu\geqq 1, \end{cases} \tag{16}$$

where $a=q, b=\nu d_\nu, b\neq 0$. Substituting the right-hand sides of (15) and (16) for G'/G and G''/G in (14), and noting that $2\nu\neq p$, it is easy to see that (14) can not hold identically for ζ near ∞, and a contradiction is obtained.

Under the assumptions of the theorem, if equation (1) has a solution $f\not\equiv 0$ which satisfies condition (6), then from Lemma 5 and Lemma 6, $f(z)$ and $f(z+\omega)$ are linearly dependent and also linearly independent. This is impossible and the proof of the theorem is completed.

Proof of the corollary

The following Lemma 7 not only shows that the corollary is true but also shows that the corollary is equivalent to the theorem.

Lemma 7. *Let $A(z)=B(\zeta)$ be a periodic entire function with period $\omega=2\pi i/\alpha$, and be transcendental in $e^{\alpha z}$, i.e. $B(\zeta)$ is transcendental and analytic on $0<|\zeta|<+\infty$. If $A(z)$ satisfies condition* (3), *then we have the representation*

$$B(\zeta)=g\left(\frac{1}{\zeta}\right)+h(\zeta), \tag{17}$$

where $g(\zeta)$ and $h(\zeta)$ are entire functions with $\sigma(g)<1$ and $\sigma(h)<1$, and at least one of $g(\zeta)$ and $h(\zeta)$ is transcendental. Furthermore if $B(\zeta)$ has a pole at $\zeta=\infty$ (resp. $\zeta=0$), then $h(\zeta)$ (resp. $g(\zeta)$) is a nonconstant polynomial. The converse is also true.

Proof. First, from the assumption, we have the expansion

$$B(\zeta)=\sum_{k=-\infty}^{+\infty} b_k\zeta^k, 0<|\zeta|<+\infty.$$

If we set

$$g\left(\frac{1}{\zeta}\right)=\sum_{k=-\infty}^{-1} b_k\zeta^k, \quad h(\zeta)=\sum_{k=0}^{+\infty} b_k\zeta^k,$$

then $g(\zeta)$ and $h(\zeta)$ are entire functions, and at least one is transcendental. And also if $B(\zeta)$ has a pole at $\zeta=\infty$ (resp. $\zeta=0$), then $h(\zeta)$ (resp. $g(\zeta)$) is a nonconstant polynomial. In Lemma 5, $\sigma(h)<1$ has been shown. Setting $\zeta=1/t, B^*(t)=B(1/t)$ and $A(z)=B^*(e^{-\alpha z})$, we can prove $\sigma(g)<1$ by the same reasoning as the proof of $\sigma(h)<1$.

Conversely, assume $B(\zeta)$ has the representation (17), where $g(\zeta)$ and $h(\zeta)$ are entire functions with $\sigma(g)<1$ and $\sigma(h)<1$, we show that $A(z)$ satisfies the condition (3) (the other properties are clear). Denote

$$M_1(r,A)=\max_{\substack{|z|=r\\ Re(\alpha z)\geqq 0}}|A(z)|, \quad M_2(r,A)=\max_{\substack{|z|=r\\ Re(\alpha z)\leqq 0}}|A(z)|.$$

For an arbitrary $r>0$, let z_r be a point with $|z_r|=r$ and $Re(\alpha z_r)\geqq 0$ at which $|A(z_r)|=M_1(r,A)$, and let $e^{\alpha z_r}=\zeta_\rho=\rho e^{i\theta_\rho}$, $|\theta_\rho|\leqq\pi$. From $\alpha z_r=\log\rho+i\theta_\rho$, $|\theta_\rho|\leqq\pi$ and $Re(\alpha z_r)\geqq 0$, it follows that $|\alpha||z_r|\geqq\log\rho$ and $\rho\to+\infty$ as $r\to+\infty$. Thus for a given $\varepsilon>0$, we have if r is sufficiently large (and ρ is also sufficiently large)

$$\begin{aligned}\log\log M_1(r,A)&=\log\log|A(z_r)|\\ &\leqq\log\log\left(\left|g\left(\frac{1}{\zeta_\rho}\right)\right|+|h(\zeta_\rho)|\right)\\ &\leqq\log\log M(\rho,h)+0(1)\\ &<(\sigma(h)+\varepsilon)\log\rho\\ &\leqq(\sigma(h)+\varepsilon)|\alpha|r.\end{aligned}$$

On the other hand, if z_r be a point with $|z_r|=r$ and $Re(\alpha z_r)\leqq 0$ at which $|A(z_r)|=M_2(r,A)$, setting $A(z)=g(e^{-\alpha z})+h(1/e^{-\alpha z})=g(t)+h(1/t)$, we have as above

$$\log\log M_2(r,A)<(\sigma(g)+\varepsilon)|\alpha|r$$

for sufficiently large r (and, for the corresponding t of z_r, $|t|$ is also sufficiently large since $Re(-\alpha z_r)\geqq 0$). From $0\leqq\sigma(g)<1$ and $0\leqq\sigma(h)<1$, we can choose $\varepsilon>0$ such that

$0<\sigma(g)+\varepsilon<1$ and $0<\sigma(h)+\varepsilon<1$. Setting $\delta_0=\max\{\sigma(g)+\varepsilon,\sigma(h)+\varepsilon\}$, we have for r near $+\infty$

$$\log\log M(r,A)=\max\{\log\log M_1(r,A),\log\log M_2(r,A)\}$$
$$<\delta_0|\alpha|r.$$

The condition (4) with $0<\delta_0<1$ has been verified, and so has the condition (3).

In addition, it is easy to prove that $\log T(r,A)=o(r)$ is equivalent to $\max\{\sigma(g),\sigma(h)\}=0$. Thus, if $\sigma(g)>0$ or $\sigma(h)>0$, we must have $\sigma(A)=+\infty$. From this and Lemma 7, we know that the family of entire functions with infinite order of growth is quite large under the condition (3).

5. Examples

The following Example 1 shows that if $\zeta=\infty$ (or $\zeta=0$) is a pole of $B(\zeta)$ with even order, the conclusion of the theorem or corollary may be false.

Example 1. Let $\phi(\zeta)$ be a transcendental entire function with $\sigma(\phi)<1$. It is easy to check that

$$f(z)=\exp\left(\phi\left(\frac{1}{e^{\alpha z}}\right)+e^{\alpha z}\right)$$

solves equation (1) in which

$$A(z)=\alpha^2\left(-\phi'^2\frac{1}{e^{2\alpha z}}+2\phi'-\phi''\frac{1}{e^{2\alpha z}}-\phi'\frac{1}{e^{\alpha z}}-e^{\alpha z}-e^{2\alpha z}\right).$$

Clearly, $\lambda(f)=0$. Setting $g(\zeta)=\alpha^2(-\phi'^2(\zeta)\zeta^2+2\phi'(\zeta)-\phi''(\zeta)\zeta^2-\phi'(\zeta)\zeta)$, it is clear that $\sigma(g)<1$ and $B(\zeta)=g(1/\zeta)-\alpha^2\zeta-\alpha^2\zeta^2$ has a pole of even order at $\zeta=\infty$.

The following Example 2 shows that if $\sigma(g)$ is an arbitrary positive integer and $\zeta=\infty$ (or $\zeta=0$) is a pole of $B(\zeta)$ with odd order, the conclusion of the theorem or corollary may also be false.

Example 2. Set $E(z)=e^{z/2}\,e^{(1/2)e^{mz}}$, where m is an arbitrary positive integer, and set

$$f_j=E^{1/2}\exp\left(\int_0^z\frac{(-1)^j}{E}\,dt\right)$$

for $j=1,2$. Then f_1 and f_2 are non-vanishing entire functions, and $f_1f_2=E$. Also it is easy to check that that Wronskian $W(f_1, f_2) = 2$ and f_1, f_2 solve equation (1) in which (from [**1**, Section 5(a)])

$$\begin{aligned}-4A &= \frac{2^2}{E^2} - \left(\frac{E'}{E}\right)^2 + 2\frac{E''}{E} \\ &= \frac{4}{E^2} + 2\left(\frac{E'}{E}\right)' + \left(\frac{E'}{E}\right)^2 \\ &= \frac{4}{e^z e^{e^{mz}}} + \tfrac{1}{4} + \left(m^2 + \frac{m}{2}\right)e^{mz} + \frac{m^2}{4}e^{2mz} \\ &= \frac{4}{\zeta e^{\zeta^m}} + \tfrac{1}{4} + \left(m^2 + \frac{m}{2}\right)\zeta^m + \frac{m^2}{4}\zeta^{2m} \\ &= \frac{4}{\zeta} + g(\zeta) = -4B(\zeta).\end{aligned}$$

$B(\zeta)$ has a pole of odd order at $\zeta = 0$, and it is easy to see that $\sigma(g) = m$.

A problem naturally arises: If $\sigma(g)$ is greater than 1 but is not a positive integer, could the theorem or corollary still hold?

Acknowledgement. The author would like to acknowledge Dr J. K. Langley for his very helpful suggestions and for going over the manuscript carefully.

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