APPROXIMATION IN BOUNDED SUMMABILITY FIELDS

J. D. HILL AND W. T. SLEDD

1. Introduction. This paper deals with several related properties of bounded summability fields of regular, real matrices. For a matrix $A = (a_{nk})$ and a sequence $x = \{x_n\}$, we write formally

$$A_n(x) = \sum_k a_{nk} x_k \qquad \text{and} \qquad A(x) = \lim_n A_n(x).$$

We denote by m the space of bounded real sequences, and by A^* the bounded summability field

$$\{x: x \in m, \quad \text{and} \quad \lim_n A_n(x) \text{ exists}\}$$

of A. The strong summability field of A is the set

$$|A| = \{x: x \in m \quad \text{and} \quad \lim_n \sum_k |a_{nk}|\, |x_k - a| = 0 \text{ for some } a\}.$$

In §2 we characterize the bounded summability fields A^* whose elements can be uniformly approximated by finite linear combinations of characteristic functions (of disjoint subsets of the natural numbers) belonging to A^*. In §3 we study the multipliers of A^*, and we show that if the elements of the matrix A are non-negative, then the multipliers of A^* coincide with the sequences that are strongly summable by A. Section 4 deals with the strong summability field of a regular matrix.

2. Approximations by characteristic functions. We denote the set of positive integers by N and the normed linear space of bounded real sequences by m. Let L be a closed linear subspace of m. A subset E of N is L-admissible if the characteristic function χ_E is a member of L. An L-admissible partition of N is a finite partition of N into L-admissible subsets and an L-admissible function is a function of the form

$$\Phi = \sum_{i=1}^{n} \lambda_i \chi_{E_i}$$

where the coefficients λ_i are real numbers and the E_i constitute an L-admissible partition of N. We obtain first a sufficient condition for L to be an algebra and then, for the case where L is a bounded summability field, a necessary condition.

THEOREM 2.1. *If L is the closure of the L-admissible functions, then L is an algebra.*

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Proof. Let G_1 and G_2 be L-admissible sets, and set

$$\psi = \chi_{G_1} + 2\chi_{G_2}.$$

Then ψ is in L and hence there exists an L-admissible function Φ such that $||\Phi - \psi|| < \frac{1}{2}$. We may write

$$N = \bigcup_{i=1}^{4} F_i.$$

where $F_1 = N \diagdown (G_1 \cup G_2)$, $F_2 = G_1 \diagdown G_2$, $F_3 = G_2 \diagdown G_1$, $F_4 = G_1 \cap G_2$. Let

$$\Phi = \sum_{i=1}^{n} \lambda_i \chi_{E_i}.$$

For each $j = 1, \ldots, n$ there is exactly one i between 1 and 4 such that $F_i \cap E_j$ is not empty. Otherwise, if $n_1 \in F_{i_1} \cap E_j$ and $n_2 \in F_{i_2} \cap E_j$, then $\Phi(n_1) = \Phi(n_2)$ while $|\psi(n_1) - \psi(n_2)| \geqslant 1$. But $|\psi(n_1) - \psi(n_2)| < 1$ since $||\Phi - \psi|| < \frac{1}{2}$.

Moreover, $F_i \supset E_j$ since

$$\bigcup_{k=1}^{4} F_k \cap E_j = E_j.$$

Thus because of the disjointness of the E_p's and of the F_q's, each F_q is the disjoint union of E_p's and consequently each F_q is an L-admissible set. This implies that the class of L-admissible sets is an algebra of sets.

But this in turn implies that any function of the form

$$\sum_{i=1}^{n} \lambda_i \chi_{G_i}$$

(where all the G_i are L-admissible sets but not necessarily disjoint) is an L-admissible function. Thus the set of L-admissible functions is an algebra, and since L is the uniform closure of the L-admissible functions, K is also an algebra.

Henriksen and Isbell **(3)** have shown that a bounded summability field is an algebra if and only if it is the strong summability field of a matrix method. Using their result, we obtain a partial converse of Theorem 2.1.

THEOREM 2.2. *If a bounded summability field A^* is a subalgebra of m, then it is the linear closure of the A^*-admissible functions.*

Proof. We first prove that A^* is closed, a fact that does not depend on A^* being an algebra.

If $\{x^{(p)}\}$ is a sequence of elements from A^* and $\lim_p x^{(p)} = x$ in the uniform norm of m, then

$$|A_n(x) - A_j(x)| \leqslant |A_n(x) - A_n(x^{(p)})| + |A_n(x^{(p)}) - A_j(x^{(p)})| + |A_j(x^{(p)}) - A_j(x)|.$$

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For sufficiently large p, the first and third terms are small. Fixing p and letting j and n become large, we see that $\{A_n(x)\}$ is a Cauchy sequence. It follows that x belongs to A^*, and hence A^* is closed.

Now we must show that each x in A^* may be approximated by an A^*-admissible function. Assume without loss of generality that $||x|| \leqslant 1$ and $A(x) = 0$. Using the above-mentioned result of Henriksen and Isbell, let $A^* = |B|$. Then given $\epsilon > 0$, let

$$F_p = \{k: p\epsilon \leqslant x_k < (p+1)\epsilon\} \qquad (p = 0, \pm 1, \pm 2, \ldots).$$

Observe that since x is bounded, all except finitely many F_p are empty. If $p \geqslant 1$, then

$$\sum_k |b_{nk}|\, \chi_{F_p}(k) \leqslant \frac{1}{p\epsilon} \sum_k |b_{nk}|\, |x_k|,$$

while if $p < -1$, then

$$\sum_k |b_{nk}|\, \chi_{F_p}(k) \leqslant \frac{1}{|p+1|\epsilon} \sum_k |b_{nk}|\, |x_k|.$$

Hence if $p \neq 0, -1$, then χ_{F_p} is strongly B-summable to 0. Thus if

$$E = N \setminus \bigcup_{P \neq 0,-1} F_p,$$

then χ_E is A-summable to 1. Therefore, if we let

$$\Phi(n) = \begin{cases} p & \text{if } n \in F_p,\ p \neq 0, -1 \text{ and } F_p \text{ is not empty,} \\ 0 & \text{if } n \in E, \end{cases}$$

then Φ is an A^*-admissible function and $||\Phi - x|| < \epsilon$.

3. Multipliers of bounded summability fields. We define a new subset of A^*. Let

$$A^{**} = \{x \in m: xy = \{x_k y_k\} \in A^* \text{ for each } y \in A^*\}.$$

Since χ_N is in A^*, we see that $A^{**} \subset A^*$ and $A^{**} = A^*$ if and only if A^* is a subalgebra of m. Moreover, A^{**} is itself a closed subalgebra of m.

Our first theorem shows that the linear functional $A(x)$ is multiplicative on A^* when A^* is an algebra. This property of $A(x)$ has been assumed in previous papers dealing with summability fields that are algebras; see **(1, 2)**.

THEOREM 3.1. *$A(x)$ is multiplicative on A^{**}.*

Proof. If x is in A^{**} and $A(x) \neq 0$, then $B = (a_{nk}\, x_k/A(x))$ is a regular matrix, and if y is in A^*, then xy belongs to A^* while $B_n(y) = A_n(xy)/A(x)$. Thus y belongs to B^* and therefore $A^* \subset B^*$. By the well-known consistency theorem of Brudno and of Mazur and Orlicz, $B(y) = A(y)$. But

$$B(y) = A(xy)/A(x);$$

hence $A(xy) = A(x)A(y)$.

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If x is in A^{**} and $A(x) = 0$, let $B = (a_{nk}x_k + a_{nk})$. Then B is regular, and if y belongs to A^*, $B_n(y) = A_n(xy) + A_n(y)$. As before, $A^* \subset B^*$, so $A(y) = B(y) = A(xy) + A(y)$. Thus $A(xy) = 0 = A(x)A(y)$.

THEOREM 3.2. *Let x belong to A^{**}, and let C denote the compact set of real numbers consisting of the closure of the range of the sequence x and the point $A(x)$. If F is a continuous real-valued function on C, and $y = \{F(x_k)\}$, then y belongs to A^{**} and $A(yz) = A(z)F(A(x))$ whenever z belongs to A^*.*

Proof. By the previous theorem, $A(zx^n) = A(z)[A(x)]^n$ whenever x is in A^{**}, z belongs to A^*, and n is a non-negative integer. Therefore the theorem holds when F is a polynomial. Using the fact that $A(x)$ is continuous in the uniform norm on m and applying the Weierstrass polynomial approximation theorem we obtain the conclusion.

THEOREM 3.3. *If $a_{nk} \geqslant 0$ for all n and k, then $A^{**} = |A|$.*

Proof. Let $F(t) = |t|$ and let $z_k = 1$ for each k. Since x is in A^{**} and

$$\lim_n \sum_k a_{nk}(x_k - A(x)) = 0,$$

it follows from Theorem 3.2 that

$$\lim_n \sum_k a_{nk}|x_k - A(x)| = 0.$$

Hence x belongs to $|A|$. The converse is clearly true.

THEOREM 3.4. *If x belongs to A^{**}, then $A(x)$ lies in the interval* $[\liminf x_k, \limsup x_k]$.

Proof. Let p be a positive integer and set

$$y_k = \begin{cases} x_k, & k \geqslant p, \\ \inf x_k, & k < p. \end{cases}$$

Since A is regular, y is in A^{**} and $A(y) = A(x)$. Let $a = \sup y_k$. Then $a - y_k = |a - y_k|$, and by Theorem 3.2 (with $F(t) = |t|$ and $z_k = 1$ for each k)

$$0 \leqslant |A(a - y)| = A(a - y) = a - A(y) = a - A(x).$$

Therefore $A(x) \leqslant a = \sup_{k \geqslant p} x_p$. Since this is true for each p, $A(x) \leqslant \lim\sup x_k$. Similarly, we see that $\liminf x_k \leqslant A(x)$.

A consequence of Theorem 3.4 is that if x belongs to A^{**}, then $A(x)$ must be a limit point of the sequence x. Suppose that $A(x) = 0$; then by Theorem 3.2, $|x|$ is in A^{**} and $|A(x)| = A(|x|) = 0$. By Theorem 3.4, the point 0 lies in $[\liminf |x_k|, \limsup |x_k|]$. Hence $\liminf |x_k| = 0$. Brauer **(1)** has proved this result when $A^{**} = A^*$.

In the next section we show that if x belongs to $|A|$, then $A(x)$ is a limit point of x in a very cogent sense.

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4. Strong summability fields.

THEOREM 4.1. *If A is a regular matrix, then the bounded sequence x is strongly summable to a if and only if there exists a subset Z of N such that $\chi_{N\setminus Z}$ is strongly A-summable to* 0, *and* $\lim_{n\in Z} x_n = a$.

Proof. Suppose that there is a such a subset Z of N. Let

$$x^{(1)} = \chi_Z \cdot x \quad \text{and} \quad x^{(2)} = \chi_{N\setminus Z}\cdot x,$$

so that $x = x^{(1)} + x^{(2)}$. Then

$$\sum_k |a_{nk}|\,|x_k - a| = \sum_{k\in Z} |a_{nk}|\,|x_k^{(1)} - a| + \sum_{k\in N\setminus Z} |a_{nk}|\,|x_k^{(2)} - a|.$$

The matrix $(|a_{nk}|\,\chi_Z(k))$ carries null sequences into null sequences, while the matrix $(|a_{nk}|\,\chi_{N\setminus Z}(k))$ carries every bounded sequence into a null sequence. Since $\lim_{k\in Z}(x_k^{(1)} - a) = 0$ and $|x_k^{(2)} - a|$ is bounded,

$$\lim_n \sum\nolimits_k |a_{nk}|\,|x_k - a| = 0.$$

Conversely, suppose that the last relation holds. Let $y_k = x_k - a$, and for each positive integer p, let

$$E_p = \{k;\, |y_k| \geqslant 1/p\}.$$

Then

$$\sum\nolimits_k |a_{nk}|\,\chi_{E_p}(k) \leqslant p\sum\nolimits_k |a_{nk}|\,|y_k|;$$

hence

$$\lim_n \sum\nolimits_k |a_{nk}|\,\chi_{E_p}(k) = 0.$$

We can now choose two sequences of positive integers $\{m_r\}$ and $\{n_r\}$ inductively so that

$$\lim_r \max_{n_r\leqslant n<n_{r+1}} \left(\sum_{k<m_r} + \sum_{k>m_{r+2}}\right) |a_{nk}| = 0.$$

and

$$\lim_r \max_{n_r\leqslant n} \sum_k |a_{nk}|\,\chi_{E_{r+1}}(k) = 0.$$

Let

$$F_r = \{k \in E_r\colon m_r \leqslant k \leqslant m_{r+2}\}$$

and let

$$N\setminus Z = \cup_r F_r.$$

Then if $n_r \leqslant n < n_{r+1}$, we have the inequality

$$\sum_k |a_{nk}|\,\chi_{N\setminus Z}(k) \leqslant \left(\sum_{k<m_r} + \sum_{k>m_{r+2}}\right) |a_{nk}| + \sum_k |a_{nk}|\,\chi_{E_{r+1}}(k).$$

Thus

$$\lim_n \sum\nolimits_k |a_{nk}|\,\chi_{N\setminus Z}(k) = 0,$$

and if k is in Z and $m_r \leqslant k \leqslant m_{r+2}$, then $|y_k| < 1/r$. Therefore

$$\lim_{n\in Z} y_n = 0.$$

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THEOREM 4.2. *If* $A^* \subset B^*$, *then* $|A| \subset |B|$.

Proof. If x belongs to $|A|$, then

$$\lim_n \sum_k |a_{nk}|\,|x_k - a| = 0$$

for some a, and hence if y belongs to m, then

$$\lim_n \sum_k |a_{nk}|\,|x_k - a|\,|y_k| = 0.$$

This implies that $(x - a)y$ belongs to A^* when y belongs to m, and since $A^* \subset B^*$, then $(x - a)y$ is in B^*. It follows from a theorem of Schur (4) that

$$\lim_n \sum_k |b_{nk}|\,|x_k - a| = 0$$

so that x is in $|B|$.

The converse of Theorem 4.2 is false. For if A is the sequence-to-sequence transformation given by

$$y_n = (x_n + x_{n+1})/2, \qquad n = 1, 2, \ldots,$$

and B is ordinary convergence, then $|B| = |A|$; yet B^* does not contain A^*.

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Michigan State University,
East Lansing, Michigan

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