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CONTINUOUS FINITE APOLLONIUS SETS IN METRIC SPACES

L. D. LOVELAND

1. Introduction. The set of all points in the Euclidean plane E^2, the ratio of whose distances from two fixed points is a constant λ, is known as the circle of Apollonius [**7**, p. 62]. This "Apollonius" set is a circle except for the degenerate cases where $\lambda = 1$ or $\lambda = 0$. In more general metric spaces the same definition applies to select certain Apollonius sets (or "λ-sets" in our terminology), but of course these sets are not always circles. For example, all λ-sets ($\lambda > 0$) relative to a circle in E^2 are two-point sets, and all λ-sets relative to E^1 are either singletons or two-point sets. This paper deals with the topological structure of a metric space when certain cardinality conditions have been imposed on its λ-sets.

Let λ be a positive real number. The *λ-set*, $\lambda(a, b)$, of two points a and b in a metric space (X, d) is the set $\{x | d(a, x) = \lambda d(b, x)\}$. No generality is lost in restricting λ to the interval $(0, 1]$ since $\lambda(a, b) = \frac{1}{\lambda}(b, a)$. In the special case where $\lambda = 1$, the λ-set is known in the literature as the *midset*, $M(a, b)$, of a and b. A metric space X is said to have the *finite λ-set property* ($F\lambda P$) if there is a number λ in $(0, 1]$ such that, for each two points a and b of X, $\lambda(a, b)$ is a finite set. The nth *order λ-set property* ($\lambda P(n)$) implies the existence of a $\lambda \in (0, 1]$ such that each λ-set in X consists of n points. The $\lambda P(1)$ and the $\lambda P(2)$ have also been called the *unique λ-set property* ($U\lambda P$) and the *double λ-set property* ($D\lambda P$), respectively, and when $\lambda = 1$ they are known as the *unique* (UMP) and *double* (DMP) *midset properties*. Similarly the $F\lambda P$ becomes the *finite midset property* (FMP) when $\lambda = 1$.

Theorem 3.1 states that a continuum is an arc if it has the $U\lambda P$. Although the converse is clearly false, a "continuity" restriction on the λ-set function, defined below, makes it true. Thus an arc is characterized among continua by the continuous unique λ-set property (Theorem 3.2). However, if an arc has a continuous λ-set function, then $\lambda = 1$ and it has the UMP (Theorem 3.6). In addition we show that arcs and simple closed curves are the only continua having the continuous nth order λ-set property ($C\lambda P(n)$); see Theorems 3.1 and 3.4. The main result of [**3**] is generalized by showing that a continuum with the continuous double λ-set property (defined below) must be a simple closed curve (Theorem 3.4). A *continuum* is a nondegenerate compact connected metric space.

Let (X, d) be a metric space, and let $P(X)$ be the set of all subsets of X. In

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the product space $X \times X$, let D be the diagonal $\{(x, y)|x = y\}$. The λ-set function $\lambda : (X \times X - D) \rightarrow P(X)$ is defined by letting $\lambda(x, y)$ be the λ-set of x and y in X. Notice that "λ" is being used in a dual role; in one sense it represents the "Apollonius" constant in the definition of the λ-set while it has now been given meaning as a function. This should cause no confusion since the λ-set $\lambda(a, b)$ is also the value of the function λ at the point (a, b). The function λ is *continuous* if $\lambda(x, y) \subset \liminf_{i\to\infty} \lambda(x_i, y_i)$ whenever $\{(x_i, y_i)\}$ converges to (x, y) in $X \times X - D$. It follows from the continuity of the metric function d that $\limsup_{i\to\infty} \lambda(x_i, y_i) \subset \lambda(x, y)$; hence, it can be proved that λ is continuous if and only if $\{\lambda(x_i, y_i)\}$ converges to $\lambda(x, y)$ whenever $\{(x_i, y_i)\}$ converges to (x, y) in $X \times X - D$. The definitions of "limit superior" and "limit inferior" can be found in [**9**, p. 209].

In the special case where $\lambda = 1$, we denote the λ-set function by M and call it the *midset function*. The reader might prefer to concentrate on midsets rather than on the more general λ-sets the first time through the paper.

A continuum with the UMP clearly has a continuous midset function, and thus it has what we call the *continuous unique midset property* ($CUMP$). More generally we use the letter "C" preceding the abbreviation for a particular midset or λ-set property to indicate that in addition to possessing that property the space also has a continuous λ-set function. For example, a metric space X has the $CDMP$ if and only if the midset function M is continuous and X has the DMP. Examples of arcs in the plane are easily found where the midset function fails to be continuous; however, every simple closed curve with the DMP can be shown to have the $CDMP$. Whether or not an arc with the DMP can exist is unknown [**6**, Question 2, p. 1005], but if there is such an arc its midset function cannot be continuous (see Theorem 3.2). Example 3.8 shows that a 1-dimensional continuum in the plane can have the $CFMP$ and still contain a triod; however, a triod itself cannot have the $CFMP$ (Theorem 3.7).

It is not difficult to show that a continuum with the $U\lambda P$ is an arc (see Theorem 3.1). Berard proved that a connected metric space with the UMP is homeomorphic to a subset of the real line [**1**]. It is also known that a complete convex metric space with the DMP is isometric to a circle with the "arc length" metric. This result first appeared in [**4**, Theorem 2] and later in [**2**]. A more recent short proof has been given [**5**]. Such an isometry need not exist when the "convex" hypothesis is replaced with "connected"; the circle with its inherited plane topology illustrates this. It is not known whether a continuum possessing the DMP must be a simple closed curve, although this has been conjectured [**6**]. A perhaps stronger midset property, the continuous double midset property, is enough to insure that a continuum is a topological simple closed curve [**3**, Theorem 3]; in fact, a continuum with the $CD\lambda P$ must be a simple closed curve (Corollary 3.5).

An *arc* is a space homeomorphic to an interval on the real line. We use $[a, b]$ to denote an arc ordered from the endpoint a to the endpoint b. An *n-frame* is homeomorphic to the union of n arcs $[v, p_i]$ in the plane which are

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pairwise disjoint except for the *vertex* v. A *triod* is a 3-frame, and the *legs* of a triod are the images of $(v, p_i]$.

2. Basic facts. A metric space X is separated by a midset $M(a, b)$ into two sets, one consisting of all points closer to a than to b and the other consisting of those points closer to b than to a. The first lemma generalizes this separation to λ-sets; the proof is a simple application of the continuity of the metric.

LEMMA 2.1. (Standard separation). *If a and b are two points of a connected metric space X and $\lambda \in (0, 1]$, then $X - \lambda(a, b)$ is the union of disjoint open sets L and R where $L = \{x | d(a, x) < \lambda d(b, x)\}$, and $R = \{x | d(a, x) > \lambda d(b, x)\}$.*

LEMMA 2.2. *If X is a continuum with the $F\lambda P$, then X is locally connected.*

A proof for Lemma 2.2 for the case $\lambda = 1$ is outlined in [**6**, Lemma 2], and the same argument works for each λ. Lemma 2.3 is Theorem 75 of [**8**, p. 218], but an easy proof is outlined in [**6**, Lemma 3] assuming it is known that a continuum is a simple closed curve if each two-point set separates it [**9**, Theorem 28.14, p. 207].

LEMMA 2.3. *If X is a locally connected continuum that contains no triod, then X is either an arc or a simple closed curve.*

LEMMA 2.4. *If x and y are two points of an arc A and A has a continuous λ-set function for some $\lambda \in (0, 1]$, then $\lambda(x, y)$ lies between x and y.*

Proof. Suppose two points x and y exist in A such that $x < y$ and $\lambda(x, y)$ is not between x and y. If a and b are the endpoints of A, we see that $\lambda(x, y)$ intersects either $[a, x)$ or $(y, b]$. We may assume for convenience that $\lambda(x, y) \cap (y, b] \neq \emptyset$, and it follows that $y \neq b$. Let $B = \{t \in [y, b] | \lambda(x, t) \cap (t, b] \neq \emptyset\}$, and note that $y \in B$ while $b \notin B$.

To show that B is closed, let $\{p_i\}$ be a sequence of points of B converging to a point p. Then $\lambda(x, p_i) \cap (p_i, b] \neq \emptyset$, for each i, and the continuity of λ forces $\lambda(x, p)$ to intersect $(p, b]$. Thus $p \in B$. On the other hand B is open since if $\{p_i\}$ converges to a point p of B, then the continuityof λ implies that $p_i \in B$ for i sufficiently large. But B cannot be open and closed since it is a nonempty proper subset of the connected set $[y, b]$.

LEMMA 2.5. *No metric space with the $C\lambda P(n)$ can contain a triod.*

Proof. Suppose X is a metric space, n is a positive integer, λ is a number in $(0, 1]$ such that X has the $C\lambda P(n)$, and X contains a triod T with vertex v. It is easy to find points a and b in different legs of T such that $v \in \lambda(a, b)$, $d(t, v) < d(a, v)$ for every $t \in (a, v)$, and $d(v, t) < d(v, b)$ for every $t \in (b, v)$. Let $\lambda(a, b) = \{v_1, v_2, v_3, \ldots, v_n\}$ where $v_1 = v$, and let $O_1, O_2, \ldots, O_n$ be pairwise disjoint open sets such that $v_i \in O_i$ for each i. Let $X - \lambda(a, b) = L \cup R$ where $a \in L$ and $b \in R$ (see Lemma 2.1), and let $[v, p] \cup [v, q] \cup [v, r]$ be a

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subtriod T' of T lying in O_1. There are two cases depending on the location of the legs of T' relative to L and R.

In the first case we assume at least two of the legs $(v, p]$, $(v, q]$, and $(v, r]$ of T', say $(v, p]$ and $(v, q]$, lie in R. The continuity of λ assures the existence of a sequence $\{a_j\}$ of points from (a, v) converging to a such that $\lambda(a_j, b)$ intersects each O_i for every j. We shall exhibit an integer J such that $\lambda(a_j, b)$ separates $\{v\}$ from $\{p, q\}$ if $j > J$. A contradiction to the $C\lambda P(n)$ ensues because $\lambda(a_j, b)$ would then contain two points of O_1, one point from (v, p) and one from (v, q), and would intersect every other O_i. To show the existence of J let $X - \lambda(a_j, b) = L_j \cup R_j$ where $a_j \in L_j$ and $b \in R_j$ as in Lemma 2.1.

Now p cannot belong to L_j for infinitely many j because $d(a_j, p) < \lambda d(b, p)$ would then imply $d(a, p) \leqq \lambda d(b, p)$, by the continuity of d, contrary to $p \in R$. The same applies to q, so there must be an integer J such that $\{p, q\} \subset R_j$ if $j > J$. By the choice of a we have $d(a, v) > d(a_j, v)$ for all j, and since $d(a, v) = \lambda d(b, v)$ it follows that $d(a_j, v) < \lambda d(b, v)$; consequently $v \in L_j$ for every j. Thus, for $j > J$, $\lambda(a_j, b)$ separates $\{v\}$ from $\{p, q\}$ as desired.

In the last case we assume at least two, say $(v, p]$ and $(v, q]$, of the three sets $(v, p]$, $(v, q]$, and $(v, r]$ lie in L. This time a sequence $\{b_j\}$ of points in (v, b) is chosen converging to b such that $\lambda(a, b_j)$ intersects each O_i for every j. As in Lemma 2.1 we let $X - \lambda(a, b_j) = L_j \cup R_j$ where $a \in L_j$ and $b_j \in R_j$. The existence of an integer J such that $\{p, q\} \subset L_j$ whenever $j > J$ follows from the continuity of d as above. The point b was chosen such that $d(v, b_j) < d(v, b)$ for every j. Thus $\lambda d(v, b_j) < \lambda d(v, b) = d(a, v)$, and it follows that $v \in R_j$ for every j. Thus if $j > J$, $\lambda(a, b_j)$ separates $\{p, q\}$ from $\{v\}$. As in the first case above, this implies $\lambda(a, b_j)$ contains at least $n + 1$ points, contrary to the $C\lambda P(n)$.

LEMMA 2.6. *If T is a triod with vertex v and endpoints a, b, and c such that T has a continuous λ-set function for some $\lambda \in (0, 1]$, then $\lambda(a, v)$ lies in the leg $[a, v)$ of T. Furthermore, if $a' \in [a, v)$, then $\lambda(a', v) \subset [a', v]$.*

Proof. Let $T = [a, b] \cup [v, c]$, and let $H = \{x \in [v, b] | \lambda(a, x) \cap (x, b] = \emptyset\}$. Notice that $b \in H$. We shall show that H is both open and closed since this implies $H = [v, b]$. It will then follow that $\lambda(a, v) \cap [v, b] = \emptyset$, and a similar argument will show that $\lambda(a, v) \cap [v, c] = \emptyset$. The conclusion of the first part of Lemma 2.6 will then follow.

To see that H is open consider a sequence $\{h_i\}$ of points converging to a point $h \in H$. Since $\{\lambda(a, h_i)\}$ converges to $\lambda(a, h)$ and $\lambda(a, h) \cap [h, b] = \emptyset$, it follows that $\lambda(a, h_i) \cap [h_i, b] = \emptyset$ for sufficiently large i. Thus some neighborhood of h must lie entirely in H, and H is open. If $\{h_i\}$ is a sequence of points of H converging to a point h, then the continuity of λ implies $\lambda(a, h) \cap [h, b] = \emptyset$. Thus $h \in H$ and H is closed.

We prove the last sentence of Lemma 2.6 by considering the set $H = \{x \in [a, v) | \lambda(x, v) \subset [x, v]\}$. Let C be the component of H containing a, and let h be the least upper bound of C in $[a, v)$. Suppose $h \neq v$, and let $\{h_i\}$ converge to h where $h_i \in C$ for each i. Then $\lambda(h_i, v) \subset [h_i, v]$ for all i, and by the continuity of λ we must have $\lambda(h, v) \subset [h, v]$. Thus $h \in C$. Since $h \neq v$, there is a sequence $\{p_i\}$ of points of (h, v) converging to h such that no p_i belongs to H and each p_i separates h from $\lambda(h, v)$. Now it is impossible for $\{\lambda(p_i, v)\}$ to converge to $\lambda(h, v)$ as required by the continuity of λ. Thus $h = v$, and the proof is complete.

3. Arcs, simple closed curves, and triods. In this section the previous lemmas are used to obtain the main results of the paper.

THEOREM 3.1. *If X is a continuum with the $U\lambda P$, then X is an arc.*

Proof. If X has the $U\lambda P$, then X is arcwise connected by Lemma 2.2. By Lemma 2.1, X is separated by a singleton set, so X is not a simple closed curve. From Lemma 2.3, X either contains a triod or X is an arc. It is easy to see that a continuum with the $U\lambda P$ has a continuous λ-set function, so Lemma 2.5 applies to rule out there being a triod in X.

THEOREM 3.2. *Let X be a continuum with a continuous λ-set function. Then X is an arc if and only if X has the $U\lambda P$.*

Proof. By Theorem 3.1, X is an arc if it has the $U\lambda P$. Suppose X is an arc and that points a and b of X exist such that $\lambda(a, b)$ contains two points. Let A denote the subarc $[a, b]$ of X. By the continuity of the metric we see that that $\lambda(a, b)$ is closed. Let p_1 and p_2 be the first and last points, respectively, of $\lambda(a, b)$. From Lemma 2.4 we have $\{a\} < \{p_1\} < \lambda(p_1, p_2) < \{p_2\} < \{b\}$, and from Lemma 2.1 we obtain open sets L and R whose union is $X - \lambda(p_1, p_2)$ where $p_1 \in L$ and $p_2 \in R$. Since the connected set $[a, p_1]$ lies in $X - \lambda(p_1, p_2)$ and intersects L, it must lie in L; and similarly $[p_2, b] \subset R$. It follows that $d(p_1, a) < \lambda d(a, p_2)$ and $d(p_1, b) > \lambda d(p_2, b)$. Substituting $d(a, p_1) = \lambda d(b, p_1)$ and $d(a, p_2) = \lambda d(b, p_2)$ in these inequalities, we obtain $d(a, p_2) < d(p_1, b)$ and $\lambda d(b, p_1) < \lambda d(a, p_2)$. Consequently we have the contradiction that $d(a, p_2) < d(a, p_2)$.

COROLLARY 3.3. *A continuum with a continuous midset function is an arc if it has the unique midset property.*

THEOREM 3.4. *If a continuum X has the $C\lambda P(n)$ for $n > 1$, then X is a simple closed curve.*

Proof. Theorem 3.4 follows directly from Lemmas 2.2, 2.3, 2.5, and Theorem 3.2.

COROLLARY 3.5. *If a continuum X has the continuous double λ-set property ($CD\lambda P$), then X is a simple closed curve.*

THEOREM 3.6. *If A is an arc with a continuous λ-set function, then $\lambda = 1$ and A has the UMP.*

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Proof. From Theorem 3.2, A has the $U\lambda P$. Let $A = [a, b]$, let $z \in A - \{a, b\}$, and let $\alpha = d(z, b)$. If $\lambda \neq 1$, then a point x exists between z and b on A such that $d(z, x) = \lambda\alpha$. Consequently $z \in \lambda(x, b)$. However $\lambda(x, b)$ separates x from b (Lemma 2.1), so $\lambda(x, b)$ also contains a point of A between x and b. Since this contradicts the $U\lambda P$, it follows that $\lambda = 1$ and A has the UMP.

THEOREM 3.7. *No triod has a continuous λ-set function.*

Proof. Suppose T is a triod with a continuous λ-set function ($\lambda \in (0, 1]$), and let A, B, and C be the legs of T. Let v be the vertex of T and choose a positive number α small enough that each leg of T intersects the boundary S of the open α-ball N centered at the vertex v of T. Order the leg B from v to its other endpoint b', and let b be the first point of B in S. Then $(v, b) \subset N$ and $d(v, b) = \alpha$. Now let $N_\lambda = \{x \in T | d(x, v) < \lambda\alpha\}$, and let $S_\lambda = BdN_\lambda$. In a similar manner we obtain a point $a \in A \cap S_\lambda$ such that $(v, a) \subset N_\lambda$ and $d(v, a) = \lambda\alpha$, where $(v, a] \subset A = (v, a']$. Now choose a point $c \in C \cap S$ and let T' be the subtriod of T such that

$$T' = \{v\} \cup (v, a] \cup (v, b] \cup (v, c].$$

By Lemma 2.6 we see that $\lambda(a', v) \subset A$ and furthermore that $\lambda(a, v) \subset [a, v]$. Let $H = \{x \in [v, b] | \lambda(a, x) \cap C = \emptyset\}$, let G be the component of H containing v, and let h be the least upper bound for G in $[a, b]$. Let $\{h_i\}$ be a sequence of points of G converging to h. Since $\{\lambda(a, h_i)\}$ converges to $\lambda(a, h)$ and no $\lambda(a, h_i)$ intersects C, it is clear that $\lambda(a, h) \cap C = \emptyset$. Thus $h \in H$. We intend to show that $h = b$; if this is not the case then there is a sequence $\{p_i\}$ of points $[v, b) - H$ converging to h. The construction of the leg $(v, b]$ of T' insures that $\lambda(a, h)$ does not contain $v(d(a, v) = \lambda\alpha = \lambda d(b, v) > \lambda d(h, v)$ for all $h \in (v, b])$. Now since $\lambda(a, p_i) \cap C \neq \emptyset$ for each i, and since $\{\lambda(a, p_i)\}$ converges to $\lambda(a, h)$, we have the contradiction that v belongs to $\lambda(a, h)$. Thus $h = b$.

The preceding paragraph shows that, for every $b^* \in [v, b]$, $\lambda(a, b^*) \cap C = \emptyset$. In much the same way we prove that $\lambda(b, a^*) \cap C = \emptyset$ for every $a^* \in [a, v)$. Now we obtain a contradiction by showing that there is no place for the leg C.

Let $T - \lambda(a, b) = L \cup R$ where $a \in L$ and $b \in R$ as in Lemma 2.1. Since $C \cap \lambda(a, b) = \emptyset$ and C is connected, we first assume $C \subset L$. Let q be a point of C, and note that $d(a, q) < \lambda d(b, q)$. Let $\{b_i\}$ be a sequence of points of (v, b) converging to b, and for each i let $T - \lambda(a, b_i) = L_i \cup R_i$ where $a \in L_i$ and $b_i \in R_i$ (see Lemma 2.1). By the previous two paragraphs we know that $\lambda(a, b_i) \cap C = \emptyset$ for each i. The continuity of the metric d insures that $q \in L_i$ for all but finitely many i, since $q \in L$. Since $d(a, v) = \lambda\alpha > \lambda d(b_i, v)$ for each i, we have $v \in R_i$. Now the contradiction is apparent since, for sufficiently large i, $\lambda(a, b_i)$ separates q from v and fails to intersect the connected set $(v, q]$. If $C \subset R$ we choose a sequence $\{a_i\}$ from (a, v) converging to a and the contradiction follows similarly.

The proof of Theorem 3.7 is easily modified to show that no n-frame ($n > 2$) has a continuous λ-set function. The interested reader should observe that Lemma 2.6 is also true for n-frames ($n > 2$).

At first glance one might expect to generalize the proof of Theorem 3.7 to show that no continuum with a continuous midset function can contain a triod. However a round disk with its inherited plane metric is a counter-example. The continuum in Example 3.8 contains a triod, has only finite midsets, and has a continuous midset function. Thus even the *CFMP* fails to eliminate triodic subsets. This same example also shows that one cannot expect the *CFMP* to select arcs and simple closed curves from the class of continua; hence there are limits to generalizing Theorem 3.4.

Example 3.8. A continuum X with the *CFMP* that contains a triod.

The continuum X lies in E^2 and is the union of a circle C and a line segment S joining two points of C such that S does not contain the center of the circle C. The space X inherits its metric from the Euclidean plane E^2. Since no line intersects X in more than three points (unless it contains S), it is easy to see that no midset in X contains more than three points. The midset function M is continuous, so X has the *CFMP*.

The hypothesis that X be connected cannot be removed in Theorems 3.1, 3.2, and 3.4. An easy example showing this is a space consisting of $n + 2$ distinct points where the pairwise distances between distinct points are always equal. Such a space is compact and has the $CMP(n)$. Also Theorems 3.1 and 3.2 become false when "compact" is removed from their hypotheses; the real line illustrates this. We do not know if "compact" is implied by the other hypotheses in Theorem 3.4; this is related to questions asked in [**2**] and [**6**].

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Utah State University,
Logan, Utah