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# MULTIPLE HILBERT AND HARDY-HILBERT INEQUALITIES WITH NON-CONJUGATE PARAMETERS

ILKO BRNETIĆ, MARIO KRNIĆ AND JOSIP PEČARIĆ

The main objective of this paper is a study of some new generalisations of Hilbert and Hardy-Hilbert type inequalities involving non-conjugate parameters. We prove general forms of multiple Hilbert-type inequalities, and we also introduce multiple inequalities of Hardy-Hilbert type with non-conjugate parameters.

## 1. INTRODUCTION

Hilbert and Hardy-Hilbert type inequalities are very significant weighted inequalities which play an important role in analysis and its applications. First, let us recall the famous Hilbert theorem for double series. Let $\{a_m\}$ and $\{b_n\}$ be two non-negative sequences and $1/p + 1/q = 1$, $p > 1$. Then the following inequality holds

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \leqslant \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} {a_m}^p\right)^{1/p}\left(\sum_{n=1}^{\infty} {b_n}^q\right)^{1/q} \tag{1}$$

Although classical, inequality (1) is of interest to numerous mathematicians and has been generalised in many different ways. Some possibilities of generalising (1) are, for example, integral versions, various choices of kernels, extension to multi-dimensional cases et cetera. Inequalities related to (1) are usually called Hilbert inequalities. We shall also investigate equivalent inequalities containing only one sequence or, in the integral case, only one function, which are usually called Hardy–Hilbert inequalities. In $n$-dimensional case such inequalities contain $n-1$ functions or sequences. For more details see [**7**].

Another way of generalising the inequality (1) is its extension to the case of non-conjugate parameters $p$ and $q$. In 1951, Bonsall investigated this problem and established the following conditions for non-conjugate parameters (see [**3**]). He considered real numbers $p$ and $q$ such that $p > 1$, $q > 1$ and $1/p + 1/q \geqslant 1$. Furthermore, he defined $\lambda$ to be $\lambda := 1/p' + 1/q'$ where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. The essence of Bonsall's paper was the apparently trivial observation that $1/q' + 1/p' + 1 - \lambda = 1$ and the application of Hölder's inequality to conjugate parameters $q'$, $p'$ and $1/(1-\lambda)$ (for more details see

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[3]). That technique was also applied to the extensions and analogues of Hilbert's double series theorem due to Hardy, Littlewood, and Pólya (see [7]).

In spite of its trivial appearance, Bonsall's idea was useful for investigating inequalities for multiple integrals involving non-conjugate parameters. So, he obtained the following inequality for $n = 3$. Let $1/p_1 + 1/p_2 + 1/p_3 \geqslant 1$ with $p_i > 1$, $1/p_i + 1/p_i' = 1$, $i = 1, 2, 3$. Let $F = \int_0^\infty f^{p_1}(x)\,dx$, $G = \int_0^\infty g^{p_2}(y)dy$ and $H = \int_0^\infty h^{p_3}(z)dz$. Then

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x)g(y)h(z)}{(x+y+z)^{2\lambda}}\,dx\,dy\,dz \leqslant kF^{1/p_1}G^{1/p_2}H^{1/p_3}, \tag{2}$$

where

$$k \leqslant \left[\Gamma\Big(\frac{1}{\lambda p_1'}\Big)\Gamma\Big(\frac{1}{\lambda p_2'}\Big)\Gamma\Big(\frac{1}{\lambda p_3'}\Big)\right]^\lambda, \quad \lambda = \frac{1}{2}(p_1' + p_2' + p_3'),$$

($\Gamma$ is the gamma function). Although Bonsall established the conditions for $n$ non-conjugate parameters (see next section), there were no results in that direction.

The main purpose of this paper is to extension of inequality (2). We obtain some general inequalities involving non-conjugate parameters for multiple integrals. Techniques that will be used in the proofs are mainly based on classical real analysis, especially the well known Hölder's inequality and on Fubini's theorem.

## 2. Multiple Hilbert and Hardy-Hilbert inequalities with non-conjugate parameters

This section is dedicated to the most general form of multiple Hilbert and Hardy-Hilbert inequalities.

In order to obtain our general results we need some definitions. Let $p_i$, $i = 1, 2, \ldots, n$, be real parameters which satisfy

$$\sum_{i=1}^n \frac{1}{p_i} \geqslant 1 \quad \text{and} \quad p_i > 1, \quad i = 1, 2, \ldots, n. \tag{3}$$

The parameters $p_i'$, $i = 1, 2, \ldots, n$ are defined by the equations

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1, \quad i = 1, 2, \ldots, n. \tag{4}$$

Since $p_i > 1$, $i = 1, 2, \ldots, n$, it is obvious that $p_i' > 1$, $i = 1, 2, \ldots, n$. We define

$$\lambda := \frac{1}{n-1}\sum_{i=1}^n \frac{1}{p_i'}. \tag{5}$$

It is easy to deduce that $0 < \lambda \leqslant 1$. Also, we introduce parameters $q_i$, $i = 1, 2, \ldots, n$, defined by the relation

$$\frac{1}{q_i} = \lambda - \frac{1}{p_i'}, \quad i = 1, 2, \ldots, n. \tag{6}$$

One easily conclude that $q_i > 1$, $i = 1, 2, \ldots, n$. The above conditions were also given by Bonsall. It is easy to see that $\lambda = \sum_{i=1}^{n}(1/q_i)$ and $1/q_i + 1 - \lambda = 1/p_i$, $i = 1, 2, \ldots, n$. Of course, if $\lambda = 1$, then $\sum_{i=1}^{n} 1/p_i = 1$, so the conditions (3)–(6) reduce to the case of conjugate parameters.

In the proof of our main result we use Hölder's inequality and also Fubini's theorem, so all the measures should be $\sigma$-finite. Further, throughout this paper, we assume that all the functions are non-negative and not identically equal to zero. Also we suppose that all the integrals in the paper converge, so we shall omit such types of conditions. Under these assumptions we have our general result.

**THEOREM 1.** *Let $n \geqslant 2$ be an integer and $p_i$, $p_i'$, $q_i$, $i = 1, 2, \ldots, n$, be the real numbers satisfying* (3), (4), (5) *and* (6). *If the functions $\phi_{ij}$, $i, j = 1, 2, \ldots, n$, satisfy condition $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$, then the following inequalities hold and are equivalent*

$$(7)\quad \int_\Omega \cdots \int_\Omega K^\lambda(x_1, \ldots, x_n) \prod_{i=1}^{n} f_i(x_i) d\mu_1(x_1) \ldots d\mu_n(x_n) \leqslant \prod_{i=1}^{n} \left[\int_\Omega (\phi_{ii} F_i f_i)^{p_i}(x_i) d\mu_i(x_i)\right]^{1/p_i}$$

*and*

$$(8)\quad \left\{\int_\Omega \left[\frac{1}{(\phi_{nn}F_n)(x_n)} \int_\Omega \cdots \int_\Omega K^\lambda(x_1, \ldots, x_n) \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \ldots d\mu_{n-1}(x_{n-1})\right]^{p_n'} \cdot d\mu_n(x_n)\right\}^{1/p_n'} \leqslant \prod_{i=1}^{n-1} \left[\int_\Omega (\phi_{ii} F_i f_i)^{p_i}(x_i) d\mu_i(x_i)\right]^{1/p_i}.$$

*where*

$$F_i(x_i) = \left[\int_\Omega \cdots \int_\Omega K(x_1, \ldots, x_n) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j) d\mu_j(x_j)\right]^{1/q_i},$$

*and $\Omega$ is interval in $(0, \infty)$.*

PROOF: The left-hand side of the inequality (7) can be transformed in the following way

$$\begin{aligned}
&\int_\Omega \cdots \int_\Omega K^\lambda(x_1, \ldots, x_n) \prod_{i=1}^{n} f_i(x_i) d\mu_1(x_1) \ldots d\mu_n(x_n) \\
&\quad = \int_\Omega \cdots \int_\Omega \prod_{i=1}^{n} \Big[K(x_1, \ldots, x_n) \phi_{ii}{}^{p_i}(x_i) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j) F_i^{p_i - q_i}(x_i) f_i{}^{p_i}(x_i)\Big]^{1/q_i} \\
&\qquad \cdot \Big[\prod_{i=1}^{n} (\phi_{ii} F_i f_i)^{p_i}(x_i)\Big]^{1-\lambda} d\mu_1(x_1) \ldots d\mu_n(x_n).
\end{aligned}$$

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Now, since $\sum_{i=1}^{n}(1/q_i) + 1 - \lambda = 1$, $q_i > 1$ and $0 < \lambda \leqslant 1$, applying Hölder's inequality we have

$$\int_\Omega \cdots \int_\Omega K^\lambda(x_1, \dots, x_n) \prod_{i=1}^{n} f_i(x_i) d\mu_1(x_1) \dots d\mu_n(x_n)$$
$$\leqslant \prod_{i=1}^{n} \left[\int_\Omega (\phi_{ii} F_i f_i)^{p_i}(x_i) d\mu_i(x_i)\right]^{1/q_i} \prod_{i=1}^{n} \left[\int_\Omega (\phi_{ii} F_i f_i)^{p_i}(x_i) d\mu_i(x_i)\right]^{1-\lambda}.$$

Since $1/q_i + 1 - \lambda = 1/p_i$, we obtain the inequality (7).

Let us show that the inequalities (7) and (8) are equivalent. Suppose that the inequality (7) is valid. If we put the function $f_n : \Omega \to \mathbf{R}$, defined by

$$f_n(x_n) = \left[\frac{1}{(\phi_{nn} F_n)^{p_n}(x_n)} \int_\Omega \cdots \int_\Omega K^\lambda(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \dots d\mu_{n-1}(x_{n-1})\right]^{p_n'/p_n}$$

into the inequality (7), we obtain

$$I(x_n)^{p_n'} \leqslant \prod_{i=1}^{n-1} \left(\int_\Omega (\phi_{ii} F_i f_i)^{p_i}(x_i) d\mu_i(x_i)\right)^{1/p_i} I(x_n)^{p_n'/p_n}$$

where $I(x_n)$ denotes the left-hand side of the inequality (8). This gives the inequality (8).

It remains to prove that inequality (7) is a consequence of inequality (8). Suppose that the inequality (8) is valid. It is obvious that

$$\int_\Omega \cdots \int_\Omega K^\lambda(x_1, \dots, x_n) \prod_{i=1}^{n} f_i(x_i) d\mu_1(x_1) \dots d\mu_n(x_n)$$
$$= \int_\Omega (\phi_{nn} F_n)(x_n) \left[\frac{1}{(\phi_{nn} F_n)(x_n)} \int_\Omega \cdots \int_\Omega K^\lambda(x_1, \dots, x_n) \cdot \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \dots d\mu_{n-1}(x_{n-1})\right] d\mu_n(x_n).$$

Applying Hölder's inequality for the conjugate parameters $p_n$ and $p_n'$ we have

$$\int_\Omega \cdots \int_\Omega K^\lambda(x_1, \dots, x_n) \prod_{i=1}^{n} f_i(x_i) d\mu_1(x_1) \dots d\mu_n(x_n)$$
$$\leqslant \left[\int_\Omega (\phi_{nn} F_n f_n)^{p_n}(x_n) d\mu_n(x_n)\right]^{1/p_n} I(x_n),$$

and the result follows from (8). That completes the proof. □

REMARK 1. Equality in the previous theorem is possible if and only if it holds in Hölder's inequality. This means that the functions

$$K(x_1,\ldots,x_n)\phi_{ii}{}^{p_i}(x_i)\prod_{j=1,j\neq i}^{n}\phi_{ij}^{q_i}(x_j)F_i^{p_i-q_i}(x_i)f_i^{p_i}(x_i),\quad i=1,2,\ldots,n$$

and $\prod_{i=1}^{n}(\phi_{ii}F_if_i)^{p_i}(x_i)$ are proportional. Hence, we see that equality in Theorem 1 can be achieved only if the functions $f_i$ and the kernel $K$ are defined by

$$f_i(x_i)=C_i\phi_{ii}(x_i)^{q_i/1-\lambda q_i}F_i(x_i)^{(1-\lambda)q_i}$$

and

$$K(x_1,x_2,\ldots,x_n)=C\prod_{i=1}^{n}F_i^{q_i}(x_i),\ i=1,2,\ldots,n,$$

where $C$ and $C_i$ are arbitrary constants. This is possible only if the functions

$$\frac{\prod_{j=1,j\neq i}^{n}\phi_{jj}{}^{\lambda q_j/1-\lambda q_j}(x_j)}{\prod_{j=1,j\neq i}^{n}\phi_{ij}{}^{\lambda q_j}(x_j)},\quad i=1,2,\ldots,n$$

are constants, and

$$\int_\Omega F_i^{q_i}(x_i)\phi_{ii}^{q_i/1-\lambda q_i}(x_i)d\mu_i(x_i)<\infty,\quad i=1,2,\ldots,n.$$

Otherwise, the inequalities in Theorem 1 are strict.

REMARK 2. If the parameters $p_i$, $i=1,2,\ldots,n$, are chosen in such a way that

$$q_j>0, j\in\{1,2,\ldots,n\},\quad q_i<0, i\neq j\quad\text{and}\quad\lambda<1 \tag{9}$$

or

$$q_i<0,\quad i=1,2,\ldots,n \tag{10}$$

then the reverse inequalities in (7) and (8) are valid.

Let us mention that the special case of Theorem 1, for $n=2$, was proved in [6]. Also, Theorem 1 is the extension of our papers [**2**] and [**8**] to the case of non-conjugate parameters.

## 3. APPLICATIONS TO HOMOGENEOUS FUNCTIONS

Our aim in this section is to apply the general results to the homogeneous function

$$K(x_1,\ldots,x_n)=\frac{1}{(\sum_{i=1}^{n}x_i)^s},$$

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of degree $-s$, in $n$ variables, and for some special choice of the functions $\phi_{ij}$, $1 \leqslant i,j \leqslant n$. We define $\phi_{ij}(x_j) := x_j^{A_{ij}}$. Further, we specialise Theorem 1 to the case of Lebesgue measures and also put $\Omega = (0,\infty)$. Then, the condition

$$\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1 \tag{11}$$

from Theorem 1, leads to

$$\prod_{i=1}^{n}\prod_{j=1}^{n} x_j^{A_{ij}} = \prod_{j=1}^{n} x_j^{\sum_{i=1}^{n} A_{ij}} = 1.$$

It is natural to set $\sum_{i=1}^{n} A_{ij} = 0$, $j = 1,2,\dots,n$, so that the condition (11) is satisfied. We also define $\alpha_i := \sum_{j=1}^{n} A_{ij}$, $i = 1,2,\dots,n$.

For our homogeneous function $K(x_1,\dots,x_n) = \left(\sum_{i=1}^{n} x_i\right)^{-s}$, we have

$$F_i(x_i) = \left[\int_0^\infty \cdots \int_0^\infty \frac{\prod_{j=1,j\neq i}^{n} x_j^{q_iA_{ij}}}{(\sum_{k=1}^{n} x_k)^s}\, dx_1 \dots dx_{i-1}\, dx_{i+1} \dots dx_n\right]^1 \Big/ q_i.$$

Now, by using the substitution $u_k = x_k/x_i$, $k = 1,\dots,n$, $k \neq i$, we can express $F_i(x_i)$ in terms of the gamma function. More precisely, we have

$$\begin{aligned} F_i(x_i) &= x_i^{(n-1-s)/q_i+\alpha_i-A_{ii}}\left[\int_0^\infty \cdots \int_0^\infty \frac{\prod_{j=1,j\neq i}^{n} u_j^{q_iA_{ij}}}{(1+\sum_{k=1,k\neq i}^{n} u_k)^s} du_1 \dots du_{i-1} du_{i+1} \dots du_n\right]^{1/q_i} \\ &= \frac{1}{\Gamma(s)^{1/q_i}} x_i^{(n-1-s)/q_i+\alpha_i-A_{ii}} \Gamma(s-n+1-q_i\alpha_i+q_iA_{ii})^{1/q_i} \prod_{j=1,j\neq i}^{n} \Gamma(q_iA_{ij}+1)^{1/q_i}, \end{aligned}$$

where we used the well known formula for gamma function (see, for instance, [**1**, Lemma 5.1]):

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{k-1} u_i^{a_i-1} du_1 \dots du_{k-1}}{(1+\sum_{i=1}^{k-1} u_i)^{\sum_{i=1}^{k} a_i}} = \frac{\prod_{i=1}^{k}\Gamma(a_i)}{\Gamma(\sum_{i=1}^{k} a_i)}.$$

Finally, for the homogeneous function $K(x_1,\dots,x_n) = \left(\sum_{i=1}^{n} x_i\right)^{-s}$, Theorem 1 reduces to

**Theorem 2.** *Let $n \geqslant 2$ be an integer, let $p_i$, $p_i'$, $q_i$, $i = 1,2,\dots,n$, be real numbers satisfying* (3), (4), (5) *and* (6) *and suppose $\sum_{i=1}^{n} A_{ij} = 0$. Then the following inequalities hold and are equivalent*

$$\begin{aligned} &\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n} f_i(x_i)}{(\sum_{j=1}^{n} x_j)^{\lambda s}}\, dx_1 \dots dx_n \\ &\qquad < K \prod_{i=1}^{n}\left[\int_0^\infty x_i^{(p_i/q_i)(n-1-s)+p_i\alpha_i} f_i^{p_i}(x_i)\, dx_i\right]^{1/p_i} \end{aligned} \tag{12}$$

and

$$(13)\quad \left[\int_0^\infty x_n^{(1-\lambda p_n')(n-1-s)-p_n'\alpha_n}\left(\int_0^\infty\cdots\int_0^\infty \frac{\prod_{i=1}^{n-1} f_i(x_i)}{(\sum_{j=1}^n x_j)^{\lambda s}}\,dx_1\ldots dx_{n-1}\right)^{p_n'} dx_n\right]^{1/p_n'} < K\prod_{i=1}^{n-1}\left[\int_0^\infty x_i^{(p_i/q_i)(n-1-s)+p_i\alpha_i} f_i^{p_i}(x_i)\,dx_i\right]^{1/p_i}.$$

*where*

$$K=\frac{1}{\Gamma(s)^\lambda}\prod_{i=1}^n \Gamma(s-n+1-q_i\alpha_i+q_iA_{ii})^{1/q_i}\prod_{i,j=1,i\neq j}^n \Gamma(q_iA_{ij}+1)^{1/q_i},$$

$A_{ij}>-1/q_i$, $i\neq j$ *and* $A_{ii}-\alpha_i>(n-s-1)/q_i$.

REMARK 3. Note that the inequalities in Theorem 2 are strict (under the assumption that the functions $f_i$ are not identically equal to zero). That follows from Remark 1.

It is interesting to consider some special cases. We can obtain some previously known results, as special cases of our inequalities.

If we put $A_{ii}=(n-s)(\lambda q_i-1)/q_i^2$ and $A_{ij}=(s-n)1/(q_iq_j)$, $i\neq j$, then the condition $\sum_{i=1}^n A_{ij}=0$ is satisfied (also $\sum_{j=1}^n A_{ij}=0$), so we obtain the following result

COROLLARY 1. *Let $n\geqslant 2$ be an integer and let $p_i$, $p_i'$, $q_i$, $i=1,2,\ldots,n$, be the real numbers satisfying* (3), (4), (5) *and* (6). *Then the following inequalities hold and are equivalent*

$$(14)\quad \int_0^\infty\cdots\int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^{\lambda s}}\,dx_1\ldots dx_n < L\prod_{i=1}^n\left[\int_0^\infty x_i^{(p_i/q_i)(n-1-s)} f_i^{p_i}(x_i)\,dx_i\right]^{1/p_i}$$

*and*

$$(15)\quad \left[\int_0^\infty x_n^{(1-\lambda p_n')(n-1-s)}\left(\int_0^\infty\cdots\int_0^\infty \frac{\prod_{i=1}^{n-1} f_i(x_i)}{(\sum_{j=1}^n x_j)^{\lambda s}}\,dx_1\ldots dx_{n-1}\right)^{p_n'} dx_n\right]^1/p_n' < L\prod_{i=1}^{n-1}\left[\int_0^\infty x_i^{(p_i/q_i)(n-1-s)} f_i^{p_i}(x_i)\,dx_i\right]^{1/p_i},$$

*where*

$$L=\frac{1}{\Gamma(s)^\lambda}\prod_{i=1}^n\Gamma\left(\frac{p_i+s-n}{p_i}\right)^{1/q_i}\prod_{i=1}^n\Gamma\left(\frac{q_i+s-n}{q_i}\right)^{\lambda-*(1/q_i)},$$

*and* $s>\max\{n-\min_{1\leqslant i\leqslant n}\{p_i\},n-\min_{1\leqslant i\leqslant n}\{q_i\}\}$.

REMARK 4. In the conjugate case, when $\lambda=1$, the constant $L$ from previous corollary reduces to $L=(1/\Gamma(s))\prod_{i=1}^n\Gamma(p_i+s-n/p_i)$ and one obtains the result from [**1**].

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Similarly as in the previous corollary, if we put $A_{ii} = (\lambda q_i - 1)/(\lambda q_i^2)$ and $A_{ij} = -1/,(\lambda q_i q_j)$, $i \neq j$, $i,j \in \{1,2,\dots,n\}$ we obtain direct generalisation of Bonsall's inequality (2) from the Introduction. Furthermore, the constant $K$ from Theorem 2 becomes

$$L = \frac{1}{\Gamma(s)^\lambda} \prod_{i=1}^{n} \Gamma\Big(s - n + 1 + \frac{1}{\lambda p_i'}\Big)^{1/q_i} \prod_{i=1}^{n} \Gamma\Big(\frac{1}{\lambda p_i'}\Big)^{\lambda-(1/q_i)}.$$

So, if

$$K(x_1,\dots,x_n) = \frac{1}{(\sum_{i=1}^{n} x_i)^{n-1}},$$

we obtain the following corollary.

**Corollary 2.** *Let $n \geqslant 2$ be an integer and let $p_i, p_i', q_i$, $i = 1,2,\dots,n$, be the real numbers satisfying* (3), (4), (5) and (6). Then the following inequalities hold and are equivalent

$$\text{(16)} \qquad \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n} f_i(x_i)}{(\sum_{j=1}^{n} x_j)^{(n-1)\lambda}}\, dx_1 \dots dx_n < M \prod_{i=1}^{n} \Big[\int_0^\infty f_i^{p_i}(x_i)\, dx_i\Big]^{1/p_i}$$

and

$$\text{(17)} \quad \Big[\int_0^\infty \Big(\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n-1} f_i(x_i)}{(\sum_{j=1}^{n} x_j)^{(n-1)\lambda}}\, dx_1 \dots dx_{n-1}\Big)^{p_n'} dx_n\Big]^1 /p_n' < M \prod_{i=1}^{n-1} \Big[\int_0^\infty f_i^{p_i}(x_i)\, dx_i\Big]^{1/p_i}.$$

where

$$M = \frac{1}{[(n-2)!]^\lambda} \Big[\prod_{i=1}^{n} \Gamma\Big(\frac{1}{\lambda p_i'}\Big)\Big]^\lambda.$$

Remark 5. Clearly, for $n = 3$, inequality (16) becomes Bonsall's result (2).

On the other hand, if we put $A_{ii} = A_i$, $A_{ii+1} = -A_{i+1}$, $A_{ij} = 0$, where $|i-j| > 1$ and the indices are taken modulo $n$, then we obtain (obviously, the condition $\sum_{i=1}^{n} A_{ij} = 0$ is satisfied) our final corollary.

**Corollary 3.** *Let $n \geqslant 2$ be an integer and let $p_i$, $p_i'$, $q_i$, $i = 1,2,\dots,n$, be the real numbers satisfying* (3), (4), (5) *and* (6). *Then the following inequalities hold and are equivalent*

$$\text{(18)} \quad \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n} f_i(x_i)}{(\sum_{j=1}^{n} x_j)^{\lambda s}}\, dx_1 \dots dx_n < N \prod_{i=1}^{n} \Big[\int_0^\infty x_i^{(p_i/q_i)(n-1-s)+p_i(A_i-A_{i+1})} f_i^{p_i}(x_i)\, dx_i\Big]^{1/p_i}$$

*and*

$$(19)\quad \Big[\int_0^\infty x_n^{(1-\lambda p'_n)(n-1-s)-p'_n(A_n-A_1)}\Big(\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n-1} f_i(x_i)}{(\sum_{j=1}^n x_j)^{\lambda s}}\,dx_1\ldots dx_{n-1}\Big)^{p'_n}dx_n\Big]^{1/p'_n} < N\prod_{i=1}^{n-1}\Big[\int_0^\infty {x_i}^{(p_i/q_i)(n-1-s)+p_i(A_i-A_{i+1})} f_i^{p_i}(x_i)\,dx_i\Big]^{1/p_i}.$$

*where*

$$N=\frac{1}{\Gamma(s)^\lambda}\prod_{i=1}^{n}\Big(\Gamma(1-q_iA_{i+1})\Gamma(s-n+1+q_iA_{i+1})\Big)^{1/q_i},$$

*for any* $s>n-2$ *and* $A_i\in\big((n-s-1/p_{i-1}),1/q_{i-1}\big)$, *where the indices* $i$ *in* $A_i$ *are taken modulo* $n$.

REMARK 6. Note that inequalities (18) and (19) are generalisations of those from [**4, 5**].

Let us also mention that the reverse inequalities in the results from this section are valid if the conditions (9) or (10) are satisfied.

## 4. Multiple Hardy-type inequality

Now we present some special cases of multiple Hardy inequalities with non-conjugate parameters. If we put

$$K(x_1,x_2,\ldots,x_n)=\begin{cases} g(x_n), & x_1,x_2,\ldots,x_{n-1}\leqslant x_n\\ 0, & \text{otherwise}\end{cases}$$

in Theorem 1, where $\Omega=[a,b]$, $a<b$, we obtain following result.

**THEOREM 3.** *Let* $n\geqslant 2$ *be an integer and* $p_i$, $p'_i$, $q_i$, $i=1,2,\ldots,n$, *be the real numbers satisfying* (3), (4), (5) *and* (6). *If the functions* $\phi_{ij}$, $i,j=1,2,\ldots,n$, *satisfy condition* $\prod_{i,j=1}^{n}\phi_{ij}(x_j)=1$, *then the following inequalities hold and are equivalent*

$$\int_a^b (g^\lambda f_n)(x_n)\Big[\prod_{i=1}^{n-1}\int_a^{x_n} f_i(x_i)d\mu_i(x_i)\Big]d\mu_n(x_n)$$
$$<\prod_{i=1}^{n-1}\Big[\int_a^b\Big(\int_{x_i}^b (g{\phi_{in}}^{q_i})(x_n)\Big(\prod_{j=1,j\neq i}^{n-1}\int_a^{x_n}{\phi_{ij}}^{q_i}(x_j)d\mu_j(x_j)\Big)d\mu_n(x_n)\Big)^{p_i/q_i}(\phi_{ii}f_i)^{p_i}(x_i)$$
$$d\mu_i(x_i)\Big]^{1/p_i}\cdot\Big[\int_a^b\prod_{j=1}^{n-1}\Big(\int_a^{x_n}{\phi_{nj}}^{q_n}(x_j)d\mu_j(x_j)\Big)^{p_n/q_n}(g^{1/q_n}\phi_{nn}f_n)^{p_n}(x_n)d\mu_n(x_n)\Big]^{1/p_n}$$

and

$$\left[\int_a^b (g\phi_{nn}{}^{-p'_n})(x_n)\prod_{i=1}^{n-1}\left(\int_a^{x_n}\phi_{ni}{}^{q_n}(x_i)d\mu_i(x_i)\right)^{1-\lambda p'_n}\left(\int_a^{x_n} f_i(x_i)d\mu_i(x_i)\right)^{p'_n} d\mu_n(x_n)\right]^{1/p'_n}$$
$$< \prod_{i=1}^{n-1}\left[\int_a^b\left(\int_{x_i}^b (g\phi_{in}{}^{q_i})(x_n)\Big(\prod_{j=1,j\neq i}^{n-1}\int_a^{x_n}\phi_{ij}{}^{q_i}(x_j)d\mu_j(x_j)\Big)d\mu_n(x_n)\right)^{p_i/q_i} (\phi_{ii}f_i)^{p_i}(x_i)d\mu_i(x_i)\right]^{1/p_i}$$

We also obtain the dual to Theorem 3 by putting

$$K(x_1,x_2,\dots,x_n) = \begin{cases} g(x_n), & x_1,x_2,\dots,x_{n-1} \geqslant x_n \\ 0, & \text{otherwise} \end{cases}$$

in Theorem 1.

**THEOREM 4.** *Let $n \geqslant 2$ be an integer and $p_i$, $p'_i$, $q_i$, $i = 1,2,\dots,n$, be the real numbers satisfying* (3), (4), (5) *and* (6). *If the functions $\phi_{ij}$, $i,j = 1,2,\dots,n$, satisfy condition $\prod_{i,j=1}^{n}\phi_{ij}(x_j) = 1$, then the following inequalities hold and are equivalent*

$$\int_a^b (g^\lambda f_n)(x_n)\left[\prod_{i=1}^{n-1}\int_{x_n}^b f_i(x_i)d\mu_i(x_i)\right]d\mu_n(x_n)$$
$$< \prod_{i=1}^{n-1}\left[\int_a^b\left(\int_a^{x_i}(g\phi_{in}{}^{q_i})(x_n)\Big(\prod_{j=1,j\neq i}^{n-1}\int_{x_n}^b\phi_{ij}{}^{q_i}(x_j)d\mu_j(x_j)\Big)d\mu_n(x_n)\right)^{p_i/q_i}(\phi_{ii}f_i)^{p_i}(x_i) d\mu_i(x_i)\right]^{1/p_i}\cdot\left[\int_a^b\prod_{j=1}^{n-1}\left(\int_{x_n}^b\phi_{nj}{}^{q_n}(x_j)d\mu_j(x_j)\right)^{p_n/q_n}(g^{1/q_n}\phi_{nn}f_n)^{p_n}(x_n)d\mu_n(x_n)\right]^{1/p_n}$$

*and*

$$\left[\int_a^b (g\phi_{nn}{}^{-p'_n})(x_n)\prod_{i=1}^{n-1}\left(\int_{x_n}^b\phi_{ni}{}^{q_n}(x_i)d\mu_i(x_i)\right)^{1-\lambda p'_n}\left(\int_{x_n}^b f_i(x_i)d\mu_i(x_i)\right)^{p'_n}d\mu_n(x_n)\right]^{1/p'_n}$$
$$< \prod_{i=1}^{n-1}\left[\int_a^b\left(\int_a^{x_i}(g\phi_{in}{}^{q_i})(x_n)\Big(\prod_{j=1,j\neq i}^{n-1}\int_{x_n}^b\phi_{ij}{}^{q_i}(x_j)d\mu_j(x_j)\Big)d\mu_n(x_n)\right)^{p_i/q_i}(\phi_{ii}f_i)^{p_i}(x_i)d\mu_i(x_i)\right]^{1/p_i}.$$

REMARK 7. It is easy to deduce that, for the kernels of Theorems 3 and 4, the inequalities are strict (see Remark 1).

Note that Theorems 3 and 4 are extension of our paper [2], to non-conjugate case.

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Faculty of Electrical Engineering
and Computing
University of Zagreb
Unska 3, Zagreb
Croatia
e-mail: ilko.brnetic@fer.hr

Department of Mathematics
University of Zagreb
Bijenička cesta 30
10000 Zagreb
Croatia
e-mail: krnic@math.hr

Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10000 Zagreb
Croatia
e-mail: pecaric@hazu.hr