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A Dynamical Proof of Pisot's Theorem

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Abstract. We give a geometric proof of classical results that characterize Pisot numbers as algebraic $\lambda > 1$ for which there is $x \neq 0$ with $\lambda^n x \to 0 \pmod 1$ and identify such x as members of $\mathbb{Z}[\lambda^{-1}] \cdot \mathbb{Z}[\lambda]^*$ where $\mathbb{Z}[\lambda]^*$ is the dual module of $\mathbb{Z}[\lambda]$.

A real number $\lambda > 1$ is called *a Pisot number* if and only if it is an algebraic integer and all its Galois conjugates (other than λ) are of modulus less than one: the golden mean $(1+\sqrt{5})/2$ is an example. Pisot's 1938 thesis [4] and, independently, Vijayaraghavan's 1941 paper [7] contain the following beautiful characterization.

Theorem 1 (Pisot,Vijayaraghavan) *Suppose that $\lambda > 1$ is an algebraic number (over the field of rational numbers $\mathbb{Q}$). The following are equivalent*

(i) *λ is a Pisot number;*

(ii) *There exists non-zero real x such that* $\lim_{n\to\infty} \lambda^n x = 0 \pmod 1$, i.e.,

$$\lim_{n\to\infty} \min\{|\lambda^n x - k| : k \in \mathbb{Z}\} = 0$$

where $\mathbb{Z}$ is the rational integers.

Moreover, any x satisfying (ii) *belongs to $\mathbb{Q}(\lambda)$, the field extension of $\mathbb{Q}$ by λ.*

The property (ii) is responsible for Pisot numbers turning up in a variety of contexts seemingly unrelated to their definition. The reader may want to savor the ensuing connections by reading [5, 2]. Our interest in Pisot's theorem stems from its role in the determination of spectrum for the translation flow on substitution tiling spaces, as exhibited by [6] and further exploited in [1]. We shall not discuss that connection here, and turn instead to our goal of supplying a proof of the theorem that offers a direct geometrical insight — something that is missing from the considerations of the classical proofs, as found in [3] or [5]. We shall also derive the following characterization of the set

$$X_\lambda := \{x \in \mathbb{R} : \lim_{n\to\infty} \lambda^n x = 0 \pmod 1\}. \tag{1}$$

In [3], this result is also attributed to Pisot and Vijayaraghavan.

Theorem 2 (Pisot,Vijayaraghavan) *Suppose $\lambda > 1$ is Pisot. Let p' be the derivative of the monic irreducible polynomial of λ over $\mathbb{Z}$, and $\mathbb{Z}[\lambda]^* := \frac{1}{p'(\lambda)}\mathbb{Z}[\lambda]$. Then $x \in X_\lambda$ if and only if $\lambda^n x \in \mathbb{Z}[\lambda]^*$ for some $n \geq 0$*, i.e.,

$$X_\lambda = \bigcup_{n\geq 0} \lambda^{-n}\mathbb{Z}[\lambda]^* = \mathbb{Z}[\lambda^{-1}] \cdot \mathbb{Z}[\lambda]^*. \tag{2}$$

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We note that $\mathbb{Z}[\lambda]^*$ is just an explicit form (as given by Euler) of *the dual* of the module $\mathbb{Z}[\lambda]$ typically defined as $\mathbb{Z}[\lambda]^* := \{x \in \mathbb{Q}(\lambda) : \operatorname{trace}(xy) \in \mathbb{Z}\ \forall y \in \mathbb{Z}[\lambda]\}$ and that $\mathbb{Z}[\lambda]^*$ is non-zero only if λ is an algebraic integer (see [8, Prop. 3-7-12]). That $x \in X_\lambda$ for $x \in \mathbb{Z}[\lambda]^*$ is clear by the following standard argument (emulating [5, Theorem 1]). Let $\lambda = \lambda_1, \lambda_2, \dots, \lambda_d$ be all the roots of p (the Galois conjugates of λ) and $x = x_1, \dots, x_d$ be the images of x under the natural isomorphisms $\mathbb{Q}(\lambda) \to \mathbb{Q}(\lambda_i)$, $x_i \in \mathbb{Q}(\lambda_i)$. Then

$$\mathbb{Z} \ni T_n := \operatorname{trace}(\lambda^n x) = \sum_{i=1}^{d} \lambda_i^n x_i = \lambda^n x + \sum_{i=2}^{d} \lambda_i^n x_i, \tag{3}$$

and so $|\lambda^n x - T_n| \to 0$ due to the Pisot hypothesis: $|\lambda_i| < 1$ for $i = 2, \dots, d$.

From now on, consider a fixed algebraic number $\lambda > 1$. Denote by p its monic minimal polynomial, which is obviously irreducible. Let $d := \deg(p)$, and fix a $d \times d$ matrix A over $\mathbb{Q}$ with eigenvalue λ. The companion matrix of p is one such A, and any other is similar to it over $\mathbb{Q}$. If λ is an algebraic integer then A can be taken over $\mathbb{Z}$. Conversely, if A preserves some lattice in $L \subset \mathbb{R}^d$, $AL \subset L$, then λ is an algebraic integer. Here by *a lattice* we understand a discrete rank d subgroup of $\mathbb{R}^d$, $\mathbb{Z}^d$ being the simplest example.

We shall frequently use the fact that A is *irreducible over* $\mathbb{Q}$: if W is a non-zero subspace of $\mathbb{Q}^d$ and $A(W) \subset W$, then $W = \mathbb{Q}^d$ (as otherwise the characteristic polynomial of $A|_W$ would divide p). Also, by irreducibility of p, A has simple eigenvalues and is diagonalizable over $\mathbb{C}$ so that we have a splitting

$$\mathbb{R}^d = E^s \oplus E^c \oplus E^u$$

where E^s, E^c, E^u are the linear spans of the real eigenspaces corresponding to the eigenvalues of modulus less, equal, and greater than 1, respectively. We shall see that, for $v \in \mathbb{R}^d \setminus \{0\}$, $A^n v \to 0$ if and only if $v \in E^s$ lies at the very heart of Pisot's theorem. Below, $\langle \cdot | \cdot \rangle$ is the standard scalar product in $\mathbb{R}^d$.

Lemma 1 *If $\langle A^n v_0 | k_0 \rangle \to 0 \pmod 1$ for some $v_0 \in \mathbb{R}^d \setminus E^s$ and $k_0 \in \mathbb{Z}^d \setminus \{0\}$, then A leaves invariant some lattice in $\mathbb{Q}^d$,* i.e., *λ is an algebraic integer.*

Lemma 2 *Suppose that A has entries in $\mathbb{Z}$ and $k_0 \in \mathbb{Z}^d \setminus \{0\}$. If $\langle A^n v_0 | k_0 \rangle \to 0 \pmod 1$ for $v_0 \in \mathbb{R}^d$, then $v_0 \in \mathbb{Q}^d + E^s$.*

Proof of Theorem 1 Taking $x = 1$ in (3) shows that (i) implies (ii), so it is left to show (i) from (ii). Pick $\omega \in \mathbb{R}^d$ to be an eigenvector of A corresponding to λ, $A\omega = \lambda\omega$. Fix $k_0 \in \mathbb{Z}^d \setminus \{0\}$. Observe that $\langle k_0 | \omega \rangle \neq 0$ by irreducibility of the transpose A^T of A (since $\{q \in \mathbb{Q}^d : \langle q | \omega \rangle = 0\}$ is A^T invariant). Thus, in the linear span $\operatorname{lin}_{\mathbb{R}}(\omega)$ of ω over $\mathbb{R}$, we can find v_0 so that $x = \langle v_0 | k_0 \rangle$. In this way,

$$\lambda^n x = \lambda^n \langle v_0 | k_0 \rangle = \langle A^n v_0 | k_0 \rangle, \quad v_0 \in \operatorname{lin}_{\mathbb{R}}(\omega). \tag{4}$$

From $x \neq 0$, $v_0 \notin E^s$ and so λ must be an algebraic integer by Lemma 1. By Lemma 2, $v_0 = q_0 + z$ for some $z \in E^s$ and $q_0 \in \mathbb{Q}^d$; and $q_0 \neq 0$ from $v_0 \notin E^s$. Consider,

$W := \mathbb{Q}^d \cap (E^s \oplus \operatorname{lin}_{\mathbb{R}}(\omega))$. Irreducibility of A, $AW \subset W$ and $q_0 \in W$ force $W = \mathbb{Q}^d$. Thus $E^s \oplus \operatorname{lin}_{\mathbb{R}}(\omega) = \mathbb{R}^d$ and λ is Pisot. ■

We turn our attention to proving the lemmas now. The two proofs will partially overlap and could be combined into a single more compact argument, but we shall keep them separate because (in applications) λ is often *a priori* known to be an algebraic integer. In that case, Pisot's theorem can be viewed as a feature of the dynamics of the endomorphism $f\colon \mathbb{T}^d \to \mathbb{T}^d$, $x \,(\mathrm{mod}\, \mathbb{Z}^d) \mapsto Ax \,(\mathrm{mod}\, \mathbb{Z}^d)$, induced by A on the d-dimensional torus, $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Besides the toral endomorphism f, our main tool will be the concept of duality of lattices. Recall that *the dual of a lattice L* is defined as $L^* := \{v \in \mathbb{R}^d : \langle v \,|\, l\rangle \in \mathbb{Z}\, \forall l \in L\}$. One easily checks that $(\mathbb{Z}^d)^* = \mathbb{Z}^d$. For any lattice L, after expressing it as $L = B\mathbb{Z}^d$ for some nonsingular matrix B, we have $L^* = (B\mathbb{Z}^d)^* = (B^T)^{-1}\mathbb{Z}^d$ where B^T is the transpose of B. In particular, L^* is also a lattice.

Proof of Lemma 1 Let $V := \{v \in \mathbb{R}^d : \langle A^n v | k_0\rangle \to 0 \,(\mathrm{mod}\, 1)\}$ and $K := \{k \in \mathbb{Q}^d : \langle A^n v | k\rangle \to 0 \,(\mathrm{mod}\, 1)\ \forall v \in V\}$. These are subgroups of $\mathbb{R}^d$, $A(V) = V$, $A^T(K) = K$, and $v_0 \in V$, $k_0 \in K$. Irreducibility of A^T forces $\operatorname{lin}_{\mathbb{Q}}(K) = \mathbb{Q}^d$ so that we can find linearly independent $k_1, \dots, k_d \in K$. Let Γ be the lattice generated by k_j's, Γ^* be its dual, and $\chi_j\colon \mathbb{R}^d/\Gamma^* \to \mathbb{C}$ be the associated basis characters on the torus $\mathbb{R}^d/\Gamma^*$, namely, $\chi_j(x \,(\mathrm{mod}\, \Gamma)) := \exp(2\pi i \langle k_j | x\rangle)$, $x \in \mathbb{R}^d$, $j = 1, \dots, d$.

The convergence $\langle A^n v_0 | k_j\rangle \to 0 \ (\mathrm{mod}\, 1)$ translates to $\chi_j(A^n v_0 \ (\mathrm{mod}\, \Gamma)) \to 1$, which (by continuity of χ_j and compactness of $\mathbb{R}^d/\Gamma^*$) is equivalent to

$$\operatorname{dist}(A^n v_0 \,(\mathrm{mod}\, \Gamma), \chi_j^{-1}(1)) \to 0.$$

Therefore, $\operatorname{dist}(A^n v_0 \,(\mathrm{mod}\, \Gamma), G) \to 0$ where $G := \bigcap_{j=1}^d \chi_j^{-1}(1) = \{0 \,(\mathrm{mod}\, \Gamma)\}$, which is to say that

$$\operatorname{dist}(A^n v_0, \Gamma^*) \to 0. \tag{5}$$

Fix $\epsilon > 0$ so that, for $x, y \in A\Gamma^* \cup \Gamma^*$, $\operatorname{dist}(x, y) < \epsilon$ forces $x = y$. (This is possible because $A\Gamma^*/\Gamma^*$ is discrete in $\mathbb{R}^d/\Gamma^*$, as can be seen by picking $a \in \mathbb{N}$ so that aA has all integer entries and observing that $A\Gamma^* \subset a^{-1}\Gamma^*$, which yields $A\Gamma^*/\Gamma^* \subset (a^{-1}\Gamma^*)/\Gamma^*$.)

From (5), there are $u_n \in \Gamma^*$, $n \in \mathbb{N}$, such that $\operatorname{dist}(A^n v_0, u_n) \to 0$. Since, $\operatorname{dist}(u_{n+1}, Au_n) \le \operatorname{dist}(u_{n+1}, A^{n+1}v_0) + \operatorname{dist}(AA^n v_0, Au_n)$, we have $\operatorname{dist}(u_{n+1}, Au_n) \to 0$ and so, as soon as $\operatorname{dist}(u_{n+1}, Au_n) < \epsilon$, it must be that $u_{n+1} = Au_n$. Therefore, for some $n_0 \in \mathbb{N}$ and all $l \ge 0$, we have $A^l u_{n_0} = u_{n_0+l} \in \Gamma^*$. Now, from $v_0 \notin E^s$, $A^n v_0 \not\to 0$ so that $u_{n_0} \ne 0$. But $u_{n_0} \in M := \{v \in \Gamma^* : A^l v \in \Gamma^*\, \forall l \ge 0\}$, which makes M a nonzero subgroup of Γ^*. Clearly $AM \subset M$. By irreducibility of A, $\operatorname{lin}_{\mathbb{Q}}(M) = \mathbb{Q}^d$ so that M is a lattice. ■

Proof of Lemma 2 Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be the toral endomorphism associated to A, $\chi\colon \mathbb{T}^d \to \mathbb{C}$ be the character associated to k_0, $\chi(x \,(\mathrm{mod}\, \mathbb{Z}^d)) := \exp(2\pi i \langle x | k_0\rangle)$, and set $p := v_0 \,(\mathrm{mod}\, \mathbb{Z}^d)$. The hypothesis $\langle A^n v_0 | k_0\rangle \to 0 \,(\mathrm{mod}\, 1)$ translates to

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$\chi(f^n(p)) \to 1$, which is equivalent to $\operatorname{dist}(f^n(p), G) \to 0$ where $G := \chi^{-1}(1)$. We claim that, in fact,

$$\operatorname{dist}(f^n(p), G_\infty) \to 0, \quad G_\infty := \bigcap_{n \geq 0} f^{-n}(G). \tag{6}$$

Indeed, otherwise $f^{n_k}(p) \to w \notin f^{-l}(G)$ for some w, $l \geq 0$, and $n_k \to \infty$, and so $f^{n_k+l}(p) \to f^l(w) \notin G$ contradicting $\operatorname{dist}(f^n(p), G) \to 0$.

To identify G_∞ as a finite subgroup of $\mathbb{T}^d$, consider its lift to $\mathbb{R}^d$,

$$\Gamma := G_\infty + \mathbb{Z}^d := \{x \in \mathbb{R}^d : x \,(\operatorname{mod} \mathbb{Z}^d) \in G_\infty\}.$$

Denote by L_{k_0} the smallest sublattice of $\mathbb{Z}^d$ containing $(A^T)^n k_0$ for all $n \geq 0$. Its dual, $L^*_{k_0}$, is a lattice in $\mathbb{Q}^d$. For $v \in \mathbb{R}^d$, we have $v \in \Gamma$ if and only if $\langle A^n v | k_0 \rangle = \langle v | (A^T)^n k_0 \rangle \in \mathbb{Z}$ for all $n \geq 0$ iff $v \in L^*_{k_0}$. Thus $G_\infty = \Gamma / \mathbb{Z}^d$ where

$$\Gamma = L^*_{k_0} \subset \mathbb{Q}^d. \tag{7}$$

Let $q_n \in G_\infty$ realize the distance in (6) so that $\operatorname{dist}(f^n(p), q_n) \to 0$ and thus also $\operatorname{dist}(f(q_n), q_{n+1}) \to 0$. Since G_∞ is discrete, there is $n_0 \in \mathbb{N}$ such that

$$q_{n+1} = f(q_n), \quad n \geq n_0. \tag{8}$$

Moreover, if we pick $\epsilon > 0$ small enough and $n_1 > n_0$ large enough, then for every $n \geq n_1$ we can write $f^n(p) = q_n + x_n + y_n + z_n$ for some unique $x_n \in E^s$, $y_n \in E^c$, $z_n \in E^u$, each of norm less than ϵ. From (8), we have $x_{n+1} = Ax_n$, $y_{n+1} = Ay_n$, $z_{n+1} = Ax_n$ for $n \geq n_1$. What is more, $\operatorname{dist}(f^n(p), q_n) \to 0$ forces $y_n \to 0$ and $z_n \to 0$, which is only possible if $y_{n_1} = 0$ and $z_{n_1} = 0$. Thus $f^{n_1}(p) = q_{n_1} + x_{n_1}$, *i.e.*, $A^{n_1} v_0 = w + x_{n_1}$ for some $w \in \Gamma$ (with $q_{n_1} = w \,(\operatorname{mod} \mathbb{Z}^d)$). To summarize, $v_0 \in A^{-n_1}\Gamma + E^s = A^{-n_1} L^*_{k_0} + E^s \subset \mathbb{Q}^d + E^s$. ■

Remark 1 (Addendum to Lemma 2) *Under the hypotheses of Lemma 2,*

$$\{v \in \mathbb{R}^d : \langle A^n v | k_0 \rangle \to 0 \pmod 1\} = \bigcup_{n \geq 0} A^{-n} L^*_{k_0} + E^s \tag{9}$$

where L_{k_0} is the smallest lattice in $\mathbb{Z}^d$ containing $(A^T)^n k_0$ for all $n \geq 0$.

Proof of Remark 1 The "$\subset$" inclusion is demonstrated in the proof of Lemma 2. To see "$\supset$", it suffices to note that, if $v \in L^*_{k_0} + E^s$, then $v = w + x$ where $w \,(\operatorname{mod} \mathbb{Z}^d) \in G_\infty$ and $x \in E^s$. Thus $\langle A^n v | k_0 \rangle$ becomes exponentially close to $\langle A^n w | k_0 \rangle \in \mathbb{Z}$ as $n \to \infty$. ■

Proof of Theorem 2 The plan is to explicitly compute the objects invloved in the preceding arguments for A that is the companion matrix of the polynomial p of λ,

$$p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0, \quad a_i \in \mathbb{Z}.$$

The eigenvectors ω and ω^* with $A\omega = \lambda\omega$, $A^T\omega^* = \lambda\omega^*$ can be found as

$$\omega^* := \frac{1}{p'(\lambda)} \cdot (a_1 + a_2\lambda + \cdots + \lambda^{d-1}, \ldots, a_{d-1} + \lambda, 1),$$
$$\omega := (1, \lambda, \lambda^2, \ldots, \lambda^{d-1}).$$

These are normalized so that $\langle\omega|\omega^*\rangle = 1$, which ensures that the projection onto $\mathrm{lin}_{\mathbb{R}}(\omega)$ along $E^s = (\omega^*)^\perp$ is given by $\mathrm{pr}^u(y) = \langle y|\omega^*\rangle\omega$, $y \in \mathbb{R}^d$. Note that the components of ω^* generate $\frac{1}{p'(\lambda)}\mathbb{Z}[\lambda]$, $\{\langle u|\omega^*\rangle | \, u \in \mathbb{Z}^d\} = \frac{1}{p'(\lambda)}\mathbb{Z}[\lambda]$.

Denote by $e_1, \ldots, e_d$ the standard basis in $\mathbb{R}^d$, and set $k_0 := e_1$. Since $e_i = (A^T)^{i-1}(e_1)$ for $i = 1, \ldots, d$, we have $L_{k_0} = \mathbb{Z}^d$. Hence, $L^*_{k_0} = \mathbb{Z}^d$.

If, as in (4) in the proof of Theorem 1, we write $x = \langle v_0|k_0\rangle$ for $v_0 \in \mathrm{lin}_{\mathbb{R}}(\omega)$, then $\lambda^n x \to 0 \,(\mathrm{mod}\, 1)$ if and only if $\langle A^n v_0|k_0\rangle \to 0 \,(\mathrm{mod}\, 1)$ if and only if $A^{n_1}v_0 \in L^*_{k_0} + E^s = \mathbb{Z}^d + E^s$ for some $n_1 \geq 0$, where the last equivalence hinges on Remark 1. Thus $x \in X_\lambda$ are of the form

$$(10)\quad x = \lambda^{-n_1}\langle A^{n_1}v_0|k_0\rangle = \lambda^{-n_1}\langle \mathrm{pr}^u(u)|k_0\rangle = \lambda^{-n_1}\langle u|\omega^*\rangle\langle\omega|k_0\rangle = \lambda^{-n_1}\langle u|\omega^*\rangle \cdot 1$$

where $u \in \mathbb{Z}^d$ and $n_1 \geq 0$. That is, $X_\lambda = \bigcup_{n_1 \geq 0} \lambda^{-n_1}\frac{1}{p'(\lambda)}\mathbb{Z}[\lambda]$, as desired. ∎

Readers accustomed to a more traditional framework will no doubt notice that, in our setting, the scalar product $\langle \cdot | \cdot \rangle$ on $\mathbb{R}^d \times \mathbb{R}^d$ serves as the completion of *the trace form* on $\mathbb{Q}(\lambda) \times \mathbb{Q}(\lambda)$, the two being related by $\langle x|y\rangle = \mathrm{trace}\big(\langle x|\omega^*\rangle \cdot \langle\omega|y\rangle\big)$ for $x, y \in \mathbb{Q}^d$. This explains our remark about the nature of $\mathbb{Z}[\lambda]^*$ from the beginning of this note.

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