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A RADIUS OF CONVEXITY PROBLEM

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The authors determine the sharp radius of convexity for functions analytic in the unit disc having power series representation of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$ where a_{n+1} is fixed and such that $zf'(z)/f(x) = \left(1+Aw(z)\right)/\left(1+Bw(z)\right)$, $-1 \le B < 0 < A \le 1$ where $w(z)$ is an analytic function satisfying the conditions of Schwarz's lemma, in the case $A + B \ge 0$. The estimate obtained is an improvement over the corresponding result obtained by Mogra and Juneja for functions analytic and starlike in the unit disc, with missing coefficients where the initial non-vanishing coefficient is fixed.

1. Introduction

Recently considerable attention has been paid to study various aspects of univalent and analytic functions $f(z)$ with power series representation $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, whose second coefficient is fixed throughout (refer, for example, [2]). Shaffer [5, 6] has studied the class $P_{\alpha,n}$ $(0 \le \alpha < 1)$ defined as follows:

$$P_{\alpha,n} = \left\{p : p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots \prec \frac{1+z}{1-(1-2\alpha)z}\right\} .$$

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The class

$$P_n(A, B) = \left\{p : p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots \prec \frac{1+Az}{1-Bz}\right\}$$

where n is any natural number and A, B are any fixed complex numbers such that $|A| \le 1$, $|B| \le 1$ has been investigated in [7]. Thus successful attempts have been made obtaining sharp estimates for $|p(z)|$, $|p'(z)|$ and $|p'(z)/p(z)|$ for functions $p(z)$ analytic in the unit disc with missing initial terms. Improving the fixed second coefficient result, Mogra and Juneja [3] have determined the sharp radius of convexity for functions analytic and starlike in the unit disc E having power series representation of the form $f(z) = z + a_{n+1}z^{n+1} + z_{n+2}z^{n+2} + \ldots$ where a_{n+1} is fixed. It is our aim in this paper to determine the sharp results of convexity for the class $S_n(A, B)$ of functions f of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$, analytic in the unit disc E satisfying $zf'(z)/f(z) = (1+Aw(z))/(1+Bw(z))$, $-1 \le B < 0 < A \le 1$ where $w(z)$ is an analytic function satisfying the conditions of Schwarz's lemma in the unit disc E. We obtain sharp estimates for the radius of convexity of the class $S_n(A, B)$ where $A + B \ge 0$. Our results immediately yield those of Mogra and Juneja [3] when $A = 1$, $B = -1$.

2. Some preliminary results

We need the following result from Goluzin [1].

LEMMA [1]. *Let* $w(z) = \sum_{k=n}^{\infty} c_k z^k$, $n = 1, 2, \ldots$, *be analytic in the unit disc* E *and satisfy* $|w(z)| < 1$ *there. Then*

(i) $|w(z)| \le \phi(r) \le r^n$,

(ii) $|w'(z)|/\left(1-|w(z)|^2\right) \le \phi'(r)/\left(1-(\phi(r))^2\right) \le nr^{n-1}/\left(1-r^{2n}\right)$

where $\phi(r) = r^n\left((r+|c_n|)/(1+r|c_n|)\right)$ *with equality in the second parts of (i) and (ii) if and only if* $|c_n| = 1$.

The proof may be found in [1].

Let $P_n(A, B)$ denote the class of analytic functions p in E of the form $p(z) = 1 + b_n z^n + \ldots$ such that $p(z) = (1+Aw(z))/(1+Bw(z))$ where $-1 \le B < 0 < A \le 1$ and w is an analytic function, of the form $w(z) = \sum_{k=n}^{\infty} c_k z^k$ satisfying $|w(z)| \le |z|^n$ in $|z| < 1$.

We establish the following results for $P_n(A, B)$.

THEOREM 1. *Let* $p(z) \in P_n(A, B)$. *Then*

$$\operatorname{Re} p(z) \ge \frac{1-A\phi(r)}{1-B\phi(r)}, \quad A + B \ge 0 ,$$

where $\phi(r) = r^n\left((r+|c_n|)/(1+r|c_n|)\right)$ *and* $|c_n| = |b_n|/(A-B)$. *The bound is sharp.*

Proof. $p(z) \in P_n(A, B)$ implies that $p(z) = (1+Aw(z))/(1+Bw(z))$, $-1 \le B < 0 < A \le 1$, $w(z) = \sum_{k=n}^{\infty} c_k z^k$ being analytic in the unit disc satisfying $|w(z)| \le |z|^n$. Now

$$\begin{aligned}\operatorname{Re} p(z) &= \frac{\operatorname{Re}(1+Aw(z))(1+B\bar{w}(z))}{|1+Bw(z)|^2} \\ &= \frac{1+(A+B)\operatorname{Re}w(z)+AB|w(z)|^2}{|1+Bw(z)|^2} \\ &\ge \frac{1-(A+B)|w(z)|+AB|w(z)|^2}{(1-B|w(z)|)^2} = \frac{1-A|w(z)|}{1-B|w(z)|}\end{aligned}$$

when $A + B \ge 0$.

The function $g(x) = (1-Ax)/(1-Bx)$ where $x = |w(z)|$ is monotonic decreasing with respect to x since $g'(x) = (-(A-B))/(1-Bx)^2 < 0$. Therefore $\operatorname{Re} p(z) \ge (1-A\phi(r))/(1-B\phi(r))$ using *(i)* of Lemma 1. Taking $w(z) = -z^n\left((|c_n|+z)/(1+|c_n|z)\right)$ we see that the bound is sharp at $z = r$ on $|z| = r < 1$.

THEOREM 2. *Let* $p(z) \in P_n(A, B)$. *Then*

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$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{r(A-B)\phi'(r)}{\left(1-A\phi(r)\right)\left(1-B\phi(r)\right)}\ , \quad A+B \ge 0\ ,$$

where $\phi(r) = r^n\left(\left(r+|c_n|\right)/\left(1+r|c_n|\right)\right)$ *and* $|c_n| = |b_n|/(A-B)$. *The bound is sharp.*

Proof. Since $p(z) \in P_n(A, B)$, $p(z) = \left(1+Aw(z)\right)/\left(1+Bw(z)\right)$, $-1 \le B < 0 < A \le 1$ where $w(z)$ is analytic in the unit disc satisfying $w(0) = 0$, $|w(z)| \le |z|^n$ there.

Therefore

$$\frac{zp'(z)}{p(z)} = \frac{(A-B)zw'(z)}{\left(1+Aw(z)\right)\left(1+Bw(z)\right)}$$

and

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{(A-B)|zw'(z)|}{\left(1-A|w(z)|\right)\left(1-B|w(z)|\right)} \quad \text{when} \quad A+B \ge 0\ .$$

Using the lemma from [1], we get

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{(A-B)r\phi'(r)}{\left(1-\phi^2(r)\right)} \frac{1-|w(z)|^2}{\left(1-A|w(z)|\right)\left(1-B|w(z)|\right)}\ .$$

Consider $h(x) = \left(1-x^2\right)/(1-Ax)(1-Bx)$, $x = |w(z)|$, $0 \le x \le 1$. Then $h'(x) = \left((A+B)x^2-2(1+AB)x+(A+B)\right)/(1-Ax)^2(1-Bx)^2$. This shows that $h'(x) \ge 0$ if $x \le x_0$ and then it becomes negative in $\left(x_0, 1\right)$ where $x_0 = \left((1+AB)-\sqrt{\left(1-A^2\right)\left(1-B^2\right)}\right)/(A+B)$. Clearly $0 \le x_0 < 1$ and $h(z)$ is increasing in $\left[0, x_0\right]$ and decreasing in $\left(x_0, 1\right)$. We can therefore use $x = |w(z)| \le \phi(r)$ in $0 \le |w(z)| \le x_0$ to get the upper bound of $|zp'(z)/p(z)|$. Thus

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{(A-B)r\phi'(r)}{\left(1-A\phi(r)\right)\left(1-B\phi(r)\right)}\ .$$

Taking $w(z) = \left[-z^n\left(z+|c_n|\right)\right]/\left(1+|c_n|z\right)$, we see that the bound is sharp at $z = r$ on $|z| = r < 1$.

3. Radius of convexity of $S_n(A, B)$

THEOREM 3. *Let* $f(z) \in S_n(A, B)$ *and* c_n *as in Lemma* 1. *Then* $f(z)$ *is convex in the disc* $|z| < r^*$ *where* $r^* = \min(r_0, r_1)$, r_0 *being the smallest positive root of* $r^{n+1} + |c_n|r^n - |c_n|x_0r - x_0 = 0$ *and* r_1 *the smallest positive root of* $(1-A\phi(r))^2 - (A-B)r\phi'(r) = 0$. *The estimate is sharp for* $r_1 < r_0$.

Proof. $f(z)$ is convex in the unit disc if and only if $\operatorname{Re}\{1+(zf''(z)/f'(z))\} \geq 0$. Now $p(z) = zf'(z)/f(z) \in P_n(A, B)$ since $f \in S_n(A, B)$. Now

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}$$

and

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \operatorname{Re}\, p(z) - \left|\frac{zp'(z)}{p(z)}\right| .$$

Using Theorems 1 and 2, we therefore get

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \frac{1-A\phi(r)}{1-B\phi(r)} - \frac{(A-B)r\phi'(r)}{(1-A\phi(r))(1-B\phi(r))}$$

whenever $A + B \geq 0$ and $|w(z)| \leq x_0$. Thus $f(z)$ is convex if and only if $\psi(r) = (1-A\phi(r))^2 - (A-B)r\phi'(r) \geq 0$ with $|w(z)| \leq x_0$. Let r_1 be the smallest positive root of the equation $\psi(r) = 0$. Further $|w(z)| \leq x_0$ implies $r^n(r+|c_n|) < x_0(1+r|c_n|)$. Let r_0 be the smallest positive root of the equation $\phi(r) = r^{n+1} + r^n|c_n| - r|c_n|x_0 - x_0 = 0$. Let $r^* = \min(r_0, r_1)$. Then $f(z)$ is convex in the disc $|z| < r^*$. The estimate is sharp can be seen by taking

$$w(z) = \frac{-z^n(z+|c_n|)}{(1+z|c_n|)} \quad \text{at} \quad z = r \quad \text{on} \quad |z| = r < 1 .$$

REMARK. Without loss of generality we can take $a_{n+1} \geq 0$ because otherwise we can consider the function

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$$w = e^{i\alpha/n} f\left(ze^{-i\alpha/n}\right) \quad \text{where} \quad \arg a_{n+1} = \alpha .$$

If we now put $A = 1$, $B = -1$ and replace $|c_n|$ by $na_{n+1}/(A-B) = na_{n+1}/2$, we get $\phi(r) = r^n\{r+(na_{n+1}/2)\}/(1+(na_{n+1}/2)r)$ and $x_0 = 1$. Therefore in $[0, 1]$, $\mathrm{Re}\{1+(zf''(z)/f'(z))\} \ge 0$ if and only if $(1-\phi(r))^2 - 2r\phi'(r) \ge 0$. That is

$$\left\{\left(1 + \frac{rna_{n+1}}{2}\right)-r^n\left(r + \frac{na_{n+1}}{2}\right)\right\}^2 - 2r^n\left\{n\left(r + \frac{na_{n+1}}{2}\right)\left(1+r\,\frac{na_{n+1}}{2}\right)+r\left(1 - \frac{n^2a_{n+1}^2}{4}\right)\right\} \ge 0 .$$

This leads to the equation

$$4r^{2n+2} + 4na_{n+1}r^{2n+1} + n^2a_{n+1}^2r^{2n} - 4n(n+1)a_{n+1}r^{n+2} - 2r^{n+1}\left(8+4n+n^3a_{n+1}^2\right) - 4n(n+1)a_{n+1}r^n + n^2a_{n+1}^2r^2 = 0 .$$

This gives us the sharp estimate obtained by Mogra and Juneja [3] for analytic starlike functions in the unit disc with missing coefficients where the initial non-vanishing coefficient is fixed.

We are at present unable to obtain sharp estimates of radius of convexity when $A + B < 0$.

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