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K-ENVELOPES FOR REAL INTERPOLATION METHODS

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Abstract. In this paper, we study the K-envelopes of the real interpolation methods with function space parameters in the sense of Brudnyi and Kruglyak [Y. A. Brudnyi and N. Ja. Kruglyak, *Interpolation functors and interpolation spaces* (North-Holland, Amsterdam, Netherlands, 1991)]. We estimate the upper bounds of the K-envelopes and the interpolation norms of bounded operators for the K_Φ-methods in terms of the fundamental function of the rearrangement invariant space related to the function space parameter Φ. The results concerning the quasi-power parameters and the growth/continuity envelopes in function spaces are obtained.

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The K-envelope of an intermediate space for a Banach couple, according to Pustylnik, is the exact upper bound of K-functionals on the unit ball of the space considered. This concept plays an important role in the interpolation theory, such as embedding properties of interpolation spaces, the generalised Lorentz and Marcinkiewicz spaces, weak-type interpolation and interpolation of operator ideals (see [**4**, **10**–**12**] for details). The K-envelopes of certain function spaces have a close connection with the growth/continuity envelopes in functions. Recently, Haroske in [**9**] studied this connection and found some estimates of the growth/continuity envelopes in the framework of the classical real interpolation with the numerical parameter $\theta \in (0, 1)$.

Our goal in this paper is to investigate the K-envelopes of the real interpolation methods with function space parameters in the sense of Brudnyi and Kruglyak [**3**]. In the first section, we give some preliminary information concerning the K_Φ interpolation methods and related topics. In Section 2, we study the K-envelopes and interpolation operators for the K_Φ-methods, and estimate the upper bounds in terms of the fundamental function of the rearrangement invariant (r.i.) space $K_\Phi(\widetilde{L}^1, \widetilde{L}^\infty)$. Section 3 concerns with the K-envelopes and some properties of the K_Φ-methods, especially Lions–Peetre's methods of constants and means, with quasi-power parameters. In the final section, we apply these estimates on the growth/continuity envelopes in function spaces, and carry over the results in [**9**] to the setting of K_Φ-methods.

1. Preliminaries. Let $\overline{X} = (X_0, X_1)$ be a Banach couple with $\Delta\overline{X} = X_0 \cap X_1$ and $\Sigma\overline{X} = X_0 + X_1$, and let X be an intermediate space for $\overline{X}$. We denote by X^0 the regularisation of X for $\overline{X}$, by X' the Banach space dual of X^0, and we write $\overline{X}' = \left(X_0', X_1'\right)$ as the dual couple of $\overline{X}$. The notation $\mathcal{B}(X, Y)$ (resp., $\mathcal{B}\left(\overline{X}, \overline{Y}\right)$) stands for the space of all bounded linear operators from Banach space X to Banach space Y (resp., from Banach couple $\overline{X}$ to Banach couple $\overline{Y}$). We simply write $\mathcal{B}(X) = \mathcal{B}(X, X)$ and $\mathcal{B}(\overline{X}) = \mathcal{B}(\overline{X}, \overline{X})$. For $T \in \mathcal{B}(X_j, Y_j)$ $(j = 0, 1)$, we denote $\|T\|_j = \|T\|_{X_j, Y_j}$ $(j = 0, 1)$. For $T \in \mathcal{B}(\overline{X}, \overline{Y})$, we denote $\|T\|_{\overline{X}, \overline{Y}} = \|T\|_0 \vee \|T\|_1$. Further information about the interpolation theory can be found in [**2**, **3**].

Let $\overline{X}$ be a Banach couple. For $t > 0$, the J- and K-functionals defined on $\Delta\overline{X}$ and $\Sigma\overline{X}$, respectively, are given by

$$J(t, x) = J(t, x; \overline{X}) = \|x\|_0 \vee (t\|x\|_1)$$

for $x \in \Delta\overline{X}$, and

$$K(t, x) = K(t, x; \overline{X}) = \inf\{\, \|x_0\|_0 + t\|x_1\|_1 \mid x = x_0 + x_1,\ x_j \in X_j\, (j = 0, 1)\,\}$$

for $x \in \Sigma\overline{X}$. Assume that X is an intermediate space for $\overline{X}$. The K-envelope of X is the function defined by

$$\kappa(t, X) = \kappa(t, X; \overline{X}) = \sup\{\, K(t, x; \overline{X}) \mid x \in X,\ \|x\|_X \le 1\,\} \tag{1.1}$$

for $t > 0$.

Let Φ be a non-zero Banach function space over $(\mathbf{R}_+, dt/t)$. We say that Φ is K-non-trivial if $1 \wedge t \in \Phi$, and define

$$K_\Phi(\overline{X}) = \{\, x \in \Sigma\overline{X} \mid \|x\|_{K_\Phi} = \|K(t, x)\|_\Phi < \infty\,\} \quad [\mathbf{3}, (3.3.1)].$$

We say that Φ is J-non-trivial if

$$\int_0^\infty 1 \wedge (1/t)\, |f(t)|\, \frac{dt}{t} < \infty \quad \text{for all } f \in \Phi,$$

and define $J_\Phi(\overline{X})$ as the space of all $x \in \Sigma\overline{X}$, which permits a canonical representation $x = \int_0^\infty u(t)dt/t$ for a strongly measurable function $u\colon \mathbf{R}_+ \to \Delta\overline{X}$, with the norm

$$\|x\|_{J_\Phi} = \inf_u \|J(t, u(t))\|_\Phi < \infty \qquad [\mathbf{3}, (3.4.3)].$$

According to [**3**, Cor. 4.1.9], K_Φ and J_Φ are exact interpolation functors for all Banach couples under the action of bounded (not necessarily linear) operators. More precisely, if $\overline{X}$ and $\overline{Y}$ are Banach couples and if $T\colon \overline{X} \to \overline{Y}$ are bounded (not necessarily linear) operators in the sense that T acting from $\Sigma\overline{X}$ to $\Sigma\overline{Y}$ such that there exists $M_j > 0$ for any $x_j \in X_j$ $(j = 0, 1)$,

$$T(x_0 + x_1) = y_0 + y_1$$

for some $y_j \in Y_j$ with

$$\|y_j\|_{Y_j} \le M_j \|x_j\|_{X_j} \qquad (j = 0, 1). \tag{1.2}$$

Thus,

$$\|Tx\|_{K_\Phi(\overline{Y})} \leq (M_0 \vee M_1)\|x\|_{K_\Phi(\overline{X})}$$

for all $x \in K_\Phi(\overline{X})$, and

$$\|Tx\|_{J_\Phi(\overline{Y})} \leq (M_0 \vee M_1)\|x\|_{J_\Phi(\overline{X})}$$

for all $x \in J_\Phi(\overline{X})$.

The function space Φ is said to be a quasi-power parameter space for real interpolation if Φ is both K- and J-non-trivial such that the Calderón operator S is bounded on Φ, where S is defined by

$$(Sf)(t) = \int_0^\infty \left(1 \wedge \frac{t}{v}\right) f(v) \frac{dv}{v}.$$

In this case, the equivalence

$$J_\Phi(\overline{X}) = K_\Phi(\overline{X}) \tag{1.3}$$

holds with the isomorphism constant depending on Φ [**3**, Corollary 3.5.15]. Let Φ' be the Köthe dual of Φ. Then Φ' is also a quasi-power parameter space for real interpolation, and the duality

$$J_\Phi(\overline{X})' = K_{\Phi'}(\overline{X}') \tag{1.4}$$

holds with the isomorphism constant depending on Φ [**3**, Theorem 3.7.2]. For a quasi-concave function $\varphi\colon \mathbf{R}_+ \to \mathbf{R}_+$, we write

$$\bar{\varphi}(t) = \sup_{s>0} \frac{\varphi(st)}{\varphi(s)}$$

for $t > 0$, and

$$\varphi(t_0, t_1) = t_0 \varphi(t_1/t_0)$$

for $t_0, t_1 > 0$. Let us now define

$$\underline{\alpha}_\varphi = \lim_{t\to 0} \frac{\ln \bar{\varphi}(t)}{\ln t} \qquad \text{and} \qquad \overline{\alpha}_\varphi = \lim_{t\to\infty} \frac{\ln \bar{\varphi}(t)}{\ln t}$$

as the lower and upper extension indices of φ. Function φ is said to be quasi-power if

$$0 < \underline{\alpha}_\varphi \leq \overline{\alpha}_\varphi < 1.$$

Furthermore, let

$$L^p_\varphi = \left\{ f \in L^0(\mathbf{R}_+, dt/t) \,\middle|\, \|f\|_{L^p_\varphi} = \left(\int_0^\infty \left| \frac{f(t)}{\varphi(t)} \right|^p \frac{dt}{t} \right)^{1/p} \right\} < \infty$$

for $1 \leq p < \infty$ with the usual modification for $p = \infty$. In particular, we denote $L^p_\theta = L^p_\varphi$ if $\varphi(t) = t^\theta$ for $0 \leq \theta \leq 1$, and $\overline{L}^p = (L^p_0, L^p_1)$. If φ is a quasi-power function and

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$1 \leq p < \infty$, then the function space L^p_φ is a quasi-power parameter for real interpolation. In case $0 < \theta < 1$ and $1 \leq p < \infty$, let

$$\overline{X}_{\theta,p} = \overline{X}_{L^p_\theta}.$$

Throughout this paper we write $\varphi \lesssim \psi$ (or $\psi \gtrsim \varphi$) if $0 \leq \varphi(t) \leq c\,\psi(t)$ for all $t > 0$ and for some constant $c > 0$, and write $\varphi \simeq \psi$ if $\varphi \lesssim \psi$ and $\psi \lesssim \varphi$. The notations $\subseteq$ and $=$ between two Banach spaces stand for continuous inclusion and isomorphic equivalence, respectively.

2. Upper bounds of K-envelopes and interpolated operators. From now on, we assume that Φ is a K-non-trivial Banach function space over $(\mathbf{R}_+, dt/t)$ such that the Hardy operator P given by

$$(Pf)(t) = \int_0^t f(v)\frac{dv}{v}$$

is bounded on Φ. We will estimate the upper bound of K-envelope given in (1.1) for the interpolation space $K_\Phi(\overline{X})$ in terms of fundamental functions of the r. i. function spaces corresponding to the parameter space Φ. Observe that if Φ is a quasi-power parameter for real interpolation, then the Hardy operator P is certainly bounded on Φ.

For an r. i. function space X over the measure space (Ω, μ), we mean the Banach function space of measurable real valued functions on (Ω, μ) which is an exact interpolation space for the Banach couple $(L^1(\Omega, \mu), L^\infty(\Omega, \mu))$. If $f \in X$, then we denote by f^* the non-increasing rearrangement of f. Let us now consider the measure space $(\mathbf{R}_+, dt)$, and write $\widetilde{L}^p = L^p(\mathbf{R}_+, dt)$ for $1 \leq p \leq \infty$. The r. i. function space X over $(\mathbf{R}_+, dt)$ is characterised by its fundamental function

$$\phi_X(t) = \|\chi_{(0,t)}\|_X \tag{2.1}$$

and dilation function

$$\sigma_X(t) = \sup\left\{ \frac{\|f(\cdot/t)\|_X}{\|f(\cdot)\|_X} \,|\, f \in X, f \neq 0 \right\}. \tag{2.2}$$

It is known that

$$\bar{\phi}_X(t) \leq \sigma_X(t) \tag{2.3}$$

for all $t > 0$. The so-called Boyd indices of space X is defined by

$$\underline{\alpha}_X = \underline{\alpha}_{\sigma_X} \quad \text{and} \quad \bar{\alpha}_X = \bar{\alpha}_{\sigma_X}. \tag{2.4}$$

We refer to [**1**] for the background of r. i. function spaces.

Lemma 2.1. *Let $\widetilde{\Phi} = K_\Phi(\widetilde{L}^1, \widetilde{L}^\infty)$ be the r. i. function space corresponding to Φ. Then*

$$\|tf^*(t)\|_\Phi \leq \|f\|_{\tilde{\Phi}} \leq \|P\|_\Phi \|tf^*(t)\|_\Phi$$

for all $f \in \widetilde{\Phi}$. In particular, if Φ is a quasi-power parameter space for real interpolation, then $0 < \underline{\alpha}_{\tilde{\Phi}} \leq \bar{\alpha}_{\tilde{\Phi}} < 1$.

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Proof. For $f \in \widetilde{\Phi}$, by [**3**, Proposition 3.1.18], we have

$$K(t, f; \widetilde{L}^1, \widetilde{L}^\infty) = \int_0^t f^*(v)dv.$$

Thus,

$$\|f\|_{\tilde{\Phi}} = \left\| \int_0^t f^*(v)dv \right\|_\Phi .$$

By the boundedness of the Hardy operator P on Φ and the inequality

$$tf^*(t) \leq \int_0^t f^*(v)dv,$$

we obtain

$$\|tf^*(t)\|_\Phi \leq \|f\|_{\tilde{\Phi}} \leq \|P\|_\Phi \|tf^*(t)\|_\Phi.$$

In particular, if Φ is a quasi-power parameter space for real interpolation, and if $f \in \widetilde{\Phi}$, then by [**1**, Proposition III.5.2], we have

$$\left\| \int_0^\infty \left(\frac{1}{t} \wedge \frac{1}{s}\right) f(v)dv \right\|_{\tilde{\Phi}} \leq \left\| \int_0^\infty \left(1 \wedge \frac{t}{v}\right) f^*(v)dv \right\|_\Phi$$
$$\leq \|S\|_\Phi \|tf^*(t)\|_\Phi \leq \|S\|_\Phi \|f\|_{\tilde{\Phi}}.$$

According to [**1**, Theorem III.5.15], $0 < \underline{\alpha}_{\tilde{\Phi}} \leq \overline{\alpha}_{\tilde{\Phi}} < 1$. □

In the rest of this section, we assume that ϕ is the fundamental function of $\widetilde{\Phi}$ given in (2.1). According to [**11**], if X is an intermediate space for $\overline{X}$, then

$$\kappa(t, X; \overline{X}) \lesssim 1 \vee t.$$

In particular, if $X_0 \subseteq X_1$, then $\kappa(t, X; \overline{X}) \simeq t$ on $(0, 1)$; and if $X_1 \subseteq X_0$, then $\kappa(t, X; \overline{X}) \simeq 1$ on $(1, \infty)$. For the interpolation space $K_\Phi(\overline{X})$, we have the following improved estimate.

PROPOSITION 2.1. *Let $c_\Phi = \|P\|_\Phi$. Then*

$$\kappa(t, K_\Phi(\overline{X}); \overline{X}) \leq c_\Phi \cdot t/\phi(t).$$

Proof. For $x \in K_\Phi(\overline{X})$ with $x \neq 0$, let $f(t) = K(t, x; \overline{X})/t$. Then $f \in \widetilde{\Phi}$ and $f^* = f$. Observe that

$$K(t, x; \overline{X}) = tf(t) \leq \int_0^t f(v)dv = \int_0^\infty f(v)\chi_{(0,t)}(v)dv \leq \|f\|_{\tilde{\Phi}} \|\chi_{(0,t)}\|_{\tilde{\Phi}'}.$$

This, together with the following estimate

$$\|x\|_{K_\Phi(\overline{X})} = \|K(t, x; \overline{X})\|_\Phi = \|tf^*(t)\|_\Phi \geq c_\Phi^{-1} \|f\|_{\tilde{\Phi}}$$

by Lemma 2.1, and the identity

$$\|\chi_{(0,t)}\|_{\tilde{\Phi}'} = t/\phi(t),$$

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implies that

$$\kappa(t, K_\Phi(\overline{X}); \overline{X}) \le c_\Phi \cdot t/\phi(t),$$

which completes the proof. □

An interpolation result of the K-envelopes is given as follows.

PROPOSITION 2.2. *Let X_0, X_1 be intermediate spaces for Banach couple $\overline{Y}$, and let $\overline{X} = (X_0, X_1)$. Then*

$$\kappa(t, K_\Phi(\overline{X}); \overline{Y}) \le c_\Phi \cdot \kappa(t, X_0; \overline{Y})/\phi(\kappa(t, X_1; \overline{Y})/\kappa(t, X_0; \overline{Y}))$$

for all $t > 0$.

Proof. Let $X = K_\Phi(\overline{X})$, $\kappa_X(t) = \kappa(t, X; \overline{Y})$, and $\kappa_j(t) = \kappa(t, X_j; \overline{Y})$ $(j = 0, 1)$. Assume that $x \in X$ with $\|x\|_X = 1$. For an arbitrary $\varepsilon > 0$, there exists a decomposition $x = x_0(s) + x_1(s)$, for which $x_j(s) \neq 0$ $(j = 0, 1)$, and

$$\|x_0(s)\|_{X_0} + s\|x_1(s)\|_{X_1} \le (1 + \varepsilon)K(s, x; \overline{X})$$

for all $s > 0$. Observe that for any $t > 0$

$$\frac{K(t, x_j(s); \overline{Y})}{\|x_j(s)\|_{X_j}} \le \kappa_j(t) \qquad (j = 0, 1).$$

Thus,

$$\frac{K(t, x_j(s); \overline{Y})}{\kappa_j(t)} \le \|x_j(s)\|_{X_j} \qquad (j = 0, 1).$$

If we choose $s = \kappa_1(t)/\kappa_0(t)$, then

$$\begin{aligned} K(t, x; \overline{Y}) &\le K(t, x_0(s); \overline{Y}) + K(t, x_1(s); \overline{Y}) \\ &\le \kappa_0(t)\left(\frac{K(t, x_0(s); \overline{Y})}{\kappa_0(t)} + s\frac{K(t, x_1(s); \overline{Y})}{\kappa_1(t)}\right) \\ &\le \kappa_0(t)(\|x_0(s)\|_{X_0} + s\|x_1(s)\|_{X_1}) \\ &\le (1 + \varepsilon)\kappa_0(t)K(\kappa_1(t)/\kappa_0(t), x; \overline{X}). \end{aligned}$$

This implies that

$$\kappa_X(t) \le \kappa_0(t)\kappa(\kappa_1(t)/\kappa_0(t), X; \overline{X}).$$

According to Proposition 2.1, we obtain

$$\kappa(t, K_\Phi(\overline{X}); \overline{Y}) \le c_\Phi \cdot \kappa(t, X_0; \overline{Y})/\phi(\kappa(t, X_1; \overline{Y})/\kappa(t, X_0; \overline{Y}))$$

for all $t > 0$. □

Let us define

$$\rho_\Phi(t) = t\bar{\phi}(1/t). \tag{2.5}$$

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It is easy to see that

$$\rho_\Phi(st) \le \rho_\Phi(s)\rho_\Phi(t)$$

for all $s, t > 0$. If Φ is a quasi-power parameter space for real interpolation, then by (2.3) and Lemma 2.1, ρ_Φ is a quasi-power function. Observe that

$$\sup_{s>0} \frac{st/\phi(st)}{s/\phi(s)} = t \sup_{s>0} \frac{\phi(s)}{\phi(st)} = t \sup_{s>0} \frac{\phi(s/t)}{\phi(s)} = t\bar{\phi}(1/t).$$

Thus,

$$\bar{\kappa}(t, K_\Phi(\overline{X}); \overline{Y}) \le c_\Phi \rho_\Phi(t) \tag{2.6}$$

by Proposition 2.1. We can now estimate the interpolation norms of bounded operators on $K_\Phi(\overline{X})$ in terms of ρ_Φ.

PROPOSITION 2.3. *If T is a non-zero bounded operator from $\overline{X}$ to $\overline{Y}$ with M_j $(j = 0, 1)$ given in (1.2), then*

$$\|T\|_{K_\Phi(\overline{X}), K_\Phi(\overline{Y})} \le c_\Phi \rho_\Phi(M_0, M_1).$$

In particular, if $T \in \mathcal{B}(\overline{X}, \overline{Y})$, then

$$\|T\|_{K_\Phi(\overline{X}), K_\Phi(\overline{Y})} \le c_\Phi \rho_\Phi(\|T\|_0, \|T\|_1).$$

Proof. If $x \in K_\Phi(\overline{X})$ with $\|x\|_{K_\Phi(\overline{X})} = \|K(t, x; \overline{X})\|_\Phi = 1$, then for $s, t > 0$, we have

$$\begin{aligned} K(st, x; \overline{X}) &\le \overline{K}(s, x; \overline{X}) K(t, x; \overline{X}) \\ &\le \bar{\kappa}(s, K_\Phi(\overline{X}); \overline{X}) K(t, x; \overline{X}) \le c_\Phi \rho_\Phi(s) K(t, x; \overline{X}) \end{aligned}$$

by (2.6). Let now $s = M_1/M_0$. For each $x \in K_\Phi(\overline{X})$ such that $x = x_0 + x_1$ with $x_j \in X_j$, and $Tx = y_0 + y_1$, for which $y_j \in Y_j$ with $\|y_j\|_{Y_j} \le M_j \|x_j\|_{X_j}$, we have

$$K(t, Tx; \overline{Y}) \le \|y_0\|_{Y_0} + t\|y_1\|_{Y_1} \le M_0(\|x_0\|_{X_0} + st\|x_1\|_{X_1}).$$

This implies that

$$K(t, Tx; \overline{Y}) \le M_0 K(st, x; \overline{X}) \le c_\Phi M_0 \rho_\Phi(s) K(t, x; \overline{X}),$$

and hence

$$\|Tx\|_{K_\Phi(\overline{Y})} = \|K(t, Tx; \overline{Y})\|_\Phi \le c_\Phi \rho_\Phi(M_0, M_1)\|x\|_{K_\Phi(\overline{X})},$$

which completes the proof. □

REMARK. For $t > 0$, let τ_s be the compression operator given by

$$(\tau_t f)(s) = f(st)$$

for $f \in L^0(\mathbf{R}_+, dt/t)$. We define the corresponding compression function by

$$\tau_\Phi(t) = \|\tau_t\|_{K_\Phi(\bar{L}^\infty)}.$$

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In particular, for $0 \le \theta \le 1$ and $1 \le p \le \infty$, we have

$$\tau_{L^p_\theta}(t) = t^\theta.$$

It is easy to see that

$$\tau_\Phi(t) \simeq t\sigma_{\tilde{\Phi}}(1/t),$$

and hence

$$\rho_\Phi(t) \lesssim \tau_\Phi(t) \tag{2.7}$$

by (2.2) and Proposition 2.1. Moreover, if ρ_Φ in Proposition 2.3 is replaced by τ_Φ, then we can show direct that

$$\|T\|_{K_\Phi(\overline{X}),K_\Phi(\overline{Y})} \lesssim \tau_\Phi(M_0, M_1).$$

The discrete version of τ_Φ and the similar norm estimate for interpolated operators are formulated in [**5**].

3. On quasi-power parameter spaces. If X is an intermediate space for $\overline{X}$, we define the dual function of the K-envelope of X as

$$\iota(t, X; \overline{X}) = \inf\{\, J(t, x; \overline{X}) \mid x \in \Delta\overline{X},\ \|x\|_X = 1\,\} = \frac{t}{\kappa(t, X'; \overline{X}')}.$$

It is known that if X is an interpolation space for $\overline{X}$, then

$$\kappa(t, X; \overline{X}) \le \iota(t, X; \overline{X}).$$

In case $X = K_\Phi(\overline{X})$ with a quasi-power parameter space Φ for real interpolation, we can improve this inequality in terms of the fundamental function ϕ of $K_\Phi(\widetilde{L}^1, \widetilde{L}^\infty)$.

PROPOSITION 3.1. *If Φ is a quasi-power parameter space for real interpolation, then for any Banach couple $\overline{X}$, we have*

$$\kappa(t, K_\Phi(\overline{X}); \overline{X}) \lesssim t/\phi(t) \lesssim \iota(t, K_\Phi(\overline{X}); \overline{X}).$$

Proof. The inequality

$$\kappa(t, K_\Phi(\overline{X}); \overline{X}) \lesssim t/\phi(t)$$

is given by Proposition 2.1. On the other hand, we have

$$\iota(t, K_\Phi(\overline{X}); \overline{X}) = \frac{t}{\kappa(t, K_\Phi(\overline{X})'; \overline{X}')} \simeq \frac{t}{\kappa(t, K_{\Phi'}(\overline{X}'); \overline{X}')}$$

by the equivalence in (1.3) and the duality in (1.4). If we apply Proposition 2.1 on $\kappa(t, K_{\Phi'}(\overline{X}'); \overline{X}')$, then we obtain

$$\kappa(t, K_{\Phi'}(\overline{X}'); \overline{X}) \lesssim \frac{t}{t/\phi(t)} = \phi(t),$$

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and hence

$$t/\phi(t) \lesssim \iota(t, K_\Phi(\overline{X}); \overline{X}),$$

which completes the proof. □

Let Φ and Ψ be two quasi-power parameter spaces for real interpolation. It is easy to see that if $K_\Phi(\overline{X}) \subseteq K_\Psi(\overline{X})$, then

$$\kappa(t, K_\Psi(\overline{X}), \overline{X}) \lesssim \kappa(t, K_\Phi(\overline{X}), \overline{X}).$$

On the other hand, by combining Proposition 3.1 with [**10**, Theorem 3], we obtain the following.

Proposition 3.2. *If Φ and Ψ are two quasi-power parameter spaces for real interpolation, for which*

$$\int_0^\infty \frac{\kappa(t, K_\Psi(\overline{X}), \overline{X})}{\kappa(t, K_\Phi(\overline{X}), \overline{X})} \frac{dt}{t} < \infty,$$

then

$$K_\Phi(\overline{X}) \subseteq K_\Psi(\overline{X}).$$

Let $\rho\colon \mathbf{R}_+ \to \mathbf{R}_+$ be a quasi-power function with $\rho(1) = 1$, and let $1 \le p_0, p_1 \le \infty$. We define K_{ρ,p_0,p_1} and J_{ρ,p_0,p_1} to be Lions–Peetre's interpolation methods of constants and means associated with the function parameter ρ, respectively. More precisely, the space $K_{\rho,p_0,p_1}(\overline{X})$ consists of all those $x \in \Sigma\overline{X}$ such that there exist strongly measurable functions $x_j\colon \mathbf{R}_+ \to X_j$ $(j = 0, 1)$ satisfying $x = x_0(t) + x_1(t)$ and $t^j\|x_j(t)\|_j/\rho(t) \in L^{p_j}(\mathbf{R}_+, dt/t)$ $(j = 0, 1)$ with the norm

$$\|x\|_{K_{\rho,p_0,p_1}} = \inf\{\|\|x_0(t)\|_0/\rho(t)\|_{L^{p_0}(dt/t)} + \|t\|x_1(t)\|_1/\rho(t)\|_{L^{p_1}(dt/t)}\};$$

and the space $J_{\rho,p_0,p_1}(\overline{X})$ consists of all those $x \in \Sigma\overline{X}$ such that there exists a strongly measurable function $u\colon \mathbf{R}_+ \to \Delta\overline{X}$ satisfying $x = \int_0^\infty u(t)\,dt/t$ and $t^j\|u(t)\|_j/\rho(t) \in L^{p_j}(\mathbf{R}_+, dt/t)$ $(j = 0, 1)$ with the norm

$$\|x\|_{J_{\rho,p_0,p_1}} = \inf\{\max_{j=0,1} \|t^j\|u(t)\|_j/\rho(t)\|_{L^{p_j}(dt/t)}\}.$$

In [**6**], the author described these interpolation methods in terms of the Brudnyi–Krugljak methods associated with the quasi-power parameters. Now we define $\varphi\colon \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ by

$$\varphi^{-1}(t) = t^{1/p_0}\rho(t^{-1/q}), \tag{3.1}$$

where $1/q = 1/p_0 - 1/p_1$. Observe that φ is a Young function satisfying both Δ_2 and ∇_2 conditions. Let Φ be the weighted Orlicz space of all measurable functions $f\colon \mathbf{R}_+ \longrightarrow \mathbf{R}$ such that

$$\int_0^\infty \varphi(t^{-q/p_0}|f(t)|)t^q\,dt/t < \infty,$$

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which is equipped with the Luxemberg norm. If $p_0 \neq p_1$, then

$$K_\Phi(\overline{X}) = K_{\rho,p_0,p_1}(\overline{X}) = J_{\rho,p_0,p_1}(\overline{X}) = J_\Phi(\overline{X}).$$

By [**7**, Proposition 2.2], if $T \in \mathcal{B}(\overline{X}, \overline{Y})$, then

$$\|T\|_{K_\Phi(\overline{X}),K_\Phi(\overline{Y}_\Phi)} \leq c\bar{\rho}(\|T\|_0, \|T\|_1), \tag{3.2}$$

where c is a positive constant depending on ρ, p_0, p_1.

LEMMA 3.1. *Let ρ and Φ be as given above, and let ρ_Φ be as given in (2.5). Then*

$$\rho_\Phi \simeq \bar{\rho}.$$

Proof. On the one hand, by (2.7) and (3.2), we have

$$\rho_\Phi(t) \lesssim \|\tau_t\|_{K_\Phi(\overline{L}^\infty)} \lesssim \bar{\rho}(\|\tau_t\|_{L_0^\infty}, \|\tau_t\|_{L_1^\infty}) = \bar{\rho}(1, t) = \bar{\rho}(t)$$

for all $t > 0$. On the other hand, let $\widetilde{L}^\varphi = L^\varphi(\mathbf{R}_+, dt)$ be the Orlicz space corresponding to the function φ given in (3.1). Then $\widetilde{L}^\varphi$ is an r.i. space over $(\mathbf{R}, dt)$ with the fundamental function $1/\varphi^{-1}(1/t)$. Moreover,

$$\widetilde{L}^\varphi = K_{\rho,p_0,p_1}(\widetilde{L}^{p_0}, \widetilde{L}^{p_1})$$

by [**7**, (2.2)]. We now have $\kappa_{L^{p_j}}(t) = t^{1/p_j'}$ $(j = 0, 1)$, and $\kappa_{\tilde{L}^\varphi}(t) = t\varphi^{-1}(1/t)$ by [**10**, Example 1]. By Proposition 2.2 and (2.6), we obtain

$$t\varphi^{-1}(1/t) \lesssim \rho_\Phi(t^{1/p_0'}, t^{1/p_1'}) = t^{1-1/p_0}\rho_\Phi(t^{1/q}),$$

and hence

$$\rho(t) = t^{q/p_0}\varphi^{-1}(t^{-q}) \lesssim \rho_\Phi(t).$$

Consequently, $\bar{\rho}(t) \lesssim \rho_\Phi(t)$. □

We can now apply Proposition 2.2 on the interpolation space $K_{\rho,p_0,p_1}(\overline{X})$ and obtain the following.

PROPOSITION 3.3. *Let X_0, X_1 be intermediate spaces for Banach couple $\overline{Y}$, and let $\overline{X} = (X_0, X_1)$. Then*

$$\kappa(t, K_{\rho,p_0,p_1}(\overline{X}); \overline{Y}) \lesssim \bar{\rho}(\kappa(t, X_0; \overline{Y}), \kappa(t, X_1; \overline{Y})).$$

In particular,

$$\kappa(t, K_{\rho,p_0,p_1}(\overline{X}); \overline{X}) \lesssim \bar{\rho}(t).$$

4. Connection with growth and continuity envelope functions. The idea of growth and continuity envelopes in function spaces comes from the classical Sobolev embedding theorems. Let X be a Banach fumction space over $\mathbf{R}^n$ equipped with the

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Lebesgue measure. The growth envelope function $\mathcal{E}_G^X : \mathbf{R}_+ \to \mathbf{R}_+ \cup \{0\} \cup \{\infty\}$ of X is defined by

$$\mathcal{E}_G^X(t) = \sup\{ f^*(t) \mid \|f\|_X \le 1 \} \tag{4.1}$$

for $t > 0$. By [**8**, Proposition 3.4], $X \subseteq L^\infty(\mathbf{R}^n)$ with the Lebesgue measure on $\mathbf{R}^n$ iff $\mathcal{E}_G^X$ is bounded.

Our first result is an extension of [**9**, Remark 3.4].

PROPOSITION 4.1. *Assume that X_0 and X_1 are intermediate spaces for the Banach couple $(L^1(\mathbf{R}^n), L^\infty(\mathbf{R}^n))$ such that $X_j \not\subseteq L^\infty(\mathbf{R}^n)$ $(j = 0, 1)$. Let $\overline{X} = (X_0, X_1)$, and let $X = K_\Phi(\overline{X})$. Then*

$$\mathcal{E}_G^X(t) \le \frac{c}{t} \int_0^t \rho_\Phi(\mathcal{E}_G^{X_0}(v), \mathcal{E}_G^{X_1}(v))dv \qquad \textit{for} \quad t > 0,$$

where c is a positive constant depending on Φ.

Proof. We denote

$$L^\infty_{\rho_\Phi} = \left\{ f \in L^0(\mathbf{R}_+, dt/t) \,\middle|\, \|f\|_{L^\infty_{\rho_\Phi}} = \sup_{t>0} \left| \frac{f(t)}{\rho_\Phi(t)} \right| < \infty \right\}.$$

Let $\lambda_j = \sup\{ s > 0 \mid \mathcal{E}_G^{X_j}(s) > 0 \}$ $(j = 0, 1)$, and assume $t < \lambda_0 \wedge \lambda_1$. By Proposition 2.1 and by a similar argument as in the proof of [**2**, Theorem 3.9.1], we have $X = K_\Phi(\overline{X}) \subseteq K_{L^\infty_{\rho_\Phi}}(\overline{X})$. Thus, there exists a positive constant c depending on Φ such that

$$K(s, f; \overline{X}) \le c\rho_\Phi(s)$$

for all $s > 0$ and for all $f \in X$ with $\|f\|_X \le 1$. Given $\varepsilon > 0$, there is a decomposition $f = f_0(s) + f_1(s)$ with

$$\|f_0(s)\|_{X_0} + s\|f_1(s)\|_{X_1} \le c(1 + \varepsilon)\rho_\Phi(s)$$

for $s > 0$. In particular, we have

$$\|f_0(s)\|_{X_0} \le c(1 + \varepsilon)\rho_\Phi(s) \quad \text{and} \quad \|f_1(s)\|_{X_1} \le c(1 + \varepsilon)\rho_\Phi(s)/s.$$

If we choose $s = \mathcal{E}_G^{X_1}(t)/\mathcal{E}_G^{X_0}(t)$, then we obtain by (4.1)

$$(f_j)^*(t) \le c(1 + \varepsilon)\rho_\Phi(s)s^{-j}\mathcal{E}_G^{X_j}(t) \le c(1 + \varepsilon)\rho_\Phi\big(\mathcal{E}_G^{X_0}(t), \mathcal{E}_G^{X_0}(t)\big)$$

$(j = 0, 1)$ for $t > 0$. By using the sub-additivity of the function

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(v)dv$$

given in [**1**, Theorem II.3.4], and the inequality $f^*(t) \le f^{**}(t)$, we obtain the result. □

Let ϕ_X be the fundamental function of X. Then by [**8**, Proposition 3.21],

$$\mathcal{E}_G^X(t) = \frac{1}{\phi_X(t)} \tag{4.2}$$

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for $t > 0$. Furthermore, the r.i. space X is an exact interpolation space for the Banach couple $\overline{Y} = (L^1(\mathbf{R}^n), L^\infty(\mathbf{R}^n))$. Observe that, for any $f \in \Sigma\overline{Y}$,

$$K(t, f; \overline{Y}) = \int_0^t f^*(v)dv \qquad \text{for} \quad t > 0. \tag{4.3}$$

By [**10**, Example 1], (4.2) and (4.3), we have

$$\mathcal{E}_G^X(t) = \frac{\kappa(t, X; \overline{Y})}{t} \tag{4.4}$$

for $t > 0$. Next result is an application of Proposition 2.2 and generalises [**9**, Proposition 3.1].

Proposition 4.2. *Let $\overline{X}$ be a Banach couple of rearrangement invariant spaces over $\mathbf{R}^n$, and let $X = K_\Phi(\overline{X})$. Then*

$$\mathcal{E}_G^X(t) \lesssim \rho_\Phi\big(\mathcal{E}_G^{X_0}(t), \mathcal{E}_G^{X_1}(t)\big) \qquad \textit{for} \quad t > 0.$$

Let $C(\mathbf{R}^n)$ be Banach space of all complex-valued bounded uniformly continuous functions over $\mathbf{R}^n$, equipped with the sup-norm. For $f \in C(\mathbf{R}^n)$ and for $t > 0$, the modulus of continuity $\omega(f, t)$ is given by

$$\omega(f, t) = \sup_{|h| \le t} \sup_{x \in \mathbf{R}^n} |f(x + h) - f(x)|.$$

Let $\mathrm{Lip}^1(\mathbf{R}^n)$ be the classical Lipschitz space of all functions $f \in C(\mathbf{R}^n)$ satisfying

$$\sup_{0<t<1} \frac{\omega(f, t)}{t} < \infty,$$

equipped with the norm

$$\|f\|_{\mathrm{Lip}^1(\mathbf{R}^n)} = \|f\|_{C(\mathbf{R}^n)} + \sup_{0<t<1} \frac{\omega(f, t)}{t}.$$

If $X \subseteq C(\mathbf{R}^n)$ is a Banach function space over $\mathbf{R}^n$, then the continuity envelope function $\mathcal{E}_C^X : \mathbf{R}^+ \to \mathbf{R}^+ \cup \{0\} \cup \{\infty\}$ of X is defined by

$$\mathcal{E}_C^X(t) = \sup\left\{ \frac{\omega(f, t)}{t} \,\middle|\, \|f\|_X \le 1 \right\} \tag{4.5}$$

for $t > 0$. By [**8**, Proposition 5.3], $X \subseteq \mathrm{Lip}^1(\mathbf{R}^n)$ iff $\mathcal{E}_C^X$ is bounded. Recall that, for any $f \in C(\mathbf{R}^n)$,

$$K(t, f; C(\mathbf{R}^n), \mathrm{Lip}^1(\mathbf{R}^n)) \simeq \omega(f, t) \qquad \text{for} \quad t > 0.$$

Thus, if X is an intermediate space for the Banach couple

$$\overline{Y} = (C(\mathbf{R}^n), \mathrm{Lip}^1(\mathbf{R}^n)),$$

then

$$\mathcal{E}_C^X(t) \simeq \frac{\kappa(t, X; \overline{Y})}{t} \qquad \text{for} \quad t > 0.$$

The counterpart of [**9**, Proposition 3.5] in our situation is formulated as follows.

Proposition 4.3. *Assume that X_0 and X_1 are intermediate spaces for the Banach couple $\left(C(\mathbf{R}^n), Lip^1(\mathbf{R}^n)\right)$ such that $X_j \nsubseteq Lip^1(\mathbf{R}^n)$ $(j = 0, 1)$. Let $\overline{X} = (X_0, X_1)$, and let $X = K_\Phi(\overline{X})$. Then*

$$\mathcal{E}_C^X(t) \lesssim \rho_\Phi\left(\mathcal{E}_C^{X_0}(t), \mathcal{E}_C^{X_1}(t)\right) \qquad \textit{for} \quad t > 0.$$

We conclude this section by a result concerning the behaviour of the growth envelope functions for the Orlicz–Besov and Orlicz–Triebel–Lizorkin spaces when $t \to \infty$. Let $\mathcal{T}(\mathbf{R}^n)$ be the Schwartz class of test functions on $\mathbf{R}^n$ with the dual space $\mathcal{T}'(\mathbf{R}^n)$. Given $f \in \mathcal{T}'(\mathbf{R}^n)$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ the Fourier transform and the inverse Fourier transform of f, respectively. Suppose that $\{\psi_\nu\}_\nu$ is a smooth dyadic resolution of unity. Let $s \in \mathbf{R}$, $1 \le r \le \infty$ and φ be a Young function satisfying both Δ_2 and ∇_2 conditions, and let $L^\varphi = L^\varphi(\mathbf{R}^n)$. The Orlicz–Besov space $B^s_{\varphi,r} = B^s_{\varphi,r}(\mathbf{R}^n)$ consists of all $f \in \mathcal{T}'(\mathbf{R}^n)$ such that

$$\|f\|_{B^s_{\varphi,r}} = \left(\sum_{\nu=0}^{\infty}\left(2^{\nu s}\left\|\mathcal{F}^{-1}\psi_\nu\mathcal{F}f\right\|_{L^\varphi}\right)^r\right)^{1/r} < \infty,$$

and the Orlicz–Triebel–Lizorkin space $F^s_{\varphi,r} = F^s_{\varphi,r}(\mathbf{R}^n)$ consists of all $f \in \mathcal{T}'(\mathbf{R}^n)$ such that

$$\|f\|_{F^s_{\varphi,r}} = \left\|\left(\sum_{\nu=0}^{\infty} 2^{\nu s}\left|\mathcal{F}^{-1}\psi_\nu\mathcal{F}f(\,\cdot\,)\right|^r\right)^{1/r}\right\|_{L^\varphi} < \infty,$$

with the usual modification if $r = \infty$ in both cases. In case $\varphi_p(t) = t^p$ with $1 < p < \infty$, we denote $B^s_{p,r} = B^s_{\varphi_p,r}$ and $F^s_{p,r} = F^s_{\varphi_p,r}$.

Proposition 4.4. *If $s > 0$, then*

$$\bar{\varphi}^{-1}(1/t) \lesssim \mathcal{E}_G^{B^s_{\varphi,r}}(t) \lesssim \varphi^{-1}(1/t)$$

and

$$\bar{\varphi}^{-1}(1/t) \lesssim \mathcal{E}_G^{F^s_{\varphi,r}}(t) \lesssim \varphi^{-1}(1/t)$$

when $t \to \infty$.

Proof. Let

$$\underline{p}_\varphi = \inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)} \quad \text{and} \quad \bar{p}_\varphi = \sup_{t>0} \frac{t\varphi'(t)}{\varphi(t)}.$$

Then $1 < \underline{p}_\varphi \le \bar{p}_\varphi < \infty$. If we choose p_0 and p_1 satisfying

$$1 < p_0 < \underline{p}_\varphi \le \bar{p}_\varphi < p_1 < \infty,$$

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and define $\rho\colon \mathbf{R}_+ \to \mathbf{R}_+$ by

$$\rho(t) = t^{q/p_0}\varphi^{-1}(t^{-q}),$$

then by [7, Proposition 3.1], we have

$$B^s_{\varphi,r} = K_{\rho,p_0,p_1}\big(B^s_{p_0,r}, B^s_{p_1,r}\big). \tag{4.6}$$

and

$$F^s_{\varphi,r} = K_{\rho,p_0,p_1}\big(F^s_{p_0,r}, F^s_{p_1,r}\big). \tag{4.7}$$

It is enough to show the estimates for $\mathcal{E}_G^{B^s_{\varphi,r}}$. According to the proof of [8, Theorem 10.19], we have

$$B^s_{p_j,r} \subseteq L^{p_j}(\mathbf{R}^n) \quad (j = 0, 1).$$

This, together with [7, (2.2)] and (4.6), implies that $B^s_{\varphi,r} \subseteq L^\varphi$, and hence

$$\mathcal{E}_G^{B^s_{\varphi,r}}(t) \lesssim \mathcal{E}_G^{L^\varphi}(t) = \varphi^{-1}(1/t)$$

as $t \to \infty$. Conversely, if $f \in B^s_{p_j,r}$ and $0 < R \le 1$, then by [8, (10.22)], we have

$$\|f(R\cdot)\|_{B^s_{p_j,r}} \lesssim R^{-n/p_j}\|f\|_{B^s_{p_j,r}} \quad (j = 0, 1),$$

and hence by (4.6) again,

$$\begin{aligned}\|f(R\cdot)\|_{B^s_{\varphi,r}} &\lesssim \bar{\rho}\big(\|f(R\cdot)\|_{B^s_{p_0,r}}, \|f(R\cdot)\|_{B^s_{p_1,r}}\big)\\ &\lesssim \bar{\rho}\big(R^{-n/p_0}\|f\|_{B^s_{p_0,r}}, R^{-n/p_1}\|f\|_{B^s_{p_1,r}}\big)\\ &\lesssim \bar{\varphi}^{-1}(R^{-n})\bar{\rho}\big(\|f\|_{B^s_{p_0,r}}, \|f\|_{B^s_{p_1,r}}\big).\end{aligned}$$

Choose the compactly supported C^∞-function f in $\mathbf{R}^n$ as given in [8, Example 2.6], and set

$$f_R(x) = \bar{\varphi}^{-1}(R^{-n})f(Rx)$$

for $x \in \mathbf{R}^n$. Then

$$\|f\|_{B^s_{\varphi,r}} \lesssim \bar{\rho}\left(\|f\|_{B^s_{p_0,r}}, \|f\|_{B^s_{p_1,r}}\right) < \infty$$

by the above estimates, and $f_R^*(t) \simeq \bar{\varphi}^{-1}(R^n)$ for $t \simeq R^{-n}$ with $0 < R \le 1$. Therefore,

$$\mathcal{E}_G^{B^s_{\varphi,r}}(t) \ge \sup_{0<R<1} f_R^*(t) \gtrsim \bar{\varphi}(1/t)$$

for large $t > 1$, as stated in the end of the proof of [8, Theorem 10.19]. By using (4.7), the estimate for $\mathcal{E}_G^{F^s_{\varphi,r}}(t)$ can be obtained in a similar way. □

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