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Semi-classical Asymptotics for the Schrödinger Operator with Oscillating Decaying Potential

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Abstract. We study the distribution of the discrete spectrum of the Schrödinger operator perturbed by a fast oscillating decaying potential depending on a small parameter h.

1 Introduction

This note is devoted to the study of the discrete spectrum of the operator

$$H(h) := -\Delta_y + V(hy, y),$$

where Δ_y is the usual Laplacian with respect to $y \in \mathbb{R}^n$ and $h > 0$. The function $(x, y) \mapsto V(x, y)$ is smooth, real-valued, and Γ- periodic on y. Suppose in addition that V is bounded with all its derivatives and satisfies

$$\lim_{|x|\to+\infty} \sup_{y\in\mathbb{R}^n/\Gamma} |V(x, y)| = 0. \tag{1.1}$$

The operator $H := -\Delta$ in $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$ is self-adjoint; its discrete spectrum is empty, while the essential one coincides with $[0, +\infty[$. Under the above hypothesis, the operator $H(h)$ admits a unique self-adjoint realization in $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$. Moreover, the essential spectrum of $H(h)$ and H are the same. In $]-\infty, 0[$ we have a discrete spectrum caused by the potential V.

There are many works on the location of the absolutely continuous spectrum of the Schrödinger operator with oscillating decaying potential (see [1, 2, 7–9, 23, 24, 33] and the references given there).

The asymptotic behaviour of the discrete spectrum of $H(1) = -\Delta + V(y, y)$ near the origin was studied in [25].

In the one-dimensional case, the existence and the asymptotic behaviour of the eigenvalues of the operator $Q(h) = -\partial_x^2 + V_0(x) + V(x, \frac{x}{h})$, tending to the border of the essential spectrum as $h \searrow 0$, were established in [5] for $V_0 = 0$, and in [15] for periodic potential V_0 (see also [4, 5, 14, 16, 17]). Our problem here is different. In fact, the scaling of $H(h)$ is that of semiclassical analysis. In particular, the number of discrete eigenvalues grows as $h \searrow 0$ and satisfies a Weyl type asymptotics. To our

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best knowledge, there has been no work so far treating the semiclassical asymptotics of the Schrödinger operator with oscillating decaying potential.

In this paper, for $f \in C_0^\infty(]-\infty, 0[;\mathbb{R})$, we give a complete asymptotic expansion of the trace of $f(H(h))$ in powers of h. We also establish a Weyl-type asymptotics formula with optimal remainder estimate. Our results depend on the Floquet eigenvalues of a periodic Schrödinger operator depending on the variable "x" (see (2.1)). The proof is similar in spirit to the one in [11] and based on the effective Hamiltonian method (see Subsection 2.2).

The paper is organized as follows: In the next section, we formulate our main results and draw conclusions and comments on it. We give an outline of the proofs in Subsection 2.2. We introduce a class of symbols and the corresponding h-Weyl operators (see Subsection 3.2). In Subsections 3.1 and 3.3 we recall the effective Hamiltonian method. The proofs of the main results are given in Section 4.

Notation We employ the following standard notations. Given a complex function f_h depending on a small positive parameter h, the relation $f_h = \mathcal{O}(h^N)$ means that there exists $C_N, h_N > 0$ such that $|f_h| \le C_N h^N$ for all $h \in]0, h_N[$. The relation $f_h = \mathcal{O}(h^\infty)$ means that, for all $N \in \mathbb{N} = \{0,1,2,\dots\}$, we have $f_h = \mathcal{O}(h^N)$. We write $f_h \sim \sum_{j=0}^\infty a_j h^j$ if, for each $N \in \mathbb{N}$, we have $f_h - \sum_{j=0}^N a_j h^j = \mathcal{O}(h^{N+1})$.

Let H be a Hilbert space. The scalar product in H will be denoted by $\langle\cdot,\cdot\rangle$. The set of linear bounded operators from H_1 to H_2 is denoted by $\mathcal{L}(H_1, H_2)$ and $\mathcal{L}(H_1)$ in the case where $H_1 = H_2$.

2 Preliminaries and Results

Let $\Gamma = \bigoplus_{i=1}^n \mathbb{Z}e_i$ be a lattice generated by the basis $e_1, e_2, \dots, e_n \in \mathbb{R}^n$. The reciprocal lattice Γ^* is defined as the lattice generated by the dual basis $\{e_1^*, \dots, e_n^*\}$ determined by $e_j \cdot e_i^* = 2\pi\delta_{ij}$, $i, j = 1, \dots, n$. Let E and E^* be fundamental domains for Γ and Γ^*, respectively. If we identify opposite edges of E (resp. E^*), then it becomes a flat torus denoted by $\mathbb{T} = \mathbb{R}^n/\Gamma$ (resp. $\mathbb{T}^* = \mathbb{R}^n/\Gamma^*$).

Le V be as above. For (x,ξ) fixed in $\mathbb{R}^{2n}$, we define

$$P(x,\xi) := (D_y + \xi)^2 + V(x,y) : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T}) \tag{2.1}$$

as unbounded operator with domain $H^2(\mathbb{T})$. The Hamiltonian $P(x,\xi)$ is semibounded and self-adjoint. Since the resolvent of $(D_y + \xi)^2$ is compact, the resolvent of $P(x,\xi)$ is also compact, and therefore $P(x,\xi)$ has a complete set of (normalized) eigenfunctions $\Phi_n(\cdot, x, \xi) \in H^2(\mathbb{T})$, $n \in \mathbb{N}$, called Bloch functions. The corresponding eigenvalues accumulate at infinity, and we enumerate them according to their multiplicities,

$$\lambda_1(x,\xi) \le \lambda_2(x,\xi) \le \cdots. \tag{2.2}$$

Since $e^{-iy\cdot\gamma^*} P(x,\xi) e^{iy\cdot\gamma^*} = P(x, \xi + \gamma^*)$, it follows that $\xi \mapsto \lambda_m(x,\xi)$ is Γ^*−periodic. The function $\xi \mapsto \lambda_m(x,\xi)$ is called the band function. Standard perturbation theory shows that $\lambda_m(x,\xi)$ is real continuous function and analytic in a neighborhood of

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any ξ_0 such that $\lambda_m(x,\xi_0)$ is simple, *i.e.*,

$$\lambda_{m-1}(x,\xi_0) < \lambda_m(x,\xi_0) < \lambda_{m+1}(x,\xi_0). \tag{2.3}$$

We are now in a position to state our main results.

Theorem 2.1 *Assume (1.1), and let $f \in C_0^\infty(]-\infty,0[;\mathbb{R})$. The operator $f(H(h))$ is of trace class, and there exists a sequence of real numbers $(a_j)_{j\in\mathbb{N}}$ such that*

$$\operatorname{tr}\big[f(H(h))\big] \sim \sum_{j=0}^{\infty} a_j h^{j-n}, \quad h \searrow 0, \tag{2.4}$$

with

$$a_0 = (2\pi)^{-n} \sum_{k\geq 1} \iint_{\mathbb{R}_x^n\times E^*} f\big(\lambda_k(x,\xi)\big)\,dx\,d\xi. \tag{2.5}$$

Let $[a,b] \subset]-\infty,0[$ be an h-independent sub-interval, and let $N([a,b];h)$ denote the number of eigenvalues of $H(h)$ in $[a,b]$ (counted with their multiplicity).

Corollary 2.2 *Under the assumption of Theorem 2.1, we have*

$$\lim_{h\searrow 0}\big[(2\pi h)^n N([a,b];h)\big] = \sum_{k\geq 1} \operatorname{vol}\big\{(x,\xi)\in\mathbb{R}^n\times E^*; \lambda_k(x,\xi)\in[a,b]\big\}. \tag{2.6}$$

Under an additional assumption, we shall improve the above corollary. Fix $b<0$, and let

$$\Sigma_b = \bigcup_{j=1}^{\infty}\big\{(x,\xi)\in\mathbb{R}^n\times E^*;\ \lambda_j(x,\xi)=b\big\}.$$

We make the following assumption :

H : for all $(x_0,\xi_0)\in\Sigma_b$, $\lambda_j(x_0,\xi_0)$ satisfies (2.3) and $\nabla_{x,\xi}\lambda_j(x_0,\xi_0)\neq 0$.

Theorem 2.3 *Under the condition stated above, we have*

$$(2\pi h)^n N(]-\infty,b];h) = \sum_{j\geq 1}\operatorname{vol}\big\{(x,\xi)\in\mathbb{R}^n\times E^*;\lambda_j(x,\xi)\leq b\big\} + \mathcal{O}(h),\ (h\searrow 0).$$

Notice that if V is positive, then the set of discrete spectrum is empty. In particular, the leading terms of the above asymptotics are all zero. The following result can be useful.

Theorem 2.4 *We suppose that there exists $x_0\in\mathbb{R}^n$ such that $\int_E V(x_0,y)dy < 0$. Then $\lambda_1(x_0,0)<0$. In particular, for b small enough, the right-hand sides of (2.5) and (2.6) are strictly positive.*

Remark 2.5

(i) Notice that only a finite number of terms in the above sums are non-zero, since $\lim_{m\to\infty}\lambda_m(x,\xi)=+\infty$. On the other hand, since $\sup_{y\in\mathbb{T}}|V(x,y)|\to 0$ as $|x|$ tends to infinity, it follows that $\lim_{|x|\to\infty}\lambda_m(x,\xi)\geq 0$. Thus, we can replace $\mathbb{R}^n\times E^*$ in (2.5) by $K\times E^*$, where K is a compact set in $\mathbb{R}^n$.

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(ii) Here is another way of stating (2.5). Let $\rho(t,x)$ be the integrated density of states corresponding to the operator $-\Delta_y + V(x,y)$ (where x is a parameter), *i.e.*,

$$\rho(t,x) := (2\pi)^{-n} \sum_{m\geq 1} \int_{\{\xi\in E^*;\ \lambda_m(x,\xi)\leq t\}} d\xi.$$

Using integration by parts in (2.5), we obtain

$$a_0 = -\int_{\mathbb{R}^n_x} \int_{\mathbb{R}} f'(t)\rho(t,x)\,dt\,dx.$$

The following result will be useful in the study of the spectral shift function and can be proved in much the same way as Theorem 2.1.

Theorem 2.6 *We assume here that $\{x \in \mathbb{R}^n, V(x,y) \neq 0\} \subset K$, for some compact $K \subset \mathbb{R}^n$ independent of $y \in \mathbb{T}$. For $f \in C_0^\infty(\mathbb{R};\mathbb{R})$, the operator $\Big(f(H(h)) - f(H_0)\Big)$ is of trace class, and there exists a sequence of real numbers $(b_j)_{j\in\mathbb{N}}$ such that*

$$\operatorname{tr}\Big[f(H(h)) - f(H_0)\Big] \sim \sum_{j=0}^{\infty} b_j h^{j-n},\ h \searrow 0,$$

with

$$b_0 = \int_{\mathbb{R}^n_x} \int_{\mathbb{R}} f'(t)\Big[\rho_0(t) - \rho(t,x)\Big]\,dt\,dx.$$

Here $\rho_0(t) = c_n(2\pi)^{-n} t_+^{n/2}$ is the integrated density of states corresponding to $-\Delta$, where c_n is the volume of the unit ball in $\mathbb{R}^n$ and $t_+ = (|t| + t)/2$.

2.1 Comments

(a) Our results remain valid for the periodic Schrödinger operator with oscillating potential. In fact, let $y \mapsto V_0(y)$ be a real-valued Γ-periodic function, and consider the operator

$$P(h) := P + V(hy,y), \quad P = -\Delta_y + V_0(y).$$

The operator P with domain $H^2(\mathbb{R}^n)$ is self-adjoint; its spectrum is the union of finite or infinite sequence of intervals $[\alpha_n,\beta_n]$ called band that are separated by gaps. Under the assumption (1.1) the essential spectra of $P(h)$ and P are the same. In $\mathbb{R}\smallsetminus\sigma(P)$ we have a discrete spectrum caused by the potential V. Let $[a,b]$ be a closed interval such that $[a,b]\cap\sigma(P) = \varnothing$. Replacing $H(h)$ by $P(h)$, Theorems 2.1–2.3 and Corollary 2.2 hold provided that we replace $\lambda_k(x,\xi)$ by $\mu_k(x,\xi)$, where now $\mu_k(x,\xi)$ are the eigenvalues of the periodic hamiltonian

$$P_1(x,\xi) = (D_y + \xi)^2 + V_0(y) + V(x,y) \colon L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T}).$$

(b) Fix $n \geq 3$, and assume for simplicity that $x \mapsto \sup_{y\in\mathbb{T}} |V(x,y)|^{n/2} \in L^1(\mathbb{R}^n)$ and that V is negative. By the Cwikel–Lieb–Rozenblum bound (see, for instance, [22, 28]) it is known that

$$N\Big(\,]-\infty,0[;h\Big) \leq L_n h^{-n} \int_{\mathbb{R}^n} \sup_{y\in\mathbb{T}} |V(x,y)|^{n/2}\,dx,$$

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where the constant L_n depends only on n. Using the above inequality we can prove that (2.6) remains true for $b = 0$. This and more precise results on the discrete spectrum of the perturbed periodic Schrödinger operator near the edges of gaps will be considered in a forthcoming paper with M. Assal.

2.2 Outline of the Proofs

By the change of variable $x = hz$, the operator $H(h)$ is unitarily equivalent to

$$\widetilde{H}(h) = -h^2\Delta_z + V\Big(z, \frac{z}{h}\Big). \tag{2.7}$$

In the case where $V(x,y) = V(x)$ is independent of the periodic variable y, the operator $\widetilde{H}(h)$ is still the semiclassical Schrödinger one, and all our results are well known in this case (see [13, 27] and the references given there).

However, there are two spatial scales in the potential $V(hx,x)$, namely x and $y = hx$, which are completely different when h tends to zero. So $H(h)$ cannot be identified with the semiclassical Schrödinger operator method, which allows us to reduce the spectral study of $H(h)$ to the one of a system of h-pseudodifferential operators $E_{-+}(z,h)$, acting on $L^2(\mathbb{T}^*;\mathbb{C}^N)$ (see Proposition 3.2). Thus, we establish a trace formula involving the effective Hamiltonian $E_{-+}(z,h)$ (see (4.6)). Now, using some standard results on h-pseudodifferential calculus, we prove our results.

3 Effective Hamiltonian Method

3.1 Grushin Problem: Brief Description

In this paragraph we review some of the standard facts on the Grushin problem. Let H_1, H_2 and H_3 be three Hilbert spaces, and let $P \in \mathcal{L}(H_1, H_3)$. Assume that there exists $R_+ \in \mathcal{L}(H_1, H_2)$ and $R_- \in \mathcal{L}(H_2, H_3)$ such that the operator

$$\mathcal{P}(z) = \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : H_1 \times H_2 \longrightarrow H_3 \times H_2$$

is bijective for $z \in \Omega$. Here, Ω is an open bounded set in $\mathbb{C}$. Let

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}$$

be its inverse. We refer to the problem $\mathcal{P}(z)$ as a *Grushin problem* and the operator $E_{-+}(z)$ is called *effective Hamiltonian*. The following properties are consequence of the identities $\mathcal{E} \circ \mathcal{P} = I$ and $\mathcal{P} \circ \mathcal{E} = I$:

$$(P - z) \text{ is invertible if and only if } E_{-+}(z) \text{ is invertible,} \tag{3.1}$$

$$\dim\ker(P - z) = \dim\ker(E_{-+}(z)), \tag{3.2}$$

$$(P - z)^{-1} = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z), \tag{3.3}$$

$$E_{-+}^{-1}(z) = R_+(z - P)^{-1}R_-. \tag{3.4}$$

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On the other hand, since $z \to (P - z)$ is holomorphic, it follows that the operators $E(z), E_\pm(z), E_{-+}(z)$ are also holomorphic in $z \in \Omega$. Moreover, we have

$$\partial_z E_{-+}(z) = E_-(z)E_+(z). \tag{3.5}$$

This identity comes from the fact that $R_\pm$ are independent of z.

3.2 Classes of Symbols and Notations

For $N \in \mathbb{N}$, we denote by $S(\mathbb{R}^{2n}; \mathcal{M}_N(\mathbb{C}))$ the space of $P \in C^\infty\big(\mathbb{R}^{2n}_{x,\xi}; \mathcal{M}_N(\mathbb{C})\big)$ such that for all α and β in $\mathbb{N}^n$ there exists $C_{\alpha,\beta} > 0$ such that

$$\|\partial_x^\alpha \partial_\xi^\beta P(x,\xi)\|_{\mathcal{M}_N(\mathbb{C})} \le C_{\alpha,\beta}, \tag{3.6}$$

where $\mathcal{M}_N(\mathbb{C})$ is the set of $N \times N$-matrices.

If P depends on a semiclassical parameter $h \in]0, h_0]$ and possibly on other parameters as well, we require (3.6) to hold uniformly with respect to these parameters. For h-dependent symbols, we say that $P(x,\xi;h)$ has an asymptotic expansion in powers of h, and we write

$$P(x,\xi;h) \sim \sum_{j=0}^{\infty} P_j(x,\xi)h^j$$

if for every $m \in \mathbb{N}$,

$$h^{-(m+1)}\Big(P - \sum_{j=0}^{m} P_j h^j\Big) \in S\big(\mathbb{R}^{2n}; \mathcal{M}_N(\mathbb{C})\big).$$

For $P \in S(\mathbb{R}^{2n}; \mathcal{M}_N(\mathbb{C}))$, the h-Weyl operator $P = P^w(x, hD_x; h)$ is defined by

$$P^w(x, hD_x; h)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} P\Big(\frac{x+y}{2}, \xi; h\Big) u(y)\, dy\, d\xi.$$

Here, $D_x = \frac{1}{i}\frac{\partial}{\partial x}$. Assume now that $P(x,\xi;h)$ is Γ^*-periodic in x. Then $P^w(x, hD_x; h)$ is well defined and bounded from $L^2(\mathbb{T}^*)$ into $L^2(\mathbb{T}^*)$. In particular, we have a global h-pseudodifferential calculus on the torus in analogy to the one in Euclidean space. In an appendix, we recall some well-known results on the h-pseudodifferential calculus.

3.3 Reduction to a Semiclassical Problem

In this subsection, we recall some results on the effective Hamiltonian method of the perturbed periodic Schrödinger operator. For the convenience of the reader we repeat the relevant material from [18] without proofs, thus making our exposition self-contained. We will only point out the main ideas of the proofs.

In the sequel we fix a compact interval $I = [a,b] \subset \mathbb{R}$, and we denote by T_Γ the distribution in $\mathcal{S}'(\mathbb{R}^{2n})$ defined by $T_\Gamma(x,y) = \sum_{\beta\in\Gamma} \delta(x - hy - h\beta)$. For $m \in \mathbb{N}$, we introduce the following Hilbert space with its natural norm

$$\mathbb{L}^m := \big\{ u(x)T_\Gamma(x,y)\,;\, \partial_x^\alpha u \in L^2(\mathbb{R}^n),\ \forall \alpha, |\alpha| \le m \big\}.$$

Using that

$$\big[(hD_x + D_y)^2 + V(x,y)\big](u(x)T_\Gamma(x,y)) = \Big[\Big(-h^2\Delta_x + V\Big(x, \frac{x}{h}\Big)\Big)u(x)\Big] T_\Gamma(x,y)$$

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and (2.7), it follows easily that the operator $H(h)$ acting on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$ is unitary equivalent to

$$\mathbb{P}(h) := \left(D_y + hD_x\right)^2 + V(x,y) : \mathbb{L}^0 \longrightarrow \mathbb{L}^0 \tag{3.7}$$

with domain $\mathbb{L}^2$. The advantage of using (3.7) lies in the fact that $\mathbb{P}(h)$ is the semi-classical Schrödinger operator with respect to x with symbol $P(x,\xi) = (D_y + \xi)^2 + V(x,y)$.

First, we work on the symbolic level. Using the Floquet theory, we construct the following Grushin problem for the symbol $P(x,\xi)$.

Proposition 3.1 ([18, Proposition 2.1]) *There exist $N \in \mathbb{N}$, a complex neighborhood Ω of I, and a bounded operator r_+ in $\mathcal{L}\left(L^2(\mathbb{T});\mathbb{C}^N\right)$ such that for all $z \in \Omega$ and $0 < h < h_0$ small enough, the operator*

$$\mathcal{P}(x,\xi,z) := \begin{pmatrix} P(x,\xi) - z & r_+^* \\ r_+ & 0 \end{pmatrix} : H^2(\mathbb{T}) \times \mathbb{C}^N \longrightarrow L^2(\mathbb{T}) \times \mathbb{C}^N,$$

is bijective with bounded two-sided inverse

$$\mathcal{E}(x,\xi,z) := \begin{pmatrix} e(x,\xi,z) & e_+(x,\xi,z) \\ e_-(x,\xi,z) & e_{-+}(x,\xi,z) \end{pmatrix}.$$

Here, $e_{-+} \in S(\mathbb{R}^{2d}_{x,\xi}; \mathcal{M}_N(\mathbb{C}))$ is Γ^*-periodic in ξ.

We now turn to the quantization of $\mathcal{P}(x,\xi,z)$ and $\mathcal{E}(x,\xi,z)$. According to Propositions A.1 and A.2, we have

$$\mathcal{P}^w(x,hD_x,z) \circ \mathcal{E}^w(x,hD_x,z) = I + h\mathcal{R}^w(x,hD_x,z;h),$$

with $\|\mathcal{R}^w\| = \mathcal{O}(1)$. By Proposition A.4, the right-hand side of the above equality is invertible for h small enough. Consequently, we have the following proposition.

Proposition 3.2 ([18, Theorem 3.7, Remark 3.9]) *There exist $N \in \mathbb{N}$, a complex neighborhood Ω of I, and a bounded operator R_+ in $\mathcal{L}\left(\mathbb{L}^0; L^2(\mathbb{T}^*;\mathbb{C}^N)\right)$ such that for all $z \in \Omega$ and $0 < h < h_0$ small enough, the operator*

$$\mathcal{P}(z,h) := \begin{pmatrix} \mathbb{P}(h) - z & R_+^* \\ R_+ & 0 \end{pmatrix} : \mathbb{L}^2 \times L^2(\mathbb{T}^*;\mathbb{C}^N) \longrightarrow \mathbb{L}^0 \times L^2(\mathbb{T}^*;\mathbb{C}^N)$$

is bijective with bounded two-sided inverse

$$\mathcal{E}(z,h) := \begin{pmatrix} E(z,h) & E_+(z,h) \\ E_-(z,h) & E_{-+}(z,h) \end{pmatrix}.$$

Here, $E_{-+} := E^w_{-+}(x,hD_x,z;h)$ is an h-pseudodifferential operator with symbol Γ^*-periodic in x and

$$E_{-+}(x,\xi,z;h) \sim \sum_{l\geq 0} E_{l,-+}(x,\xi,z)\, h^l,$$

where $E_{0,-+}(x,\xi,z) = e_{-+}(\xi,-x,z)$ is given in Proposition 3.1.

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For simplicity of notation we ignore the dependence of $E, E_\pm, E_{-+}$ on (z,h). From (2.1), (2.2), (3.2), (3.1), (3.3), (3.4), (3.5), and the above propositions, it follows that

$$\big(z-\mathbb{P}(h)\big)^{-1} = -E + E_+E_{-+}^{-1}E_-, \tag{3.8}$$

$$E_{-+}^{-1} = R_+\big(z-\mathbb{P}(h)\big)^{-1}R_+^*, \tag{3.9}$$

and

$$\partial_z E_{-+} = E_-E_+, \tag{3.10}$$

$$\begin{gathered}\det\big(e_{-+}(x,\xi,z)\big) = 0 \text{ iff } \exists k\in\mathbb{N} \text{ such that } z=\lambda_k(x;\xi),\\ \|(e_{-+}(x,\xi,z))^{-1}\|_{\mathcal{L}(\mathcal{M}_N(\mathbb{C}))} \le \frac{C}{|\Im z|},\\ \dim\ker(P(x,\xi)-z) = \dim\ker(e_{-+}(x,\xi,z)).\end{gathered} \tag{3.11}$$

Remark 3.3 Let $z_0\in\mathbb{R}$, $d=\dim\ker(e_{-+}(x,\xi,z))$ for a fixed (x,ξ). By ordinary perturbation theory (see Kato [21]) we can reorder the eigenvalues $(\lambda_j(z))_{1\le j\le N}$ of $e_{-+}(x,\xi,z)$ to be holomorphic in a neighborhood of $z_0\in\mathbb{R}$ and $\lambda_1(z_0)=\cdots=\lambda_d(z_0)=0$. Using (3.11) we see that $|\lambda_j(z)|\ge C_j|\Im z|$, so $\lambda_j'(z_0)\ne 0$ for all $1\le j\le N$. Hence, $z\mapsto \det e_{-+}(x,\xi,z)$ has a root z_0 of multiplicity d.

4 Proof of the Results

4.1 Proof of Theorem 2.1

Fix $a<b<0$ such that $\operatorname{supp} f\subset\,]a,b[\, =: I$. Let $\varphi(x)\in C^\infty(\mathbf{R}^n_x;[0,1])$ be equal to one for $|x|>2R$ and $\varphi(x)=0$ for $|x|<R$. We fix R large enough such that

$$\sup_{(x,y)\in\mathbb{R}^{2n}} |\varphi(x)V(x,y)| \le \frac{|b|}{2}. \tag{4.1}$$

Let $\widehat{e}_{-+}(x,\xi,z)$ be the effective Hamiltonian given by Proposition 3.1 associated with

$$\widehat{P}(x,\xi) = (D_y+\xi)^2 + \varphi(x)V(x,y),$$

and put

$$\begin{aligned}\widehat{E}_{-+}(x,\xi,z;h) &= \widehat{e}_{-+}(\xi,-x,z) + E_{-+}(x,\xi,z;h) - E^0_{-+}(x,\xi,z).\\ &= \widehat{e}_{-+}(\xi,-x,z) + \sum_{j\ge1} h^j E_{j,-+}(x,\xi,z).\end{aligned} \tag{4.2}$$

By (4.1), we have

$$\langle(\widehat{P}(x,\xi)-z)u,u\rangle \ge \frac{|b|}{2}\|u\|^2, \quad \forall u\in C_0^\infty(\mathbb{T}^*;\mathbb{C}^n),$$

uniformly on $z\in[a,b]$. Combining this with (3.10), we deduce that

$$|\det\widehat{e}_{-+}(x,\xi,z)| \ge \frac{1}{C} \text{ uniformly on } (x,\xi,z)\in\mathbb{R}^n\times\mathbb{T}^*\times[a,b],$$

which together with (4.2) yield, for h small enough,

$$|\det\widehat{E}_{-+}(x,\xi,z;h)| \ge \frac{1}{2C} \text{ uniformly on } (x,\xi,z)\in\mathbb{T}^*\times\mathbb{R}^n\times[a,b]. \tag{4.3}$$

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On the other hand, from the properties of φ, we have

$$E_{-+}(x,\xi,z;h) = \widehat{E}_{-+}(x,\xi,z;h) \text{ for large } \xi.$$

It follows from (4.3) and Proposition A.4 that for h small enough, $(\widehat{E}_{-+})^{-1}$ is well defined and holomorphic for z near $[a,b]$ and

$$\|(\widehat{E}_{-+})^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^*;\mathbb{C}^N))} = \mathcal{O}(1).$$

Let $\widetilde{f} \in C_0^\infty((a,b)+i[-1,1])$ be an almost analytic extension of f, *i.e.*, $\widetilde{f} = f$ on $\mathbb{R}$ and $\overline{\partial}_z \widetilde{f}$ vanishes on $\mathbb{R}$ to infinite order, *i.e.*, $\overline{\partial}_z \widetilde{f}(z) = \mathcal{O}_N(|\Im z|^N)$ for all $N \in \mathbb{N}$. Then the functional calculus due to Helffer-Sjöstrand (see *e.g.*, [13, Chapter 8]) yields

$$f(\mathbb{P}) = -\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)(z-\mathbb{P})^{-1} L(dz).$$

Here $L(dz) = dxdy$ is the Lebesgue measure on the complex plane $\mathbb{C} \sim \mathbb{R}^2_{x,y}$. The identity

$$E_{-+}^{-1} = \widehat{E}_{-+}^{-1} - E_{-+}^{-1}(E_{-+} - \widehat{E}_{-+})\widehat{E}_{-+}^{-1},$$

combined with (3.8) and the fact that $\widehat{E}_{-+}^{-1}, E, E_+, E_-$ are holomorphic in z near $[a,b]$, give

$$f(\mathbb{P}) = -\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)\Big(E_+ E_{-+}^{-1}(\widehat{E}_{-+} - E_{-+})\widehat{E}_{-+}^{-1}E_-\Big)L(dz). \tag{4.4}$$

In the above equality we have used the fact that $\int \overline{\partial}_z \widetilde{f}(z)K(z)L(dz) = 0$ provided that $K(z)$ is holomorphic in a neighborhood of $\operatorname{supp}\widetilde{f}$.

By Proposition A.3, $(E_{-+} - \widehat{E}_{-+})$ is of trace class and we can take the trace and permute integration and the operator tr in (4.4). The identity $\partial_z E_{-+} = E_- E_+$ shows that for $\Im z \neq 0$,

$$\operatorname{tr}\Big(E_+ E_{-+}^{-1}(\widehat{E}_{-+} - E_{-+})\widehat{E}_{-+}^{-1}E_-\Big) = \operatorname{tr}\Big(E_{-+}^{-1}(\widehat{E}_{-+} - E_{-+})\widehat{E}_{-+}^{-1}\partial_z E_{-+}\Big). \tag{4.5}$$

Let $\chi \in C_0^\infty(\mathbb{R}^n_\xi)$ be equal to 1 in a neighborhood of

$$\Pi_\xi\Big(\operatorname{supp}(E_{-+}^0(x,\xi,z) - \widehat{e}_{-+}(\xi,-x,z))\Big),$$

and denote by $\widehat{\chi} = \chi^w(hD_x)$ the corresponding operator on $L^2(\mathbb{T}^*;\mathbb{C}^N)$. Since

$$\Pi_\xi\Big(\operatorname{supp}(E_{0,-+}(x,\xi,z) - \widehat{e}_{-+}(\xi,-x,z))\Big) \cap \operatorname{supp}(1-\chi) = \varnothing,$$

it follows from Proposition A.5 that

$$\|(\widehat{E}_{-+} - E_{-+})\widehat{E}_{-+}^{-1}\partial_z E_{-+}(1-\widehat{\chi})\|_{\mathrm{tr}} = \mathcal{O}(h^\infty).$$

On the other hand, (3.9) yields $\|E_{-+}^{-1}\| = \mathcal{O}(|\Im z|^{-1})$. Hence

$$\|E_{-+}^{-1}(\widehat{E}_{-+} - E_{-+})\widehat{E}_{-+}^{-1}\partial_z E_{-+}(1-\widehat{\chi})\|_{\mathrm{tr}} = \mathcal{O}(h^\infty|\Im z|^{-1}).$$

Combining this equality with (4.4) and (4.5) we obtain

$$\operatorname{tr}\big[f(\mathbb{P})\big] = -\frac{1}{\pi}\operatorname{tr}\Big[\int \overline{\partial}_z \widetilde{f}(z)E_{-+}^{-1}(\widehat{E}_{-+} - E_{-+})\widehat{E}_{-+}^{-1}\partial_z E_{-+}\widehat{\chi}L(dz)\Big] + \mathcal{O}(h^\infty).$$

Splitting the integral into two terms and using the fact that $\widehat{E}_{-+}^{-1}\partial_z\widehat{E}_{-+}$ is holomorphic in z, we get

$$\operatorname{tr}\big[f(\mathbb{P})\big] = -\frac{1}{\pi}\operatorname{tr}\Big[\int \overline{\partial}_z\widetilde{f}(z)E_{-+}^{-1}\partial_z E_{-+}\widehat{\chi}L(dz)\Big] + \mathcal{O}(h^\infty). \tag{4.6}$$

The proof of the following lemma is similar to the one in [11].

Lemma 4.1 *There exists* $r(x,\xi;h) \in S(\mathbb{R}^{2n}, \mathcal{M}_N(\mathbb{C}))$ *such that*

$$r(x,\xi;h) \sim \sum_{j\ge 0} h^j r_j(x,\xi)$$

and

$$\operatorname{Op}_h^w(r(x,\xi;h)) = -\frac{1}{\pi}\int_{|\Im z|\ge h^\delta} \overline{\partial}_z\widetilde{f}(z)(E_{-+})^{-1}\partial_z E_{-+}L(dz).$$

Moreover, r_j *is* Γ^**-periodic in* x *for all* $j \ge 0$ *with:*

$$r_0(x,\xi) = -\frac{1}{\pi}\int \overline{\partial}_z\widetilde{f}(z)(E_{0,-+}(x,\xi,z))^{-1}\partial_z E_{0,-+}(x,\xi,z)L(dz).$$

If we restrict the integral in the right-hand side of (4.6) to the domain $|\Im z| \le h^\delta$, then we get a term $\mathcal{O}(h^\infty)$ in trace norm. Here we have used the fact that $\overline{\partial}_z\widetilde{f}(z) = \mathcal{O}_N(|\Im z|^N)$ for all $N \in \mathbb{N}$. If we restrict our attention to the domain $|\Im z| \ge h^\delta$, then by Lemma 4.1 and Proposition A.3 we get (2.4). To finish the proof let us compute a_0. We have

$$\begin{aligned} a_0 &= \iint_{E^*\times\mathbb{R}^n} \widehat{\operatorname{tr}}[r_0(x,\xi)]dxd\xi = \iint_{E^*\times\mathbb{R}^n} \widehat{\operatorname{tr}}[r_0(x,\xi)]dxd\xi \\ &= \iint_{E^*\times\mathbb{R}^n}\Big(-\frac{1}{\pi}\int \overline{\partial}_z\widetilde{f}(z)\widehat{\operatorname{tr}}\big[(E_{0,-+}(x,\xi,z))^{-1}\partial_z E_{0,-+}(x,\xi,z)\big]L(dz)\Big)dxd\xi. \end{aligned}$$

Here $\widehat{\operatorname{tr}}$ denotes the trace in the set of square matrices. Thanks to Liouville's formula (*i.e.*, $\widehat{\operatorname{tr}}(\partial_z A(z)A^{-1}(z)) = \frac{\partial_z \det A(z)}{\det A(z)}$ in the sense of matrices), we get

$$a_0 = \iint_{E^*\times\mathbb{R}^n}\Big(-\frac{1}{\pi}\int \overline{\partial}_z\widetilde{f}(z)\frac{\partial_z \det E_{-+}^0(x,\xi,z)}{\det E_{-+}^0(x,\xi,z)}L(dz)\Big)dxd\xi.$$

To prove (2.5) we use Remark 3.3 and the following lemma.

Lemma 4.2 *Let* g *be an analytic function. Let* $(z_k)_{k\ge1}$ *be the roots (counted with their multiplicity) of* g *in* $\operatorname{supp}(\widetilde{f})$. *We have:*

$$\frac{-1}{\pi}\int \overline{\partial}_z\widetilde{f}(z)\frac{g'(z)}{g(z)}L(dz) = \sum_{k\ge1} f(z_k).$$

Proof This follows from the formula $\frac{1}{\pi}\overline{\partial}_z(\frac{1}{z-z_0}) = \delta(\cdot - z_0)$ and the fact that

$$\frac{g'(z)}{g(z)} = \sum_{k\ge1}\frac{1}{z-z_k} + k(z),$$

where k is holomorphic for z in a small neighborhood of $\operatorname{supp}\widetilde{f}$. ∎

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4.2 Proof of Corollary 2.2

For every small $\epsilon > 0$, choose $\overline{f_\epsilon}, \underline{f_\epsilon} \in C_0^\infty(\mathbb{R}; [0,1])$ with

$$1_{[a+\epsilon, b-\epsilon]} \le \underline{f_\epsilon} \le 1_{[a,b]} \le \overline{f_\epsilon} \le 1_{[a-\epsilon, b+\epsilon]}.$$

It then suffices to observe that

$$\operatorname{tr}\big[\underline{f_\epsilon}(H(h))\big] \le N([a,b];h) \le \operatorname{tr}\big[\overline{f_\epsilon}(H(h))\big],$$

which yields

$$\begin{aligned}\lim_{\epsilon\searrow 0}\lim_{h\searrow 0}\Big((2\pi h)^n \operatorname{tr}\big[\underline{f_\epsilon}(H(h))\big]\Big) &\le \lim_{h\searrow 0}(2\pi h)^n N([a,b];h)\\ &\le \lim_{\epsilon\searrow 0}\lim_{h\searrow 0}\Big((2\pi h)^n \operatorname{tr}\big[\overline{f_\epsilon}(H(h))\big]\Big),\end{aligned}$$

and to apply Theorem 2.1.

4.3 Proof of Theorem 2.3

To prove this theorem one needs a more precise trace formula than Theorem 2.1. Let $\theta \in C_0^\infty(\mathbb{R})$, and put

$$\check{\theta}_h(\tau) := \frac{1}{2\pi h}\int e^{it\tau/h}\theta(t)dt.$$

Analysis similar to that in the proof of (4.6) shows that

$$\begin{aligned}(4.7)\quad \operatorname{tr}\big[f(H(h))\check{\theta}_h(t-H(h))\big] &=\\ \operatorname{tr}\Big[-\frac{1}{\pi}\int \overline{\partial}_z\widetilde{f}(z)\check{\theta}_h(t-z)(E_{-+})^{-1}\partial_z E_{-+}\widehat{\chi}L(dz)\Big] &+ \mathcal{O}(h^\infty),\end{aligned}$$

In the first equality we have used the fact that $\widetilde{f}(z)\check{\theta}_{h^2}(t-z)$ is an almost analytic extension of $f(x)\check{\theta}_h(t-x)$, since $z \mapsto \check{\theta}_h(t-z)$ is analytic. Here, the support of $\widetilde{f}$ is in a small neighborhood of $z = b$. Trace formulas involving effective Hamiltonians like (4.7) were studied in [11].

According to the definition of Σ_b and (3.10) we have

$$\Sigma_b = \{(x,\xi) \in \mathbb{R}^{2n};\ e_{-+}(x,\xi,b) = 0\}.$$

Fix $(x_0,\xi_0) \in \Sigma_b$. Under the assumption of Theorem 2.3 we can choose

$$e_{-+}(x,\xi,z) = \begin{pmatrix} \lambda_{j_i}(x,\xi)-z & 0 & \cdot & \cdot & \cdot & 0\\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & g(x,\xi,z) & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where $\det(g(x,\xi,z)) \ne 0$ for all (x,ξ,z) in in a small neighborhood $\mathcal{W}$ of (x_0,ξ_0,b).

The assumption **H** implies that the principal symbol $e_{-+}(\xi,-x,b)$ of $E_{-+}(b)$ is micro-hyperbolic at every point $(x,\xi) \in \Sigma_b$.

Thus, applying [11, Theorem 1.8] to the left-hand side of (4.7), we obtain

$$\operatorname{tr}\left[f(H(h))\check{\theta}_h(t - H(h))\right] \sim \sum_{j=0}^{\infty} \beta_j h^{j-n}, \quad (h \searrow 0). \tag{4.8}$$

Now Theorem 2.3 follows from Theorem 2.1 and (4.8) by tauberian arguments (see [27, Theorem V-13]).

4.4 Proof of Theorem 2.4

According to (2.1) and (2.2), $\lambda_1(x_0, 0)$ is the first eigenvalue of the operator $P(x_0, 0) : -\Delta + V(x, y) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$. Let $\psi_0(y) = 1$ be the constant function on the torus. By the min-max principle, we have

$$\lambda_1(x_0, 0) = \inf_{\psi \in H^2(\mathbb{T})} \langle P(x_0, 0)\psi, \psi\rangle \le \langle P(x_0, 0)\psi_0, \psi_0\rangle = \int_E V(x_0, y)\, dy,$$

which yields Theorem 2.4.

A Appendix

In this appendix, we recall some well-known results on the h-pseudodifferential calculus. For the proofs we refer to [13].

By X we denote either $\mathbb{R}^{2n}$ or $\mathbb{T}^* \times \mathbb{R}^n$. We recall that

$$S(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})) = \{P \in S(\mathbb{R}^{2n}; \mathcal{M}_N(\mathbb{C}));\ \Gamma^* - \text{ periodic in } x\}.$$

Put $Y = \Pi_x X$ (i.e., $Y = \mathbb{R}^n$ (resp. $\mathbb{T}^*$) for $X = \mathbb{R}^{2n}$ (resp. $\mathbb{T}^* \times \mathbb{R}^n$)).

Proposition A.1 (Composition formula) *Let $a_i \in S(X; \mathcal{M}_N(\mathbb{C}))$, $i = 1, 2$. Then $b^w(y, hD_y; h) = a_1^w(y, hD_y) \circ a_2^w(y, hD_y)$ is an h-pseudo-differential operator, and*

$$b(y, \eta; h) \sim \sum_{j=0}^{\infty} b_j(y, \eta) h^j, \text{ in } S(X; \mathcal{M}_N(\mathbb{C})).$$

Proposition A.2 (L^2– boundedness) *Let $a = a(x, \xi; h) \in S(X; \mathcal{M}_N(\mathbb{C}))$. Then $a^w(x, hD_x; h)$ is bounded : $L^2(Y; \mathbb{C}^N) \to L^2(Y; \mathbb{C}^N)$, and there is a constant C independent of h such that*

$$\|a^w(x, hD_x; h)\| \le C.$$

Proposition A.3 (trace) *Let $a = a(x, \xi; h) \in S(X; \mathcal{M}_N(\mathbb{C}))$. We assume that $\partial_x^\alpha \partial_\xi^\beta a \in L^1(X)$, for all $|\alpha| + |\beta| \le 2n + 2$. Then $a^w(x, hD_x; h)$ is trace class operator and*

$$\operatorname{tr}(a^w(x, hD_x; h)) = \frac{1}{(2\pi h)^n} \iint_Y \widehat{\operatorname{tr}}\big(a(x, \xi; h)\big)\, dx\, d\xi,$$

$$\|a^w(x, hD_x; h)\|_{\operatorname{tr}} \le C_n h^{-n} \sum_{|\alpha|+|\beta|\le 2n+1} \iint_Y \|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\|_{\mathcal{M}_N(\mathbb{C})}\, dx\, d\xi.$$

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Proposition A.4 (invertibility) *Let* $a = a(x, \xi; h) \in S(X; \mathcal{M}_N(\mathbb{C}))$. *We assume that there exists* $C > 0$ *(independent of* h*) such that*

$$|\det a(x, \xi; h)| \geq C.$$

Then, for h *small enough, the operator* $a^w(x, hD_x; h) : L^2(Y) \to L^2(Y)$ *is invertible with uniformly bounded inverse.*

Proposition A.5 *Let* $Q_1, Q_2, Q_3 \in S(X; \mathcal{M}_N(\mathbb{C}))$. *We assume that*

$$\Pi_\xi Q_1 := \overline{\{\xi \in \mathbb{R}^n; Q(x, \xi) \neq 0\}}$$

is compact and $\Pi_\xi Q_1 \cap \Pi_\xi Q_3 = \varnothing$. *Then*

$$\|Q_1^w(x, hD_x) \circ Q_2^w(x, hD_x) \circ Q_3^w(x, hD_x)\|_{\mathrm{tr}} = \mathcal{O}(h^\infty).$$

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