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ON THE MONODROMY GROUPS OF RIEMANN SURFACES OF GENUS $\geqq 1$

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It is well-known [**5**, **19**] that every finite group can appear as a group of automorphisms of an algebraic Riemann surface. Hurwitz [**9**, **10**] showed that the order of such a group can never exceed $84(g-1)$ provided that the genus g is $\geqq 2$. In fact, he showed that this bound is the best possible since groups of automorphisms of order $84(g-1)$ are obtainable for some surfaces of genus g. The problems considered by Hurwitz and others can be considered as particular cases of a more general question: Given a finite group G, what is the minimum genus of the surface for which it is a group of automorphisms? This question has been completely answered for cyclic groups by Harvey [**7**]. Wiman's bound $2(2g+1)$, the best possible, materializes as a consequence. A further step was taken by Maclachlan who answered this question for non-cyclic Abelian groups. He showed [**15**] that if $A = \prod_{i=1}^{n} Z_{m_i}$ with invariants $m_1, \ldots, m_n$, $n > 1$, $m_i | m_{i+1}$ and $|A| > 4$, then the minimum genus g of a surface for which A is a group of automorphisms is given as:

(a) n even and $\neq 2$:

$$\frac{2(g-1)}{|A|} = \min_{n \geqq 2\gamma \geqq 0} \left[2(\gamma - 1) + \sum_{i=1}^{n-2\gamma} \left(1 - \frac{1}{m_i}\right) + \left(1 - \frac{1}{m_{n-2\gamma}}\right) \right]$$

(b) $n = 2$:

$$\frac{2(g-1)}{|A|} = 1 - \frac{1}{m_1} - \frac{2}{m_2}$$

(c) n odd:

$$\frac{2(g-1)}{|A|} = \min_{n > 2\gamma \geqq 0} \left[2(\gamma - 1) + \sum_{i=1}^{n-2\gamma} \left(1 - \frac{1}{m_i}\right) + \left(1 - \frac{1}{m_{n-2\gamma}}\right) \right]$$

(d) 2, 2; 2, 4; 2, 2, 2; 3, 3: $g = 2, 3, 3, 4$ respectively.

The methods used to obtain the above results are combinatorial and involve the representation of compact Riemann surfaces and their automorphism groups as quotient spaces and groups of Fuchsian groups. These methods are fully explained by Macbeath [**13**].

In an earlier paper [**12**], the author has investigated the question: Which finite groups G can appear as monodromy groups of surfaces of

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genus 0? We showed that there are infinitely many such groups which are not solvable and only rather obvious cases of Abelian groups with this property. In this continued investigation, we show that if a Riemann surface has genus 1, then an infinite class of non-solvable (and even simple) groups can appear and very few cyclic and non-cyclic Abelian groups can appear. When the genus is >1, our results are related to the above stated results in the following way: A necessary and sufficient condition for a covering to be Galois is that the order of the monodromy group be equal to n (the number of sheets) [**16**]. Thus, in this instance, the monodromy group must necessarily be represented using the right-regular representation. In fact, for regular coverings the monodromy group $M(R_l)$ must be isomorphic to F_{l-1}/C, where F_{l-1} is the fundamental group of the 2-sphere with l punctures P_λ and where C is the subgroup of F_{l-1} arising from the projection onto S^2 of the closed curves on R_l going through a fixed reference point Q' on $R_{l'}$ which, in turn, is isomorphic [**17**] to the group of deck transformations (a transitive permutation group if and only if the covering is regular) for R_l. In this subinstance, we provide bounds for the maximum order of cyclic and non-cyclic, Abelian groups of deck transformations for each fixed g. These bounds, which are, at times, more limiting since we are in the specific instance $f:R_l \rightarrow N$ with the genus g of $R_l \geqq 2$ and the genus g' of $N = 0$, are recorded in Theorems 6 and 10 so that they can be compared to and contrasted with those above giving a maximum order for any cyclic or Abelian automorphism group with $g \geqq 2$ and $0 \leqq g' \leqq g$.

Galois coverings are theoretically relatively easy to handle. However, they are not primarily the coverings that one encounters since they are usually only of very high degree. As a consequence, they are not the only case of interest. The methods in this paper might ultimately permit a classification of monodromy groups of non-Galois coverings as well. In this case, the monodromy group is no longer isomorphic to the group of deck transformations and the presently known methods will not apply to that situation.

We proceed now with a detailed description of our findings.

The following explicit and well-known construction (for results, see [**6**]) permits a faithful representation of every finite group as a transitive permutation group: Given a group G of finite order and a subgroup H, there is a permutation of the set of distinct right cosets of H

$$(1) \qquad \pi(g) = \begin{pmatrix} Hx \\ Hxg \end{pmatrix}, \quad x \in G \text{ for each } g \in G$$

so that $g \rightarrow \pi(g)$ is a representation of G as a transitive permutation group on these distinct right cosets of H and so that $\pi(g)$ fixes H if and only if $g \in H$. In fact, we can speak of any transitive permutation representation of G as the representation on a subgroup H. (1) will be faithful if and only if H contains no normal subgroup (not excluding H) of G larger than the identity subgroup. This fact demands that the regular representation be the only faithful, transitive representation of an Abelian group.

A second explicit and well-known construction by A. Hurwitz [**9**] shows that a topological n-sheeted Riemann surface R_l having the branch points P_λ, $\lambda = 1, 2, \ldots, l$, is defined by the following information: the points P_λ and a set of permutations π_λ acting on n symbols (the sheets) in such a way that the π_λ generate a transitive permutation group $M(R_l)$ and satisfy the relation

(2) $\quad \prod \pi_i = 1$

where each generator π_i must appear at least once (and possibly more than once) in this product and where 1 denotes the identity permutation. Letting h denote the total number of cycles (including 1-cycles) occurring in all of the π_λ and g denote the genus of R_l, then $\beta = nl - h = 2g + 2n - 2$ is the branching number of R_l. The group $M(R_l)$ generated by the π_λ is called the *ordinary Riemann monodromy group of the surface* R_l.

Coupling the above two constructions, we can determine whether or not a specified finite group G appears as the monodromy group of a Riemann surface of genus g, $g \geqq 0$, by first faithfully representing G as a transitive permutation group using (1) and by then checking to see whether or not Conditions A and B stated below are compatible.

Condition A. The product of the generating permutations $\pi(g_\nu)$ corresponding to a suitably selected set of generators g_ν of G, in some appropriately arranged order, is the identity permutation 1.

Condition B. β_C is equal to $\beta_T = 2g + 2n - 2$ where β_C is obtained by summing all $\sum_{i=1}^{m} (l_i - 1)$ with l_i representing the length of each cycle in the disjoint product of cycles of each $\pi(g_\nu)$ in Condition A and observing m varies with each ν.

If (A) and (B) can be simultaneously satisfied using some subgroup H of index n in G for some genus $g \geqq 0$, then we will say that G is of *type* $MR(g)$. Otherwise, we will say that G is of type $NMR(g)$.

We have shown [**12**], on the one hand, that symmetric, alternating, cyclic and dihedral groups of all orders as well as $PSL(2, 7)$ classify as type $MR(0)$ and, on the other hand, that the quaternion group Q_2, the generalized quaternion group Q_4 of order 16, a particular non-Abelian group T of order 27 as well as all direct products $\prod_{i=1}^{n} \mathbf{Z}_{m_i}$ with $m_i | m_{i+1}$ and $n > 1$ of cyclic groups of order >4 classify as type $NMR(0)$.

In the present investigation, we make a further search in order to determine which among these above cited groups classify as type $MR(1)$ and which classify as type $NMR(1)$. In the course of this investigation, we uncovered the result that all cyclic and non-cyclic, Abelian groups are of type $NMR(1)$ with the exceptions of C_2, C_3, C_4, C_6, $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{Z}_2 \times \mathbf{Z}_4$ and $\mathbf{Z}_3 \times \mathbf{Z}_3$ which are of type $MR(1)$. We also uncovered bounds much like those in [**7**] and [**15**] for each fixed genus $g > 1$, giving a more restricted class of such groups which are possibly of type $MR(g)$. We noticed that Q_2, Q_4 and T first make their appearance as type $MR(g)$ for $g = 2, 4, 2$ respectively.

The fact that all symmetric groups S_n, $n \geqq 2$, on n symbols are of type $MR(g)$, $g \geqq 1$, is a long-standing one. This fact materializes if we simply let $(1\ 2), (1\ 3), \ldots, (1\ n)$ be the generators of the faithful representation of S_n of order $n!$ on its maximal non-normal subgroup of order $(n-1)!$ and then observe that

$$\left[\prod_{i=2}^{n} (1\ i)(1\ i)\right]\left[\prod_{j=1}^{l} (1\ k)(1\ k)\right] = 1$$

where k is an integer $2 \leqq k \leqq n$ and $l = g$, the genus of R_l, and that $\beta_C = 2(n-1) + 2l = \beta_T$ so that (A), (B) are consistent and the conclusion is immediate. Thus, we can state:

THEOREM 1. *Symmetric groups S_n, $n \geqq 2$, are of type $MR(g)$ for each $g \geqq 0$.*

The following calculations permit a large class of simple groups to be classified as type $MR(1)$.

Let

$$s = (3\ 4 \cdots n), \quad t = (1\ 2\ 3) \quad (n \text{ odd})$$

or

$$s = (1\ 2)(3\ 4 \cdots n), \quad t = (1\ 2\ 3) \quad (n \text{ even})$$

be the generators of the faithful representation of order $n!/2$ on its maximal non-normal subgroup of order $(n-1)/2!$. When n is odd,

$$stt = (3\ 4 \cdots n)(1\ 2\ 3)^2 = (3\ 4 \cdots n-1\ n\ 2\ 1)$$

and

$$(stt)^{-1} = (1\ 2\ n\ n-1 \cdots 5\ 4\ 3)$$

so that

$$(st^2)(st^2)^{-1} = (3\ 4 \cdots n)(1\ 2\ 3)^2(1\ 2\ n\ n-1 \cdots 5\ 4\ 3) = 1$$

and

$$\beta_C = (n-3) + 2(2) + (n-1) = \beta_T.$$

When n is even,

$$stt = [(1\ 2)(3\ 4 \cdots n)](1\ 2\ 3)^2 = (1)(2\ 3\ 4 \cdots n)$$

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and

$$(stt)^{-1} = (1)(2\ n\ n-1\cdots 5\ 4\ 3)$$

so that

$$(st)^2(st^2)^{-1} = [(1\ 2)(3\ 4\cdots n)](1\ 2\ 3)^2[(1)(2\ n\ n-1\cdots 5\ 4\ 3)] = 1$$

and

$$\beta_C = 1 + (n-3) + 2(2) + (n-2) = 2n = \beta_T.$$

Thus, (A) and (B) hold for $n \geqq 3$ and we can conclude:

THEOREM 2. *Alternating groups A_n of every possible order $n!/2$, $n \geqq 2$, are of type $MR(1)$.*

The presentation [**8**] of the (simple) projective special linear group of degree 2 over the field of integers mod 7 is

$$PSL(2,7): \left\langle A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \right.$$
$$\left. A^7 = (AB)^3 = B^2 = (A^2BA^4B)^3 = I \right\rangle.$$

Let H be the cyclic subgroup of order 2 in $PSL(2,7)$ generated by B. If $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, is any element in $PSL(2,7)$, then the cosets of H in $PSL(2,7)$ assume the form

$$HC = \left\{ \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}, \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \right\}.$$

Furthermore, by (1), every $D \in PSL(2,7) \rightarrow \begin{pmatrix} HC \\ HCD \end{pmatrix}$. If D has order 3, 4 or 7, then $HC \neq HCD$. For suppose the contrary. Then $CD = (\pm F)C$ for some $F \neq I \in H$ implying that $D = C^{-1}(\pm F)C$ and thus contradicting the fact that elements belonging to the same conjugacy class must have the same order. Thus, every coset is moved upon right multiplication by elements of orders 3, 4 or 7. If $D = B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then it can easily but tediously be shown that exactly four cosets stay fixed while those remaining are moved under the action of (1).

Since $i_G(H) = 84$, $B_T = 168$. Moreover, the above comments force elements of orders 3, 4, 7 to be respectively mapped by (1) into permutations consisting of twenty-eight 3-cycles, twenty-one 4-cycles and twelve 7-cycles with respective contributions of 56, 63, 72 to β_C and B to be mapped into a permutation consisting of forty 2-cycles with contribution of 40 to β_C.

Therefore, A, B and $C = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ generate $PSL(2, 7)$ and

$$\pi(C)\pi(B)\pi(A) = 1 \quad \text{with} \quad \beta_C = 168.$$

(A), (B) apply and we have:

THEOREM 3. *The simple group $PSL(2, 7)$ is of type $MR(1)$.*

Although the above conclusion was formulated in terms of the particular subgroup H of order 2, it could have equally well been formulated in terms of various other subgroups of $PSL(2, 7)$; e.g., the cyclic subgroups $H_1 = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle$ or $H_2 = \left\langle \begin{pmatrix} 3 & 2 \\ 3 & 0 \end{pmatrix} \right\rangle$ of orders 3 and 4, etc. would have led to the above theorem.

The next two results show that although dihedral groups (which are extensions of degree two of cyclic groups) are always of type $MR(1)$, cyclic groups are generally barred from being of this type.

Dihedral groups of order $2n$ have presentation

$$D_n : \langle s, t; s^n = 1, t^2 = 1, tst = s^{-1} \rangle.$$

If we choose $H = \{e\}$ of index $2n$ in D_n, then st and t (both of order 2) will generate D_n and will each be mapped by (1) into n 2-cycles and will each contribute n to $\beta_T = 4n$. Since

$$[\pi(st)]^2[\pi(t)]^2 = 1 \quad \text{and} \quad \beta_C = 4n = \beta_T,$$

we have:

THEOREM 4. *Dihedral groups of every possible order are of type $MR(1)$.*

We give now the following:

THEOREM 5. *Cyclic groups C_n of all orders n, $n \geqq 2$, are of type $NMR(1)$ excepting C_2, C_3, C_4 and C_6 which are of type $MR(1)$.*

Proof. We show easily that each of the exceptionally cited groups is of type $MR(1)$ as follows:

$C_2 = \langle a \rangle$ with $\pi(a) = (1\ 2)$ so that $[\pi(a)]^4 = 1$ and $\beta_C = 4 = \beta_T$.

$C_3 = \langle a \rangle$ with $\pi(a) = (1\ 2\ 3)$ so that $[\pi(a)]^3 = 1$ and $\beta_C = 6 = \beta_T$.

$C_4 = \langle a \rangle$ with $\pi(a) = (1\ 2\ 3\ 4)$, $\pi(a^2) = (1\ 3)(2\ 4)$ so that $[\pi(a)]^2\ \pi(a^2) = 1$ and $\beta_C = 8 = \beta_T$.

$C_6 = \langle a \rangle$ with $\pi(a) = (1\ 2\ 3\ 4\ 5\ 6)$, $\pi(a^2) = (1\ 3\ 5)(2\ 4\ 6)$, $\pi(a^3) = (1\ 4)(2\ 5)(3\ 6)$ so that $\pi(a)\pi(a^2)\pi(a^3) = 1$ and $\beta_C = 12 = \beta_T$.

Since (A), (B) are consistent in each of these instances, the last claim is justified.

We now justify the first claim that no groups among C_5 and C_n, $n > 6$, are of type $MR(1)$. Clearly, C_5 cannot be of type $MR(1)$ since $C_5 = \langle a \rangle$ so that each $\pi(a^i)$, $i = 1, \ldots, 4$, contributes 4 to $\beta_T = 10$ and

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so that $\beta_C \neq \beta_T$ provided that (A) holds. Precisely three observations can be made which will eliminate the possibility $n > 6$:

1. Using one element or the combination of exactly one generator with its inverse, two non-generating elements, a non-generator with a generator or two (or more) not necessarily distinct generators leads to immediate contradictions.

2. Select one generator and two elements of order the least prime p_i in $O(G)$ since the selection of three non-generators leads to a contradiction as well and since any other combination of three elements produces a larger β_C. If $O(G)$ is even, then the elements of order 2 make the smallest contribution to β_C so that

$$\beta_C \geqq [O(G) - 1] + 2[O(G)/2] = 2O(G) - 1$$

and so that β_C cannot be equal to β_T since any other combination of three such elements would never permit the needed contribution of 1 to make $\beta_C = \beta_T$. If $O(G)$ is odd, then elements of order the least prime p_i appearing in $O(G)$ contribute the least to β_C so that

$$\beta_C \geqq [O(G) - 1] + 2\left[O(G) - \prod_{i=1}^{n} \hat{p}_i^{\alpha_i}\right]$$

where $\prod p_i^{\alpha_i}$ is the prime factorization of $O(G)$ and where $\wedge$ indicates to once delete p_i. Thus, $\beta_C > \beta_T = 2O(G)$ since division of both expressions by $O(G)$ yields

$$1 > \frac{1}{O(G)} + \frac{2}{p_i}$$

which is valid for $O(G) > 6$ and $p_i \geqq 3$.

3. The selection of four or more elements yields a β_C which will be larger than any β_C obtained using three elements. Never can $\beta_C = \beta_T$ without violating another needed condition.

Since 1–3 are impossibilities, we can conclude that each C_n, $n = 5$ and $n > 6$, is of type $NMR(1)$.

Although no cyclic group of order larger than $2(2g + 1)$ can be an automorphism group of any surface of genus $\geqq 2$, bounds are now produced to restrict the class of cyclic groups which can possibly be the group of deck transformations for some surface of a fixed genus $g \geqq 2$ over a surface of genus $g' = 0$.

THEOREM 6. *Let C_n, $n \geqq 2$, $n \neq 2, 3, 4, 6$, be given. Let $g \geqq 2$ represent the genus and let p be an odd prime. If*

(i) $O(C_n) = p^\alpha, \alpha > 1,$ *and* $p^\alpha > 2g + p^{\alpha-1}$

or

$$O(C_n) = 2^\alpha, \alpha > 1, \text{ and } 2^{\alpha-2} > g$$

or

(ii) $O(C_n) = p, p > 3$, *and* $p > 2g + 1$

or

$$\text{(iii) } O(C_n) = \prod_{i=1}^{m} p_i^{\alpha_i}, \quad p_1 < \ldots < p_m, m \geqq 2,$$

and either

$$\prod_{i=1}^{m} p_i^{\alpha_i} > 2g\left[\frac{p_1}{p_1 - 1}\right], \alpha_1 \neq 1$$

or

$$\prod_{i=1}^{m} p_i^{\alpha_i} > p_1\left[\frac{2g}{p_1 - 1} + 1\right], \alpha_1 = 1,$$

then C_n *is of type* $NMR(g)$.

Proof. It becomes necessary to choose some set of generators a_i of C_n so that

$$(3) \quad \prod_{i=1}^{m} \pi(a_i) = 1$$

and

$$(4) \quad \beta_C = 2n + 2t = \beta_T, \quad t = g - 1, t > 0$$

are simultaneously satisfied. We show, by case argument, that these restrictions provide the claimed bounds.

(i) $O(C_n) = p^\alpha, p \geqq 3, \alpha > 1$: Elements of orders $p^\alpha, p^\alpha - 1, \ldots, p$ are respectively mapped onto permutations consisting of one p^α-cycle, p $p^{\alpha-1}$-cycles, $\ldots, p^{\alpha-1}$ p-cycles with respective contributions of $p^\alpha - 1$, $p^\alpha - p, \ldots, p^\alpha - p^{\alpha-1}$ to β_T.

Neither (3) nor (4) can be satisfied using exactly one element of C_n.

If two elements occur in product (3), then one of the two must be a generator and the other its inverse. The total contribution to β_T would be $2p^\alpha - 2$ so that (4) would become impossible.

If three elements occur in product (3), then one of the three must be a generator. Consider

$$(5) \quad \pi(a^k)\pi(a^{t_1})\pi(a^{t_2})$$

where a^k is a generator and where each a^{t_i} is either generating or non-generating. We can then conclude that

$$(6) \quad \beta_C \geqq (p^\alpha - 1) + (p^\alpha - 1) + (p^\alpha - p^{\alpha-1}).$$

We need only determine when $\beta_C > \beta_T$ which is equivalent to determining what restrictions must be imposed on α so that the inequality

$$(7) \quad (p^\alpha - 1) + (p^\alpha - 1) + (p^\alpha - p^{\alpha-1}) > 2\,p^\alpha + 2t$$

holds. However, (7) is equivalent to

(8) $p^\alpha > 2g + p^{\alpha-1}$.

If four or more elements appear in product (3), then the total contribution to β_C would be strictly larger than that in (6).

If $O(C_n) = 2^\alpha$, then (7) reduces to

(9) $2^{\alpha-2} > g$.

(ii) $O(C_n) = p, p > 3$: We observe that three generating elements produce the smallest possible β_C so that we must determine when

(10) $(p - 1) + (p - 1) + (p - 1) > 2p + 2t = \beta_T$

holds. However, (10) is equivalent to the claimed result

(11) $p > 2g + 1$.

(iii) $O(C_n) = \prod_{i=1}^m p_i^{\alpha_i}$, $m \geqq 2$, p_i primes with $p_1 < p_2 < \ldots < p_m$ and $\alpha_i \in \mathbf{Z}^+$: Now, all elements are of orders $\prod_{i=1}^m p_i^{\beta_i}$ where

$$\prod_{i=1}^m p_i^{\beta_i} \Big| \prod_{i=1}^m p_i^{\alpha_i}.$$

Elements of order p_1 make the least possible contribution,

$$\prod_{i=2}^m p_i^{\alpha_i} - p_1^{\alpha_1-1}\left[\prod_{i=2}^m p_i^{\alpha_i}\right],$$

to β_T while elements of order p_1^2 ($\alpha > 1$) or p_i ($i = 2, \ldots, m$) make contributions of either

$$\prod_{i=1}^m p_i^{\alpha_i} - p_1^{\alpha_1-2}\left[\prod_{i=2}^m p_i^{\alpha_i}\right] \quad \text{or} \quad \prod_{i=1}^m p_i^{\alpha_i} - p_j^{\alpha_j-1}\left[\prod_{\substack{i=1\\ i\neq j}}^m p_i^{\alpha_i}\right]$$

respectively to β_C.

As in (i) and (ii) above, one or two elements appearing in product (3) is ruled out. Using three (or more) elements, we will be able to determine what restrictions must be placed on the α_i so that $\beta_C > \beta_T$. This amounts to determining for which α_i either

(12) $$\left[\prod_{i=1}^m p_i^{\alpha_i} - 1\right] + \left[\prod_{i=1}^m p_i^{\alpha_i} - 1\right] + \prod_{i=1}^m p_i^{\alpha_i} \Big/ p_1[(p_1 - 1)]$$

or

(13) $$\left[\prod_{i=1}^m p_i^{\alpha_i} - 1\right] + \prod_{i=2}^m p_i^{\alpha_i}(p_1 - 1) + p_1\left[\prod_{i=2}^m p_i^{\alpha_i} - 1\right]$$

is strictly larger than

(14) $$\beta_T = 2\prod_{i=2}^m p_i^{\alpha_i} + 2t$$

corresponding to whether $\alpha_1 \neq 1$ or $\alpha_1 = 1$ respectively. We notice that β_C will always be larger than or equal to either number in (13), (14). These inequalities lead to the respective inequalities

$$(15)\quad \prod_{i=1}^{m} p_i^{\alpha_i} > 2g\left[\frac{p_1}{p_1 - 1}\right] \quad \text{and} \prod_{i=1}^{n} p_i^{\alpha_i} > p_1\left[\frac{2g}{p_1 - 1} + 1\right]$$

which are again imposing definite and needed restrictions on all of the α_i. (8), (11), (15) exhibit the desired bounds.

Wiman determined which C_n are possible groups of automorphisms for a surface of fixed genus $\geqq 2$. Harvey showed [**7**] that Wiman's bound is deducible from the following bound: The minimum genus g of the surface which admits an automorphism of order $N = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ is given by:

$$g = \max\left[2, \frac{p_1 - 1}{2}\frac{N}{p_1}\right], \quad \alpha_1 > 1, N \text{ prime}$$

or

$$g = \max\left[2, \left(\frac{p_1 - 1}{2}\right)\left(\frac{N}{p_1} - 1\right)\right], \quad \alpha_1 = 1.$$

Easy algebraic manipulations show that these bounds are equivalent to those of Theorem 6 above, even though they were derived using a totally disparate technique. Further, while this theorem does not determine specifically which cyclic groups are deck transformation groups, it does eliminate all cyclic groups which cannot possibly be such. A simple calculation, for example, shows that C_2 (actually $MR(g)\ \forall g$) C_3, C_4, C_6 ($n = 3, 4, 6$ to be checked separately), C_8, C_{10} are the only cyclic groups which qualify as deck transformation groups for a surface of genus 2 over a surface of genus 0.

Recognizing that certain dihedral groups are non-cyclic groups of prime power order p^n, $n \geqq 2$, we question whether or not all non-cyclic groups of prime power order p^n, $n \geqq 2$, are of type $MR(1)$. The answer is provided by the following straightforward:

THEOREM 7. *The groups*
(i) $Q_2 = \langle a, b; a^4 = 1, a^2 = (ab)^2 = b^2\rangle$
(ii) $Q_4 = \langle a, b; a^4 = b^2, bab^{-1} = a^{-1}\rangle$
(iii) $T = \langle a, b; a^9 = b^3 = 1, b^{-1}ab = a^4\rangle$
are all of type $NMR(1)$.

Proof. (i) Every subgroup of the quaternion group Q_2 is a normal subgroup with implication that it can only be faithfully represented on the identity subgroup e of index 8 in Q_2. Two elements of order four must be used to generate Q_2 with corresponding permutation generating elements contributing 6 each to β_C. An element of order two is the only element

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which will provide the needed contribution of 4 to $\beta_T = 16$. However, the product of these three elements would not yield (B). This fact bars Q_2's appearance.

(ii) The subgroup $\{1, a^4\}$ of order two is normal in Q_4 and is contained in every subgroup of order 4. This means that Q_4 can be faithfully represented only on e. Elements of orders 2, 4, 8 are respectively mapped by (1) onto permutations consisting of eight 2-cycles, four 4-cycles, two 8-cycles which, in turn, contribute 8, 12, 14 to $\beta_T = 32$. The only possible combinations of elements which will produce β_T are those using either four elements of order 2 or two elements of order 4 and one of order 2. Since a^4 is the only element of order 2 which alone cannot generate Q_4, the first combination is eliminated instantly. If a^4 is coupled with any two elements of order 4, i.e., with any two of a^2, a^6 or a^ib, $0 \leqq i \leqq 7$, then again Q_4 cannot be generated. Q_4 is thus typed as $NMR(1)$.

(iii) The subgroups $\{b^ia^3\}$ $(i = 1, 2)$, $\{b\}$ of index 9 in T and the trivial subgroup e of index 27 in T are the only subgroups which provide faithful representations. For subgroups of index 9, $\beta_T = 18$ while for subgroups of index 27, $\beta_T = 54$.

For the subgroups of index 27, elements of orders 9, 3 respectively map onto permutations consisting of three 9-cycles, nine 3-cycles and, in turn, contribute 24, 18 to β_T. Thus, $\beta_C \neq \beta_T$, a contradiction of (B).

For subgroups of index 9, elements of order 9 map onto permutations consisting of one 9-cycle contributing 8 to β_C while elements of order 3 map onto either two 3-cycles or three 3-cycles with respective contributions of 4, 6 to β_C. The only possible combinations of elements allowing $\beta_C = \beta_T$ are

(i) $O(3)_6$, $O(3)_6$, $O(3)_6$

(ii) $O(9)_8$, $O(3)_4$, $O(3)_6$

(iii) $O(9)_8$, $O(3)_6$, $O(3)_6$

(iv) $O(3)_6$, $O(3)_4$, $O(3)_4$, $O(3)_4$

where $O(t)_i$ means an element of order t contributing i to β_T and where the elements of type $O(3)_4$ are b, b^2, ba^3, ba^6, b^2a^3, b^2a^6 and the elements of type $O(9)_8$ are a, a^2, a^4, a^5, a^7, a^8, ba, ba^4, ba^5, ba^7, ba^8, b^2a^2, b^2a, b^2a^3, b^2a^7, b^2a^8, ba^2, b^2a^4 and the elements of order $O(3)_6$ are a^3, a^6. (i) and (iv) are not possible since such elements cannot generate T and (ii), (iii) are not possible since the products of such elements can never be e. No subgroup of index 9 provides consistency of (A), (B).

T is of type $NMR(1)$.

Knowing that none of Q_2, Q_4, T is of type $MR(1)$, we ask: When can each of these groups first appear as type $MR(g)$? We answer this question in

THEOREM 8. *Q_2, Q_4, T first appear as type $MR(g)$ for $g = 2, 4, 2$ respectively.*

Proof. Q_2: For genus 2, $\pi(b)\pi(ab)\pi(a^3) = 1$ where b, ab, a^3 generate Q_2 and where $\beta_C = \beta_T = 18$. Q_2 is of type $MR(2)$.

Q_4: For genus 4, $\pi(a)\pi(ab)\pi(a^3) = 1$ where b, ab, a^3 generate Q_4 and where $\beta_C = \beta_T = 38$. Q_4 is of type $MR(4)$. We show that Q_4 is not of type $MR(2)$ or $MR(3)$. For genus 2, $\beta_T = 34$ so that β_T can be realized only by using one element of each of the orders 2, 4, 8. No such combination allows both (A), (B) to be the case. Q_4 is of type $NMR(2)$. For genus 3, $\beta_T = 36$. Three elements of order 4 or two elements of order 8 with one element of order 2 alone can produce β_T. Such combinations cannot generate Q_4. Q_4 is typed as $NMR(3)$.

T: For genus 2 and $i_G(H) = 9$, $\beta_T = 20$. The elements b, a, $(ba)^{-1}$ are elements of orders 3, 9, 9 respectively and map onto two 3-cycles, one 9-cycle, one 9-cycle contributing 4, 8, 8 to β_T. $\pi(b)\pi(a)[\pi(ba)^{-1}] = 1$ as well. (A), (B) hold. T is of type $MR(2)$.

In Theorem 5, we established that very few cyclic groups are typed as $MR(1)$. We now prove the same result for non-cyclic Abelian groups.

THEOREM 9. *All direct products* $A = \prod_{i=1}^n \mathbf{Z}_{m_i}$, $m_i | m_{i+1}$, $n > 1$, $m_1 \neq 1$ *of cyclic groups are of type* $NMR(1)$ *with the exceptions of the groups* $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{Z}_2 \times \mathbf{Z}_4$ *and* $\mathbf{Z}_3 \times \mathbf{Z}_3$ *which are of type* $MR(1)$.

Proof. A minimal set of generators for A consists of n elements and can be taken to be the set

$$(16) \quad \{a_1 = (g_1, 1, \ldots, 1), a_2 = (1, g_2, \ldots, 1), \ldots, a_n = (1, 1, \ldots, g_n)\}$$

where g_i is some generator of $\mathbf{Z}_{m_i}$. A can be faithfully represented only on the subgroup 1 of index $\prod_{i=1}^n m_i$ in A so that the number of sheets of the corresponding surface must be $\prod_{i=1}^n m_i$ and β_T must be $2[\prod_{i=1}^n m_i]$.

We determine when (A), (B) will be contradictory which will yield the desired conclusion. In other words, we determine when β_C obtained by the counting argument of (B) will be strictly greater than $\beta_T = 2[\prod_{i=1}^n m_i]$ as predicted by the Riemann-Hurwitz relation for any selected set of generators $\pi(k_i)$ of $\pi(A)$ satisfying the condition $\prod \pi(k_i) = 1$, i.e., we determine when

$$(17) \quad n\left[\prod_{i=1}^n m_i\right] - \sum_{i=1}^n m_1 m_2 \ldots \widehat{m}_i \ldots m_n > 2\prod_{i=1}^n m_i = \beta_T$$

holds where $\wedge$ indicates that m_i is to be deleted from the shown product for each i for any such set $\pi(k_i)$. For then β_C must be $> \beta_T$. Equivalently, we determine when

$$(18) \quad n > 2 + \sum_{i=1}^n \frac{1}{m_i}.$$

Let $m_i = 2$ for each $i = 1, \ldots, n$. Then, (18) $\Leftrightarrow n > 4$ with implication that if all $m_i \geqq 2$ and if n is fixed >4, then (17) is valid and (A),

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(B) are contradictory. Therefore, all $\prod_{i=1}^{n} \mathbf{Z}_{m_i}$, $m_i \geqq 2$, $n > 4$ are immediately typed as $NMR(1)$. It remains to investigate the instances $n \leqq 4$.

When $n = 4$, each $\prod_{i=1}^{n} \mathbf{Z}_{m_i}$ is of type $NMR(1)$ provided that all m_i are not 2. For (17) becomes equivalent to

$$(19) \quad 4 > 2 + \sum_{i=1}^{4} \frac{1}{m_i}.$$

Letting $m_i = 3$ for each $i = 1, \ldots, 4$, (19) $\Leftrightarrow 4 > 10/3$, a validity implying the truth of (17) for all $m_i \geqq 3$. Letting $m_1 = m_2 = m_3 = 2$ and $m_4 = 4$, we find that (19) $\Leftrightarrow 4 > 15/4$, another validity implying the truth of (17) for $m_1 = 2$, $m_2, m_3 \geqq 2$, $m_4 \geqq 4$. $\prod_{i=1}^{4} \mathbf{Z}_2$ is the only such product which still might appear, and it does not appear since $\beta_T = 32$ and since each generator in (16) contributes 8 to β_T with the product of these generators $\neq 1$.

When $n = 3$, (17) reduces to determining when

$$(20) \quad \beta_C \geqq 4\left[\prod_{i=1}^{3} m_i\right] - 2m_1m_2 - m_2m_3 - m_1m_3 > 2\left[\prod_{i=1}^{3} m_i\right] = \beta_T$$

or when

$$(21) \quad 2 > \frac{2}{m_3} + \frac{1}{m_1} + \frac{1}{m_2}.$$

Let $m_i = 3$ for each i. Then (21) $\Leftrightarrow 2 > 4/3$. This means that (20) will be true for all $m_i \geqq 3$. Let $m_1 = m_2 = 2$ and $m_3 = 6$. Then (21) $\Leftrightarrow$ $2 > 4/3$. (20) is thus true for all $m_i = 2$, $m_2 \geqq 2$, $m_3 \geqq 6$. For $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4$, each generator of order 2 contributes 8 and each generator of order 4 contributes 12 to $\beta_T = 32$. No element in $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4$ can make up the slack of 4 in β_T. $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4$ is of type $NMR(1)$. Moreover, $\prod_{i=1}^{3} \mathbf{Z}_2$ is of type $MR(1)$. Simply observe that the three generators together with the inverse of their product satisfy (A), (B).

When $n = 2$, we notice first that

$$\pi[(g_1, 1)]\pi[(1, g_2)]\pi[((g_1)^{-1}, (g_2)^{-1})] = 1$$

where the g_i's are generators of $\mathbf{Z}_3$ each contributing 6 to $\beta_T = 18$ so that (A), (B) hold implying that $\mathbf{Z}_3 \times \mathbf{Z}_3$ is of type $MR(1)$. If $m_1 = m_2 = 2$, then

$$[\pi[(g_1, 1)]]^2[\pi[(1, g_2)]]^2 = 1$$

and $\beta_C = 8 = \beta_T$. Thus, $\mathbf{Z}_2 \times \mathbf{Z}_2$ is also typed as $MR(1)$. If $m_1 = 2$ and $m_2 = 4$, then

$$\pi[(g_1, 1)]\pi[(1, g_2)]\pi[(g_1, g_2)^{-1}] = 1$$

and $\beta_C = 16$ so that $\mathbf{Z}_2 \times \mathbf{Z}_4$ is typed as $MR(1)$. Otherwise, (17) reduces to determining when

$$(22)\quad \beta_C \geqq 2\left[\prod_{i=1}^{2} m_i\right] - \left[\sum_{i=1}^{2} \hat{m}_i\right] + m_1[m_2 - 1] > 2\prod_{i=1}^{2} m_i = \beta_T$$

or equivalently when

$$(23)\quad 1 > \frac{2}{m_2} + \frac{1}{m_1}$$

is true. Let $m_1 = 2$ and $m_2 = 6$. Then, (23) $\Leftrightarrow 1 > 5/6$ and $\mathbf{Z}_2 \times \mathbf{Z}_6$ is of type $NMR(1)$, the implication being that no $\mathbf{Z}_{m_1} \times \mathbf{Z}_{m_2}$ can be of type $MR(1)$ when $m_1 \geqq 2$ and $m_2 \geqq 6$.

Extension to genus 1 from genus 0 only permits the appearance of three new groups, namely $\mathbf{Z}_3 \times \mathbf{Z}_3$, $\mathbf{Z}_2 \times \mathbf{Z}_4$ and $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

We record the following result which can be immediately deduced from the above proof.

THEOREM 10. *If*

(i) $n = 2$ *and* $1 - \dfrac{2}{m_2} - \dfrac{1}{m_1} > \dfrac{2(g-1)}{m_1 m_2}$

or

(ii) $n > 2$ *and*

$$\prod_{i=1}^{n-1} m_i(m_n - 1) + n\prod_{i=1}^{n} m_i - \sum_{i=1}^{n} m_1 \dots \hat{m}_i \dots m_n > 2\prod_{i=1}^{n} m_i + 2(g-1)$$

where $\wedge$ *denotes that* m_i *is to be deleted from the shown product, then* $\prod_{i=1}^{n} \mathbf{Z}_{m_i}$, $m_i | m_{i+1}$, $n > 1$, $m_1 \neq 1$ *cannot be of type* $MR(g)$, $g \geqq 2$.

Again, as expected, elementary algebra shows that the above bounds coincide identically with those of Maclachlan in Theorem 4 of [**15**] provided that we adhere to the absolutely imposed restriction that $\gamma = 0$.

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