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A CHARACTERIZATION OF SOFT HYPERGRAPHS

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ABSTRACT. A hypergraph $H=(X, \mathscr{E})$ is a subtree of a tree (SOFT) hypergraph if there exists a tree T such that $X=V(T)$ and for each $E_i \in \mathscr{E}$ there is a subtree T_i of T such that $E_i = V(T_i)$. It is shown that H is a SOFT hypergraph if and only if $\mathscr{E}$ has the Helly property and $\Omega(\mathscr{E})$, the intersection graph of $\mathscr{E}$, is chordal. Results of Berge and Gavril have previously shown these to be necessary conditions.

The couple $H=(X, \mathscr{E})$ is a hypergraph if X is a finite set and $\mathscr{E}=\{E_i: i \in I\}=\{E_1, E_2, \ldots, E_m\}$ is a family of subsets of X (called edges) with each $E_i \neq \phi$ and $\bigcup_{i \in I} E_i = X$. A family $\mathscr{E}_0 \subseteq \mathscr{E}$ is a *matching* if the edges of $\mathscr{E}_0$ are pairwise disjoint, and $\upsilon(H)$ denotes the maximum cardinality of a matching of H; a subset $X_0 \subseteq X$ is a *transversal* if $X_0 \cap E_i \neq \phi$ for each $i \in I$, and $\tau(H)$ denotes the minimum cardinality of a transversal of H. Clearly $\upsilon(H) \leq \tau(H)$, and if $\upsilon(H)=\tau(H)$ then H is said to be a *Menger system.*

THEOREM 1. [2] *Let $V(T)$ be the vertex set of a tree T and H be a hypergraph with $\mathscr{E}=\{S_1, \ldots, S_m\}$, where each S_i is the vertex set of a subtree of T, and with $X=\bigcup_{i=1}^{t} S_i$. Then H is a Menger system.*

Call hypergraph $H=(X, \mathscr{E})$ a *subtree of a tree* (SOFT) hypergraph if there exists a tree T such that $X=V(T)$ and for each $E_i \in \mathscr{E}$ there is a subtree T_i of T such that $E_i = V(T_i)$. The objective here is a characterization of SOFT hypergraphs. If the connected components of H are $H_1=(X_1, \mathscr{E}_1), \ldots, H_c=(X_c, \mathscr{E}_c)$ then clearly $\upsilon(H)=\upsilon(H_1)+\cdots+\upsilon(H_c)$, $\tau(H)=\tau(H_1)+\cdots+\tau(H_c)$, H is a Menger system if and only if each H_i is, and H is a SOFT hypergraph if and only if each H_i is.

It is said that $\mathscr{E}$ satisfies the *Helly property* if $J \subseteq I$ and $E_i \cap E_j \neq \phi$ for all $i, j \in J$ implies that $\bigcap_{j \in J} E_j \neq \phi$.

THEOREM 2. [1, *p.* 399, *Example* 3]. *If H is a SOFT hypergraph then $\mathscr{E}$ satisfies the Helly property.*

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Let $\Omega(\mathscr{E})$ denote the intersection graph on $\mathscr{E}$, that is, the vertex set of $\Omega(\mathscr{E})$ is $\{e_1, e_2, \ldots, e_m\}$ with (e_i, e_j) an edge of $\Omega(\mathscr{E})$ if and only if $E_i \cap E_j \neq \phi$. A graph is called a *chordal graph* if every circuit with more than three vertices has a chord (an edge connecting two non-consecutive vertices of the circuit), and a graph is called a *subtree graph* if it is the intersection graph of a family of subtrees of a tree. Gavril [4] has shown that a graph is a subtree graph if and only if it is a chordal graph. For example, if $H=(X, \mathscr{E})$ with $X=\{1,2,3,4,5,6\}$ and $\mathscr{E}=\{E_1, \ldots, E_6\}$ where $E_1=\{1,2\}$, $E_2=\{1,3\}$, $E_3=\{1,5\}$, $E_4=\{3,4\}$, $E_5=\{3,5\}$ and $E_6=\{5,6\}$, then $\Omega(\mathscr{E})$ is chordal and hence a subtree graph. Note that being a chordal graph is the property of an unlabelled graph. Indeed, no tree can have subtrees whose vertex sets are $E_1, \ldots, E_6$, and H is not a SOFT hypergraph, as can be seen by observing that $\{\{1,3\},\{1,5\},\{3,5\}\}$ shows that $\mathscr{E}$ does not have the Helly property. One does, however, obtain the following theorem as a corollary of Gavril's result.

THEOREM 3. *If H is a* SOFT *hypergraph, then $\Omega(\mathscr{E})$ is chordal.*

THEOREM 4. *Hypergraph $H=(X, \mathscr{E})$ is a* SOFT *hypergraph if and only if $\Omega(\mathscr{E})$ is chordal and $\mathscr{E}$ satisfies the Helly property.*

Proof. By Theorems 2 and 3 the conditions are necessary if H is a SOFT hypergraph.

For the converse, induct on $m=|\mathscr{E}|$. Clearly if $m=1$, one can select any tree T with $|E_1|$ vertices labelled with the elements of E_1. For $m=2$ one can form trees T_1, T_2 and T_3 with vertex sets $E_1 \cap E_2$, E_1-E_2 and E_2-E_1 (any one, or the last two, of which may be empty). Select vertices v_1, v_2 and v_3 in T_1, T_2 and T_3, respectively, and form tree T by adding edges (v_1, v_2) and (v_1, v_3). If $E_1 \cap E_2=\phi$, then one adds edge (v_2, v_3).

Suppose $m \geq 3$ and, by induction, that the conditions are sufficient if $|\mathscr{E}| \leq m-1$. Since $\Omega(\mathscr{E})$ is chordal it has a vertex for which any two vertices adjacent to it are adjacent to each other (see [3] or [5]). Thus it can be assumed that e_1 is such a vertex, e_i is adjacent to e_1 (where $2 \leq i \leq m$) if and only if $2 \leq i \leq k$ (if e_1 is an isolated vertex then one is clearly done by induction, and so one assumes $2 \leq k$), and any two of $e_1, e_2, \ldots, e_k$ are adjacent. Since $\mathscr{E}$ satisfies the Helly property, one has $\bigcap_{i=1}^k E_i \neq \phi$, say $a \in \bigcap_{i=1}^k E_i=E$. Let $E_1'=E_1-a$. For $2 \leq i \leq m$, let $E_i'=E_i-E_1'$.

Now suppose $2 \leq h<j \leq m$. Since $E_i' \subseteq E_i$, if $E_h' \cap E_j' \neq \phi$ then $E_h \cap E_j \neq \phi$. Assume $E_h \cap E_j \neq \phi$. If $j \geq k+1$ then, since $E_j \cap E_1=\phi$, one has $E_h' \cap E_j'=E_h' \cap E_j=E_h \cap E_j \neq \phi$; if $j \leq k$ then $a \in E_h' \cap E_j'$. Thus $E_h \cap E_j \neq \phi$ if and only if $E_h' \cap E_j' \neq \phi$. This implies that $\mathscr{E}'=\{E_2', \ldots, E_m'\}$ has the Helly property and that $\Omega(\mathscr{E}')$, which is isomorphic to $\Omega(\mathscr{E})-e_1$, is chordal. By induction, $H'=(X-E_1', \mathscr{E}')$ is a SOFT hypergraph of some tree T'.

Let T be obtained from T' by adding $|E_1'|$ vertices, labelled with the elements of E_1', each of which is made adjacent to a. It is straightforward to see that E_i is the vertex set of a subtree of tree T for $1 \leq i \leq m$ and that the vertex set of T is X.

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