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ISOMETRIES OF MEASURABLE FUNCTIONS

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Let (X, Σ, μ) be a σ-finite measure space and denote by $L^{\infty}(X, K)$ the Banach space of essentially bounded, measurable functions F defined on X and taking values in a separable Hilbert space K. In this article a characterization is given of the linear isometries of $L^{\infty}(X, K)$ onto itself. It is shown that if T is such an isometry then T is of the form $\big(T(F)\big)(x) = U(x)\big(\Phi(F)\big)(x)$, where Φ is a set isomorphism of Σ onto itself, and U is a measurable operator-valued function such that $U(x)$ is almost everywhere an isometry of K onto itself. It is a consequence of the proof given here that every isometry of $L^{\infty}(X, K)$ is the adjoint of an isometry of $L^{1}(X, K)$.

1. Introduction

Throughout this article (X, Σ, μ) will denote a σ-finite measure space, and the letter K will represent a separable Hilbert space which may be either real or complex. We denote by $\langle \cdot, \cdot \rangle$ the inner product in K, and by S the one-dimensional Hilbert space which is the scalar field associated with K.

A function F from X to K will be called measurable if the scalar function $\langle F, e \rangle$ is measurable for each $e \in K$. Then for $1 \le p \le \infty$, we denote by $L^{p}(X, K)$ the Banach space of (equivalence classes of) measurable functions F from X to K for which the norm

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$$\|F\|_p = \left\{\int \|F(x)\|^p d\mu\right\}^{1/p} \quad , \quad p < \infty \; ,$$

$$\|F\|_\infty = \text{ess sup}\|F(x)\|$$

is finite. ($\|\cdot\|_p$ will denote the norm in $L^p(X, K)$ and $\|\cdot\|$ that in K .) If $F \in L^p(X, K)$, we define the support of F , which will be denoted by $\text{supp}(F)$, to be the set $\{x \in X : F(x) \neq 0\}$.

Let $\{e_1, e_2, \ldots\}$ be some orthonormal basis for K . For $F \in L^p(X, K)$, we define the measurable coordinate functions f_n by $f_n(x) = \langle F(x), e_n \rangle$. Then almost everywhere we have $\sum_n |f_n(x)|^2 < \infty$, and $F(x) = \sum_n f_n(x)e_n$. Moreover, it is clear that each f_n belongs to $L^p(X, S)$.

In [1, p. 178], Banach determined the isometries of $L^p(X, S)$, $1 \leq p < \infty$, $p \neq 2$, for the case in which X is the unit interval and μ Lebesgue measure. Lamperti later obtained a complete description of the isometries of $L^p(X, S)$ for an arbitrary σ-finite measure space (X, Σ, μ) , and the same values of p , [8]. The first such result for vector-valued functions was obtained in [2], where the surjective isometries of $L^p(X, K)$, for $1 \leq p < \infty$, $p \neq 2$, were characterized. This result was also established, *via* quite different methods, in [6] by Fleming and Jamison, and strengthened in [9] by Sourour, who replaced the Hilbert space K by a separable Banach space E having only trivial L^p-summands. In this article we investigate the surjective isometries of $L^\infty(X, K)$.

If K is the one-dimensional Hilbert space S , then since $L^\infty(X, S)$ is isometrically isomorphic to $C(Y)$, where Y is the maximal ideal space of $L^\infty(X, S)$, a description of the isometries can be obtained through an application of the Banach-Stone theorem. And if K is finite dimensional, it can be shown that $L^\infty(X, K)$ is isometrically isomorphic to $C(Y, K)$, the space of continuous functions on Y to K , under the map

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$\sum_{n=1}^{N} f_n e_n \to \sum_{n=1}^{N} \hat{f}_n e_n$, where $f \to \hat{f}$ is the Gelfand representation of $L^\infty(X, S)$. In this case the description we give in this article can also be obtained from what is known about isometries of spaces of continuous vector-valued functions [3]. However when K is infinite dimensional, the continuity on Y of the coordinate functions $\hat{f}_n$ no longer implies continuity for $\sum_n \hat{f}_n e_n$, and thus the problem requires different methods of approach.

A mapping Φ of Σ onto itself, defined modulo null sets, is called a *regular set isomorphism* if it satisfies the properties

$$\Phi(A') = [\Phi(A)]' ,$$

$$\Phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \Phi(A_n) ,$$

and

$$\mu[\Phi(A)] = 0 \text{ if, and only if, } \mu(A) = 0 \text{ for all sets } A, A_n \text{ in } \Sigma .$$

(Throughout, A' will denote the complement of A .) A regular set isomorphism induces a linear transformation, also denoted by Φ , on the space of measurable scalar functions defined on X , which is characterized by $\Phi(\chi_A) = \chi_{\Phi(A)}$, where χ_A is the characteristic function of the measurable set A . This process is described by Doob in [5, pp. 453-454].

Given a regular set isomorphism Φ of Σ onto itself, and $F = \sum_n f_n e_n$ in $L^p(X, K)$, we define $\Phi(F)$ by the equation

$$\big(\Phi(F)\big)(x) = \sum_n \big(\Phi(f_n)\big)(x) e_n .$$

The fact that, in the case in which K is infinite dimensional, the series on the right is indeed convergent in K for almost all x was established in [2, p. 10]. One readily verifies that the definition of $\Phi(F)$ is independent of the choice of orthonormal basis for K .

We will use the fact that the set of extreme points of the unit ball in $L^\infty(X, K)$ consists of those elements F such that $\|F(x)\| = 1$ almost

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everywhere on X . Throughout the article, given $e \in K$, we denote by **e** that element of $L^{\infty}(X, K)$ which is constantly equal to e .

2. Isometries

Throughout, T will denote a fixed isometry of $L^{\infty}(X, K)$ onto itself.

LEMMA 1. *Let* E *be an element of* $L^{\infty}(X, K)$ *with* $\|E(x)\| = 1$ *almost everywhere. If* $A \in \Sigma$ *then* $\operatorname{supp}\left(T\left(\chi_A \cdot E\right)\right)$ *and* $\operatorname{supp}\left(T\left(\chi_{A'} \cdot E\right)\right)$ *are disjoint measurable sets whose union is almost everywhere equal to* X *. Moreover,* $\left\|\left(T\left(\chi_A \cdot E\right)\right)(x)\right\|$ *is equal to one almost everywhere on* $\operatorname{supp}\left(T\left(\chi_A \cdot E\right)\right)$.

Proof. Note that since E is an extreme point of the unit ball in $L^{\infty}(X, K)$, so is $T(E)$, and thus we have, almost everywhere,

$$(1) \qquad 1 = \|(T(E))(x)\| = \left\|\left(T\left(\chi_A \cdot E\right)\right)(x) + \left(T\left(\chi_{A'} \cdot E\right)\right)(x)\right\| .$$

First suppose that $\left\|\left(T\left(\chi_A \cdot E\right)\right)(x)\right\| \neq 1$ almost everywhere on $\operatorname{supp}\left(T\left(\chi_A \cdot E\right)\right)$; that is, there is a measurable subset $B \subseteq \operatorname{supp}\left(T\left(\chi_A \cdot E\right)\right)$ with $\mu(B) > 0$ and $\left\|\left(T\left(\chi_A \cdot E\right)\right)(x)\right\| < 1 - \varepsilon_1$ for some $\varepsilon_1 > 0$ on B .

The set $\left\{x : \left\|\left(T\left(\chi_{A'} \cdot E\right)\right)(x)\right\| = 1\right\}$ cannot intersect $\operatorname{supp}\left(T\left(\chi_A \cdot E\right)\right)$ in a set of positive measure. For $\left\|\chi_A \cdot E \pm \chi_{A'} \cdot E\right\|_{\infty} = 1$ gives $\left\|\left(T\left(\chi_A \cdot E\right)\right)(x) \pm \left(T\left(\chi_{A'} \cdot E\right)\right)(x)\right\| \leq 1$ almost everywhere, and $\left(T\left(\chi_{A'} \cdot E\right)\right)(x)$ is an extreme point of the unit ball of K for all $x \in \left\{x : \left\|\left(T\left(\chi_{A'} \cdot E\right)\right)(x)\right\| = 1\right\}$. Thus $\left\|\left(T\left(\chi_{A'} \cdot E\right)\right)(x)\right\| < 1$ almost everywhere on $\operatorname{supp}\left(T\left(\chi_A \cdot E\right)\right)$, so that we can find a subset C of B with $\mu(C) > 0$ and $\left\|\left(T\left(\chi_{A'} \cdot E\right)\right)(x)\right\| < 1 - \varepsilon_2$ on C , for some $\varepsilon_2 > 0$.

Let $\varepsilon = \min\left\{\varepsilon_1, \varepsilon_2\right\}$. Then we certainly have $\left\|T\left(\chi_A \cdot E\right) \pm \varepsilon \cdot \chi_C \cdot E\right\|_{\infty} \leq 1$ and $\left\|T\left(\chi_{A'} \cdot E\right) \pm \varepsilon \cdot \chi_C \cdot E\right\|_{\infty} \leq 1$, so that $\left\|\chi_A(x) \cdot E(x) \pm \left[T^{-1}\left(\varepsilon \cdot \chi_C \cdot E\right)\right](x)\right\| \leq 1$ and $\left\|\chi_{A'}(x) \cdot E(x) \pm \left[T^{-1}\left(\varepsilon \cdot \chi_C \cdot E\right)\right](x)\right\| \leq 1$ almost everywhere. But since $T^{-1}\left(\varepsilon \cdot \chi_C \cdot E\right)$ is not the zero element of $L^{\infty}(X, K)$, its support must meet

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either A or A' in a set of positive measure, contradicting the fact that $\chi_A(x) \cdot E(x)$ is an extreme point of the unit ball of K almost everywhere on A, and $\chi_{A'}(x) \cdot E(x)$ is an extreme point of the unit ball of K almost everywhere on A'.

This contradiction shows that $\|(T(\chi_A \cdot E))(x)\| = 1$ almost everywhere on $\operatorname{supp}(T(\chi_A \cdot E))$. Similarly, $\|(T(\chi_{A'} \cdot E))(x)\| = 1$ on $\operatorname{supp}(T(\chi_{A'} \cdot E))$. It is clear that $\operatorname{supp}(T(\chi_A \cdot E))$ and $\operatorname{supp}(T(\chi_{A'} \cdot E))$ are measurable. The fact that these sets are disjoint again follows since

$$\|(T(\chi_A \cdot E))(x) \pm (T(\chi_{A'} \cdot E))(x)\| \le 1$$

almost everywhere, and $(T(\chi_A \cdot E))(x)$ is an extreme point of the unit ball in K for all $x \in \operatorname{supp}(T(\chi_A \cdot E))$. Finally, the union of $\operatorname{supp}(T(\chi_A \cdot E))$ and $\operatorname{supp}(T(\chi_{A'} \cdot E))$ is, by (1), equal to X.

LEMMA 2. *With* E *as in Lemma* 1 *define, for* $A \in \Sigma$, $\Phi(A) = \operatorname{supp}(T(\chi_A \cdot E))$. *Then* Φ *is a regular set isomorphism of* Σ *onto itself.*

Proof. It follows immediately from Lemma 1 that for $A \in \Sigma$,

$$\Phi(A') = [\Phi(A)]' .$$

Note that $\mu(A) \neq 0$ if and only if $\chi_A \cdot E \neq 0$ in $L^\infty(X, K)$ which is true if and only if $T(\chi_A \cdot E) \neq 0$ in $L^\infty(X, K)$ and this holds if and only if $\mu[\Phi(A)] > 0$. Thus trivially we have

$$\mu[\Phi(A)] = 0 \text{ if, and only if, } \mu(A) = 0 .$$

Now suppose that A and B are disjoint measurable sets. Since $\|(T(\chi_A \cdot E))(x) \pm (T(\chi_B \cdot E))(x)\| \le 1$ almost everywhere and $(T(\chi_A \cdot E))(x)$ is an extreme point of the unit ball in K for all $x \in \operatorname{supp}(T(\chi_A \cdot E))$, $\Phi(A)$ and $\Phi(B)$ must be almost everywhere disjoint. Thus if B is a measurable subset of the measurable set A, then B and A' are disjoint so that $\Phi(B)$ and $\Phi(A')$ are disjoint. Hence $B \subseteq A$ implies that $\Phi(B) \subseteq \Phi(A)$. It is easily seen that the reverse implication is also true: $\Phi(B) \subseteq \Phi(A)$ implies that $B \subseteq A$. The sentence before last also implies that A and

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B are disjoint if and only if $\Phi(A)$ and $\Phi(B)$ are disjoint.

Next assume that $\{A_1, A_2, \ldots\}$ is a pairwise disjoint sequence of measurable sets and let $A = \bigcup_{n=1}^{\infty} A_n$. Then since $A_n \subseteq A$ for all n, we have $\Phi(A_n) \subseteq \Phi(A)$ for all n, so that $\bigcup_{n=1}^{\infty} \Phi(A_n) \subseteq \Phi(A)$. Set $B = \Phi(A) - \bigcup_{n=1}^{\infty} \Phi(A_n)$. We would like to show that $\mu(B) = 0$. To this end we first show that Φ maps Σ onto itself.

Thus suppose that B is any measurable set. We have

$$T(E) = \chi_B \cdot T(E) + \chi_{B'} \cdot T(E)$$

so that

$$E = T^{-1}[\chi_B \cdot T(E)] + T^{-1}[\chi_{B'} \cdot T(E)] .$$

By Lemma 1 (interchanging the roles of T and T^{-1}, A and B, E and $T(E)$), the two elements on the right in the last equation have disjoint supports, and have norm equal to one almost everywhere on their respective supports. Thus $T^{-1}[\chi_B \cdot T(E)]$ is of the form $\chi_C \cdot E$ for some $C \in \Sigma$, and hence

$$T(\chi_C \cdot E) = \chi_B \cdot T(E) \tag{2}$$

which says that $\Phi(C) = B$. Thus Φ is onto.

Now with $B = \Phi(A) - \bigcup_{n=1}^{\infty} \Phi(A_n)$, take $C \in \Sigma$ with $\Phi(C) = B$. By what was established in the second paragraph of this proof, we must have $C \subseteq A$ in this instance. Thus if we suppose that B, hence C, has positive measure, then, for some n, C meets A_n in a set of positive measure. But $\Phi(C)$ and $\Phi(A_n)$ are disjoint, and this contradiction shows that we must have $\mu(B) = 0$. Thus

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$$\Phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \Phi(A_n) ,$$

completing the proof of the lemma.

NOTE. Equation (2) tells us that for each $A \in \Sigma$,

$$T(\chi_A \cdot E) = \chi_{\Phi(A)} \cdot T(E) .$$

From this it follows that for every $f \in L^\infty(X, S)$ we have

$$T(f \cdot E) = \Phi(f) \cdot T(E) .$$

LEMMA 3. *If* E_1 *and* E_2 *are two pointwise orthogonal elements of* $L^\infty(X, K)$ *with* $\|E_j(x)\| = 1$ *almost everywhere on* X , $j = 1, 2$, *then* $T(E_1)$ *and* $T(E_2)$ *are pointwise orthogonal for almost all* x .

Proof. Using the polarization identity [7, p. 274] (and assuming the space is complex), we have

$$\langle (T(E_1))(x), (T(E_2))(x) \rangle = \frac{1}{4} \sum_{n=1}^{4} i^n \left\| (T(E_1))(x) + i^n (T(E_2))(x) \right\|^2$$
$$= \frac{1}{4} \sum_{n=1}^{4} i^n \left\| \left(T\left(E_1 + i^n E_2\right)\right)(x) \right\|^2 .$$

But the elements $E_1 + i^n E_2$, $n = 1, 2, 3, 4$, are extreme points of the ball of radius $\sqrt{2}$ centered at the origin in $L^\infty(X, K)$. Hence $\left\| \left(T\left(E_1 + i^n E_2\right)\right)(x) \right\| = \sqrt{2}$ almost everywhere on X , so the inner product $\langle (T(E_1))(x), (T(E_2))(x) \rangle$ is almost everywhere equal to zero.

If K is a real Hilbert space, the corresponding polarization identity also gives the desired result.

LEMMA 4. *Let* E_1 *and* E_2 *be as in Lemma* 3, *and for* $j = 1, 2$, *let* Φ_j *be the regular set isomorphism of* Σ *onto itself that is associated with* E_j *as in Lemma* 2. *Then* $\Phi_1 = \Phi_2$.

Proof. Let $A \in \Sigma$ and consider

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$$(3)\quad \left(T\left(\chi_A\cdot\left(E_1+E_2\right)/\sqrt{2}\right)\right)(x) = (1/\sqrt{2})\left(T\left(\chi_A\cdot E_1\right)\right)(x) + (1/\sqrt{2})\left(T\left(\chi_A\cdot E_2\right)\right)(x)$$
$$= (1/\sqrt{2})\chi_{\Phi_1(A)}(x)\left(T\left(E_1\right)\right)(x) + (1/\sqrt{2})\chi_{\Phi_2(A)}(x)\left(T\left(E_2\right)\right)(x) .$$

Now the left hand side of (3) is equal to

$$\chi_{\Phi_{1,2}(A)}(x)\left(T\left(E_1+E_2\right)/\sqrt{2}\right)(x)$$
$$= (1/\sqrt{2})\chi_{\Phi_{1,2}(A)}(x)\left(T\left(E_1\right)\right)(x) + (1/\sqrt{2})\chi_{\Phi_{1,2}(A)}(x)\left(T\left(E_2\right)\right)(x) ,$$

where $\Phi_{1,2}$ is the regular set isomorphism associated with $\left(E_1+E_2\right)/\sqrt{2}$ *via* Lemma 2. Thus, by the almost everywhere linear independence of $\left(T\left(E_1\right)\right)(x)$ and $\left(T\left(E_2\right)\right)(x)$ we have

$$\chi_{\Phi_1(A)}(x) = \chi_{\Phi_{1,2}(A)}(x) = \chi_{\Phi_2(A)}(x)$$

almost everywhere, and hence $\Phi_1 = \Phi_2$.

LEMMA 5. *Suppose that* F_1, F_2 *are elements of* $L^\infty(X, K)$ *such that* $F_1(x)$ *and* $F_2(x)$ *are orthogonal in* K *for almost all* x . *Then* $\langle \left(T\left(F_1\right)\right)(x), \left(T\left(F_2\right)\right)(x)\rangle = 0$ *almost everywhere.*

Proof. First suppose that $F_1 = g_1G_1$ and $F_2 = g_2G_2$, where the g_j are elements of $L^\infty(X, S)$ with $\operatorname{supp}\left(g_j\right) = \operatorname{supp}\left(F_j\right)$ and the G_j are measurable vector functions such that $\left\|G_j(x)\right\| = 1$ on $\operatorname{supp}\left(F_j\right)$, $j = 1, 2$. Let $A = \operatorname{supp}\left(F_2\right) - \operatorname{supp}\left(F_1\right)$. On A we have $F_2(x) = \sum_n f_n(x)e_n$, the convergence being almost everywhere pointwise in K . Since $A = \bigcup_n A_n$, where $A_n = \{x \in A : f_n(x) \neq 0\}$ we can define a measurable vector function H_1 on A by $H_1(x) = \overline{f_2(x)}e_1 - \overline{f_1(x)}e_2$, for $x \in A_1 \cup A_2$, and for $n > 2$, $H_1(x) = \overline{f_n(x)}e_1 - \overline{f_1(x)}e_n$ for $x \in A_n - \left(A_1 \cup \ldots \cup A_{n-1}\right)$. Then $H_1(x)$ is almost everywhere orthogonal to $F_2(x)$ on A , and the vector function J_1 defined on A by $J_1(x) = H_1(x)/\left\|H_1(x)\right\|$, $x \in A$, is such that $\left\|J_1(x)\right\| = 1$ for $x \in A$ and

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$J_1(x)$ is pointwise orthogonal to $F_2(x)$ almost everywhere on A.

Similarly we can find a measurable vector function J_2 defined on $B = \operatorname{supp}(F_1) - \operatorname{supp}(F_2)$ such that $\|J_2(x)\| = 1$ on B and $\langle F_1(x), J_2(x)\rangle = 0$ almost everywhere on B. We now define measurable vector functions E_1 and E_2 by

$$E_1(x) = \begin{cases} G_1(x) , & x \in \operatorname{supp}(F_1) , \\ J_1(x) , & x \in A , \\ e_1 , & x \in X - (\operatorname{supp}(F_1) \cup \operatorname{supp}(F_2)) , \end{cases}$$

$$E_2(x) = \begin{cases} G_2(x) , & x \in \operatorname{supp}(F_2) , \\ J_2(x) , & x \in B , \\ e_2 , & x \in X - (\operatorname{supp}(F_1) \cup \operatorname{supp}(F_2)) . \end{cases}$$

Then $\|E_j(x)\| = 1$ and $\langle E_1(x), E_2(x)\rangle = 0$ almost everywhere.

We can write $F_j = g_jE_j$, $j = 1, 2$, so that by Lemma 4 and the note following Lemma 2, there exists a regular set isomorphism Φ of Σ onto itself such that $T(F_j) = \Phi(g_j)T(E_j)$ for $j = 1, 2$. We thus have

$$\begin{aligned}\langle (T(F_1))(x), (T(F_2))(x)\rangle &= \langle (\Phi(g_1))(x)(T(E_1))(x), (\Phi(g_2))(x)(T(E_2))(x)\rangle \\ &= (\Phi(g_1))(x)\overline{(\Phi(g_2))(x)}\langle (T(E_1))(x), (T(E_2))(x)\rangle = 0\end{aligned}$$

almost everywhere by Lemma 3.

Now since every $F \in L^\infty(X, K)$ can be written in the form $F = gG$, where $g \in L^\infty(X, S)$ with $\operatorname{supp}(g) = \operatorname{supp}(F)$ and G is measurable with $\|G(x)\| = 1$ almost everywhere on $\operatorname{supp}(F)$ (that is, let $g(x) = \|F(x)\|$ and $G(x) = F(x)/\|F(x)\|$ for $x \in \operatorname{supp}(F)$) the proof of the lemma is complete.

LEMMA 6. *Let* Φ *be the regular set isomorphism of* Σ *onto itself determined by* $T(\chi_A\cdot e_n) = \chi_{\Phi(A)}T(e_n)$ *, for* $n = 1, 2, \ldots$ *and* $A \in \Sigma$ *, and denote also by* Φ *the corresponding linear transformation of measurable scalar functions. Then the map defined for* $F \in L^\infty(X, K)$ *by* $F \to \Phi(F)$ *is an isometry of* $L^\infty(X, K)$ *onto itself.*

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Proof. It follows from Lemma 4, the separability of K, and the note following Lemma 2 that there exists a well defined regular set isomorphism Φ of Σ onto itself such that for all $n = 1, 2, \ldots$ and all bounded measurable scalar functions f, $T(f \cdot e_n) = \Phi(f) \cdot T(\mathrm{e}_n)$. It is obvious that $\|\Phi(f)\|_\infty = \|f\|_\infty$ holds for all scalar simple functions f, and the fact that the same equality then holds for an arbitrary element f of $L^\infty(X, S)$ follows easily from the fact that Φ preserves the almost everywhere convergence of sequences of measurable functions.

Now using [2, p. 10] and the fact that Φ preserves the set of positive elements of $L^\infty(X, S)$, it follows that for $F \in L^\infty(X, K)$, $\|(\Phi(F))(\cdot)\| = \Phi(\|F(\cdot)\|)$, where $\|F(\cdot)\|$ denotes the $L^\infty(X, S)$ element g defined by $g(x) = \|F(x)\|$ for $x \in X$. Thus

$$\|\Phi(F)\|_\infty = \text{ess sup}\|(\Phi(F))(x)\| = \text{ess sup } \Phi(\|F(x)\|) .$$

But by the previous paragraph this last quantity is equal to ess sup $\|F(x)\| = \|F\|_\infty$ so that the map is norm preserving, and it is clear that it is also surjective.

We now define the measurable operator function $U(x)$ by first defining $U(x)$ on the basis vectors e_n of K *via* the equation $U(x)e_n = (T(\mathrm{e}_n))(x)$, and then extending $U(x)$ linearly to K. By Lemma 3, $\{(T(\mathrm{e}_1))(x), (T(\mathrm{e}_2))(x), \ldots\}$ is almost everywhere an orthonormal set in K, so that $U(x)$ is an isometry of K into itself almost everywhere. It thus follows from Lemma 6 that for $F \in L^\infty(X, K)$, the map $F(\cdot) \to U(\cdot)(\Phi(F))(\cdot)$ is an isometry of $L^\infty(X, K)$ into itself.

In the following lemma we use the fact that, modulo the usual conjugate-linear identification of K with its own dual space, $L^\infty(X, K)$ is the dual space of $L^1(X, K)$ [4, p. 282].

LEMMA 7. *For* $F = \sum_n f_n e_n \in L^\infty(X, K)$ *define, for* $N = 1, 2, \ldots$, $F_N = \sum_{n=1}^{N} f_n e_n$. *Then* $U(\cdot)(\Phi(F_N))(\cdot)$ *tends to* $U(\cdot)(\Phi(F))(\cdot)$ *in the weak* $*$ *topology of* $L^\infty(X, K)$.

Proof. The sequence $\{\Phi(F_N)\}$ converges almost everywhere to $\Phi(F)$

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and is uniformly bounded in norm by $\|\Phi(F)\|_\infty = \|F\|_\infty$. Hence the sequence $\{U(\cdot)(\Phi(F_N))(\cdot)\}$ converges almost everywhere to $U(\cdot)(\Phi(F))(\cdot)$, and is also uniformly bounded in norm by $\|F\|_\infty$. It follows that if $G \in L^1(X, K)$, then $G(\cdot)U(\cdot)(\Phi(F_N))(\cdot)$ converges almost everywhere to $G(\cdot)U(\cdot)(\Phi(F))(\cdot)$, and is dominated by $\|F\|_\infty\|G(\cdot)\|$. Thus, by the dominated convergence theorem,

$$\left|\int G(x)U(x)(\Phi(F_N))(x)d\mu - \int G(x)U(x)(\Phi(F))(x)d\mu\right|$$
$$\leq \int \|G(x)U(x)(\Phi(F_N))(x)-G(x)U(x)(\Phi(F))(x)\|d\mu$$
$$\to 0 \text{ as } N \to \infty .$$

LEMMA 8. *Let* ν *be the measure defined on* Σ *by* $\nu(A) = \mu[\Phi^{-1}(A)]$, *for* A Σ, *and let* $h = \frac{d\nu}{d\mu}$. *Define* R *mapping* $L^1(X, K)$ *to itself by* $(R(G))(x) = U(x)h(x)(\Phi(G))(x)$, *for* $G \in L^1(X, K)$. *Then* R *maps* $L^1(X, K)$ *isometrically onto itself and* $T = R^{*-1}$.

Proof. First note that, by the definition of a regular set isomorphism, ν is absolutely continuous with respect to μ and that, for $A \in \Sigma$, we have

$$(4) \qquad \mu(A) = \nu[\Phi(A)] = \int_{\Phi(A)} d\nu = \int_{\Phi(A)} \frac{d\nu}{d\mu}\, d\mu = \int_{\Phi(A)} h d\mu .$$

Moreover, by [9, p. 283] the mapping R defined above is an isometry of $L^1(X, K)$ into itself.

If $G = \sum_n g_n e_n \in L^1(X, K)$, then the sequence $\{G_N\}$, where for $N = 1, 2, \ldots$ and $x \in X$, $G_N(x) = \sum_{n=1}^{N} g_n(x)e_n$, converges almost everywhere to G and is dominated by $\|G\|$. Thus, by the dominated convergence theorem, finite sums of the form $G_N = \sum_{n=1}^{N} g_n e_n$ are dense in $L^1(X, K)$, and it is clear that we still have a dense set if we restrict the coordinate functions g_n to be scalar simple functions. Moreover, it is also clear that $L^\infty(X, K)$ elements of the form $H(x) = \sum_n s_n(x)e_n$, where the s_n are scalar simple functions, are dense in $L^\infty(X, K)$.

Thus suppose we have two such elements $G_N \in L^1(X, K)$ and $H \in L^\infty(X, K)$. We may suppose that $G_N = \sum_{i=1}^{N} \sum_{j=1}^{n} c_{ij}\chi_{A_j}e_i$ and $H = H_1 + H_2$, where $H_1 = \sum_{i=1}^{N} \sum_{j=1}^{n} d_{ij}\chi_{A_j}e_i$, $H_2 = \sum_{i=N+1}^{\infty} s_i e_i$, the c_{ij}, d_{ij} are scalars, and the A_j are pairwise disjoint measurable sets. We thus have

$$\int \langle G_N, H\rangle d\mu = \int \left(\sum_{i=1}^{N} \sum_{j=1}^{n} c_{ij}\bar{d}_{ij}\chi_{A_j}\right) d\mu = \sum_{i=1}^{N} \sum_{j=1}^{n} c_{ij}\bar{d}_{ij}\mu(A_j) ,$$

But by (4) this latter quantity is equal to

$$\sum_{i=1}^{N} \sum_{j=1}^{n} c_{ij}\bar{d}_{ij} \int_{\Phi(A_j)} h d\mu$$

$$= \int \left\langle \sum_{i=1}^{N} \sum_{j=1}^{n} hc_{ij}\chi_{\Phi(A_j)}T(\mathrm{e}_i), \sum_{i=1}^{N} \sum_{j=1}^{n} d_{ij}\chi_{\Phi(A_j)}T(\mathrm{e}_i)\right\rangle d\mu$$

$$= \int \langle U(x)h(x)(\Phi(G_N))(x), (T(H_1))(x)\rangle d\mu$$

and since by Lemma 5, $(T(H_2))(x)$ is almost everywhere orthogonal in K to $T(\mathrm{e}_i)$, $1 \le i \le N$, this last integral equals

$$\int \langle U(x)h(x)(\Phi(G_N))(x), (T(H_1))(x)+(T(H_2))(x)\rangle d\mu$$

$$= \int \langle R(G_N), T(H)\rangle d\mu = \int \langle G_N, R^* \circ T(H)\rangle d\mu .$$

Here R^* is, again modulo the conjugate-linear identification of K with its own dual, the Banach space adjoint of R [2, p. 11].

Thus for each H in $L^\infty(X, K)$ of the form considered, we have shown that the linear functionals on $L^1(X, K)$ determined by H and $(R^* \circ T)(H)$ agree when evaluated at all elements G_N belonging to a dense subset of $L^1(X, K)$. Hence, for such H, we have $(R^* \circ T)(H) = H$; and since this equality holds for all H belonging to a dense subset of $L^\infty(X, K)$, we have $R^* = T^{-1}$ and $R^{*-1} = T$. Note that this implies that R is actually an isometry of $L^1(X, K)$ onto itself [*10*, p. 226].

THEOREM. *For* $F \in L^\infty(X, K)$, $\big(T(F)\big)(x) = U(x)\big(\Phi(F)\big)(x)$.

Proof. Let $F = \sum_n f_n e_n$ belong to $L^\infty(X, K)$, and for $N = 1, 2, \ldots$ set $F_N = \sum_{n=1}^{N} f_n e_n$. Then by an argument exactly analogous to that given in the proof of Lemma 7, $F_N \to F$ in the weak $*$ topology of $L^\infty(X, K)$. Since adjoints of maps continuous with respect to the norm topology remain continuous with respect to the weak $*$ topology, we have $T(F_N) = R^{*-1}(F_N) \to R^{*-1}(F) = T(F)$ weak $*$. But by Lemma 7, $T(F_N) = U(\cdot)\big(\Phi(F_N)\big)(\cdot)$ tends weak $*$ to $U(\cdot)\big(\Phi(F)\big)(\cdot)$, and the proof of the theorem is complete.

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