Proceedings of the Edinburgh Mathematical Society (2001) **44**, 351–363 ©

THE HYPERELLIPTIC MAPPING CLASS GROUP OF KLEIN SURFACES

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(Received 24 March 1999)

Abstract In this paper we study the algebraic structure of the hyperelliptic mapping class group of Klein surfaces, which is closely related to the mapping class group of punctured discs. This group plays an important role in the study of the moduli space of hyperelliptic real algebraic curves. Our main result provides a presentation by generators and relations for the hyperelliptic mapping class group of surfaces of prescribed topological type.

Keywords: Klein surface; hyperelliptic Klein surface; mapping class group; real algebraic curve; moduli space

AMS 2000 *Mathematics subject classification:* Primary 14H10; 20H10; 30F50

1. Introduction

The study of the mapping class group goes back to the classical works of Fricke, Artin, Magnus, Bohnenblust, Markoff and others. Most of this work concerns the braid group and the mapping class group of punctured spheres. This group has been extremely useful in understanding the moduli space of hyperelliptic complex algebraic curves because it is intimately related with the hyperelliptic mapping class group of the hyperelliptic Riemann surfaces by virtue of the papers by Birman and Hilden [**4**], Harvey and Maclachlan [**7**] and Zieschang [**13**]. Our goal in this paper is to study the algebraic structure of the hyperelliptic mapping class group of Klein surfaces, which, as we will see, is closely related to the mapping class group of punctured discs. This group plays an important role, analogous to the one in the complex case, in the study of the moduli space of hyperelliptic real algebraic curves.

Our main result provides a presentation by generators and relations for the hyperelliptic mapping class group of surfaces of prescribed topological type (for the mapping class group of Riemann surfaces, such a presentation was obtained in [**8**]).

The article is divided into the following sections. Section 2 is devoted to introducing the notions and notation we are going to use. In § 3 we establish the relationship between the

https://doi.org/10.1017/S0013091599000322 Published online by Cambridge University Press

subset of the moduli space, determined by the hyperelliptic Klein surfaces of prescribed topological type, and its hyperelliptic mapping class group. In § 4 we get a presentation of the modular group of the group uniformizing the quotient of the Klein surface under its hyperelliptic involution. This is useful to get a presentation, in § 5, of the relative modular group or hyperelliptic mapping class group of hyperelliptic Klein surfaces. Finally, we obtain some applications in § 6; in particular we determine under what conditions the hyperelliptic mapping class group is finite.

2. Preliminaries

A *Klein surface* is a surface (orientable or not, with or without boundary) endowed with a dianalytic structure. Let X be a Klein surface of algebraic genus $p \geqslant 2$ and non-empty boundary. The surface X is said to be *hyperelliptic* if it admits a dianalytic involution ϕ such that the quotient $X/\langle\phi\rangle$ has algebraic genus zero. It is well known that such an involution is unique, and it is called the *hyperelliptic involution*.

Notice that there exists a functorial equivalence between compact hyperelliptic Klein surfaces and hyperelliptic real algebraic curves (see [**1**, **5**]) analogous to the classical one, and so we can express the obtained results in terms of hyperelliptic real algebraic curves.

The surface X can be written as a quotient $X = U/\Gamma$ of the hyperbolic plane U under the action of a surface non-Euclidean crystallographic (NEC) group Γ. The hyperellipticity of X means that the quotient $X/\langle\phi\rangle$ can be uniformized by an NEC group Γ' containing Γ as a subgroup of index 2, i.e. $X/\langle\phi\rangle = U/\Gamma'$.

The algebraic presentation of the group Γ'—or, in other words, the topological data of the covering $X \to X/\langle\phi\rangle$—is determined by a symbol, the *signature* of Γ', which has the following form:

$$\sigma(\Gamma') = (0; +; [2, \overset{r}{\dots}, 2]; \{(2, \overset{s}{\dots}, 2)\}), \tag{2.1}$$

where r and s are non-negative integers, and s is even. That means that $X/\langle\phi\rangle$ is a topological disc and the covering $X \to X/\langle\phi\rangle$ ramifies at r inner points of $X/\langle\phi\rangle$ and s points in the boundary. In other words, $X/\langle\phi\rangle$ has an orbifold structure (see [**12**]) with r conic points and s corner points, and Γ' can be considered as the fundamental group of the orbifold $X/\langle\phi\rangle$.

Let g be the topological genus of X, k the number of connected components of its boundary, and $\varepsilon = 2$ if X is orientable and $\varepsilon = 1$ otherwise. We call the triple (g, k, ε) the *topological type* of X, which determines the integers r, s in (2.1). In fact, it is known (see [**5**, ch. 2]) that

$$\left.\begin{array}{ll} \text{if } g = 0, & \text{then } r = 0 \text{ and } s = 2k; \\ \text{if } g \neq 0 \text{ and } \varepsilon = 2 \text{ (and so } k < 3), & \text{then } r = 2g + k,\ s = 0; \\ \text{if } g \neq 0 \text{ and } \varepsilon = 1, & \text{then } r = g \text{ and } s = 2k. \end{array}\right\} \tag{2.2}$$

Note that we always assume that the algebraic genus $p = \varepsilon g + k - 1$ of X is greater than or equal to 2.

For all these general results concerning hyperelliptic Klein surfaces, the interested reader is referred to [**5**].

In classical terms, the *mapping class group* of an orientable topological surface without boundary is defined as the group of isotopy classes of orientation-preserving autohomeomorphisms of the surface. We define the *mapping class group* of the Klein surface X as the group of isotopy classes of autohomeomorphisms of the underlying topological surface. This is isomorphic to the so-called *modular group* of $\varGamma$, defined as

$$\operatorname{mod}(\varGamma) = \frac{\operatorname{aut}(\varGamma)}{\operatorname{inn}(\varGamma)},$$

where $\operatorname{aut}(\varGamma)$ is the group of automorphisms of $\varGamma$ and $\operatorname{inn}(\varGamma)$ is the subgroup consisting of its inner automorphisms (see [**13**, Corollary 8.8]).

If X is hyperelliptic, the group

$$\operatorname{mod}(\varGamma') = \frac{\operatorname{aut}(\varGamma')}{\operatorname{inn}(\varGamma')}$$

is called the *modular group* of $\varGamma'$. Since $\varGamma$ is a normal subgroup of $\varGamma'$, the group $\operatorname{aut}(\varGamma', \varGamma)$, automorphisms f of $\varGamma'$ such that $f(\varGamma) = \varGamma$, contains $\operatorname{inn}(\varGamma')$. The quotient

$$\operatorname{mod}(\varGamma', \varGamma) = \frac{\operatorname{aut}(\varGamma', \varGamma)}{\operatorname{inn}(\varGamma')}$$

is called the *relative modular group*, or the *hyperelliptic mapping class group* of fixed topological type (g, k, ε), and we shall denote it by $\operatorname{map}^h(g, k, \varepsilon)$. The group $\operatorname{map}^h(g, k, \varepsilon)$ is isomorphic to the mapping class group of the orbifold $X/\langle\phi\rangle$, i.e. the group of isotopy classes of homeomorphisms preserving the singular points and its type. This approach will be used in § 4.

3. Moduli of hyperelliptic Klein surfaces

We are mainly interested in determining the group $\operatorname{map}^h(g, k, \varepsilon)$, since it plays a central role in describing the moduli space of hyperelliptic Klein surfaces. Let us determine the relation between these two objects, but first we introduce some more notation. Let G be the group of maps from $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ to $\hat{\boldsymbol{C}}$ having the form

$$z \mapsto \frac{az + b}{cz + d} \quad \text{or} \quad z \mapsto \frac{a'\bar{z} + b'}{c'\bar{z} + d'},$$

where $\bar{z}$ is the complex conjugate of z; a, b, c, d, a', b', c' and d' are real numbers;

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \quad \text{and} \quad \det\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = -1.$$

For each NEC group $\varDelta$, let $M(\varDelta, G)$ be the set of type-preserving group monomorphisms $\varDelta \hookrightarrow G$ with discrete image (see [**7**] for the analogous notion for Fuchsian groups). The *Teichmüller space of* $\varDelta$ is the quotient

$$T(\varDelta) = \frac{M(\varDelta, G)}{\sim},$$

where $\sim$ is the equivalence relation given by conjugation by elements of G. In particular, the inclusion $i : \Gamma \hookrightarrow \Gamma'$ between the NEC groups Γ and Γ' uniformizing X and $X/\langle\phi\rangle$, induces a map

$$i_* : M(\Gamma', G) \to M(\Gamma, G) : \alpha \longmapsto \alpha \circ i,$$

which is compatible with the equivalence relation $\sim$, and so one gets a map

$$m : T(\Gamma') \longrightarrow T(\Gamma) : [\alpha] \longmapsto [\alpha \circ i],$$

which is an isometric embedding with respect to the Teichmüller metric (see [**10**]).

The group $\operatorname{aut}(\Gamma)$ acts in a natural way on $T(\Gamma)$, and this action induces the following map:

$$\operatorname{mod}(\Gamma) \times T(\Gamma) \to T(\Gamma) : ([\varphi], [\alpha]) \longmapsto [\alpha \circ \varphi].$$

Then, as in the case of Riemann surfaces (see [**7**, § 4]), the *moduli space* of hyperelliptic Klein surfaces of topological type (g, k, ε) is defined as

$$M_h(g, k, \varepsilon) = \Lambda(\Gamma)/\operatorname{mod}(\Gamma),$$

where

$$\Lambda(\Gamma) = \bigcup_{[\varphi]\in\operatorname{mod}(\Gamma)} [\varphi](m(T(\Gamma'))).$$

On the other hand, the action of $\operatorname{mod}(\Gamma')$ on $T(\Gamma')$ restricts to an action of $\operatorname{mod}(\Gamma', \Gamma)$ on $T(\Gamma')$, and so the quotient $T(\Gamma')/\operatorname{mod}(\Gamma', \Gamma)$ makes sense. We are now in a position to prove the main result of this section.

Theorem 3.1. *There exists a homeomorphism between the moduli space $M_h(g, k, \varepsilon)$ and the quotient $T(\Gamma')/\operatorname{mod}(\Gamma', \Gamma)$.*

So, as in the classical case of Riemann surfaces, the relative modular group $\operatorname{mod}(\Gamma', \Gamma)$ or the hyperelliptic mapping class group $\operatorname{map}^h(g, k, \varepsilon)$ allows us to present the moduli space of hyperelliptic Klein surfaces of given topological type as a quotient of a Teichmüller space.

Proof of Theorem 3.1. The map $\phi : T(\Gamma') \to M_h(g, k, \varepsilon)$ defined as $\phi([\alpha]) = [m([\alpha])]_{\operatorname{mod}(\Gamma)}$ is obviously surjective. Moreover, if $\phi([\alpha]) = \phi([\beta])$ for some $[\alpha]$ and $[\beta]$ in $T(\Gamma')$, then $[\beta \circ i] = [\alpha \circ i \circ \varphi]$ for some $\varphi \in \operatorname{aut}(\Gamma)$, and so there exists $g \in G$ such that $\alpha \circ i \circ \varphi = c_g \circ \beta \circ i$, where c_g denotes conjugation by g. Hence, if $j = c_g \circ \beta$, then $\alpha(i(\Gamma)) = j(i(\Gamma))$. This implies, by the uniqueness of the hyperelliptic involution, that the groups $\alpha(\Gamma')$ and $j(\Gamma')$ coincide, since both are isomorphic to Γ' and contain $\alpha(i(\Gamma)) = j(i(\Gamma))$ as a subgroup of index 2. Thus $\psi = \alpha^{-1} \circ j \in \operatorname{aut}(\Gamma', \Gamma)$ and $[\alpha \circ \psi] = [j] = [\beta]$.

From the above and Corollary 8.9 and Theorem 9.12 of [**10**], it follows that the map

$$\hat{\phi} : \frac{T(\Gamma')}{\operatorname{mod}(\Gamma', \Gamma)} \longrightarrow M_h(g, k, \varepsilon),$$

induced by ϕ, is a homeomorphism. □

4. The modular group mod(Γ')

As we have already said, our goal in this paper is to compute the group mod(Γ', Γ). This will be done in the next section and, as a first step in this direction, we now compute mod(Γ').

Let D be the closed unit disc, and let us fix two sets, $A_1 \subset D^o$ = interior of D and $A_2 \subset \partial D$, of cardinality r and s, respectively. Obviously, the set $\Sigma = \Sigma_{r,s}$ of autohomeomorphisms φ of D, such that $\varphi(A_1) = A_1$ and $\varphi(A_2) = A_2$, is a group under composition. Two elements φ and ψ in Σ are said to be *equivalent*, and denoted $\varphi \sim \psi$, if there exists a homotopy

$$H : I \times D \to D,$$

where I is the closed interval $[0,1]$, such that $H_0 = \varphi$, $H_1 = \psi$, and for each $t \in I$:

$$H_t(\partial D) = \partial D, \quad H_t(A_1) = A_1 \quad \text{and} \quad H_t(A_2) = A_2.$$

The quotient $\Sigma/\sim$ is the modular group $\mathrm{mod}_{r,s}(D)$ of the punctured disc, and the following proposition is an easy consequence of the results of Zieschang [**13**].

Proposition 4.1. *The groups* $\mathrm{mod}_{r,s}(D)$ *and* $\mathrm{mod}(\Gamma')$ *are isomorphic.*

For every $s \geqslant 2$, let us label $A_2 = \{b_0, \dots, b_{s-1}\}$, and let G_s be the group of contiguity-preserving permutations of A_2 (of course, b_0 and b_{s-1} are contiguous). Clearly, G_2 is the cyclic group of order 2 generated by $\sigma : b_i \longmapsto b_{1-i}$, and for $s > 2$, G_s is the dihedral group D_s of order $2s$ generated by the rotation $\rho : b_i \longmapsto b_{i-1}$, $1 \leqslant i \leqslant s-1$; $b_0 \longmapsto b_{s-1}$, and the symmetry $\sigma : b_i \longmapsto b_{s-i-1}$, $0 \leqslant i \leqslant s-1$.

From Macbeath [**9**], the restriction $\varphi \mid A_2$ of each $\varphi \in \Sigma$ occurs in G_s, and in fact one gets a group epimorphism

$$\pi : \mathrm{mod}_{r,s}(D) \to G_s : [\varphi] \longmapsto \varphi \mid A_2.$$

To see this, we only need to observe that for a homotopy $H : I \times D \longrightarrow D$ with $H_0 = \varphi$ and $H_1 = \psi$, and every point $x \in A_2$, the image $H(I \times \{x\})$ is a connected subset of A_2, i.e. a unique point $y_x \in A_2$, and so $\varphi(x) = H(0,x) = y_x = H(1,x) = \psi(x)$. Consequently, G_s is isomorphic to the quotient $\mathrm{mod}_{r,s}(G)/\ker \pi$.

Proposition 4.2. *If* $s \geqslant 2$, *the group* $\mathrm{mod}(\Gamma') = \mathrm{mod}_{r,s}(D)$ *is a semidirect product* $G_s \propto \ker \pi$.

Proof. In order to show that $\mathrm{mod}_{r,s}(D)$ is a semidirect product of G_s and $\ker \pi$ it suffices to define a section $\varphi : G_s \to \mathrm{mod}_{r,s}(D)$ of π. Since ρ and σ are the generators of G_s (or just σ if $s = 2$) it is enough to construct $\varphi(\rho) = \hat{\rho}$ and $\varphi(\sigma) = \hat{\sigma}$. In fact, using the isomorphism between $\mathrm{mod}_{r,s}(D)$ and $\mathrm{mod}(\Gamma')$, we shall first construct $\hat{\rho}$ and $\hat{\sigma}$ as elements in $\mathrm{mod}(\Gamma')$. For this purpose, let us denote by

$$\{x_1, \dots, x_r, e, c_0, \dots, c_s\}$$

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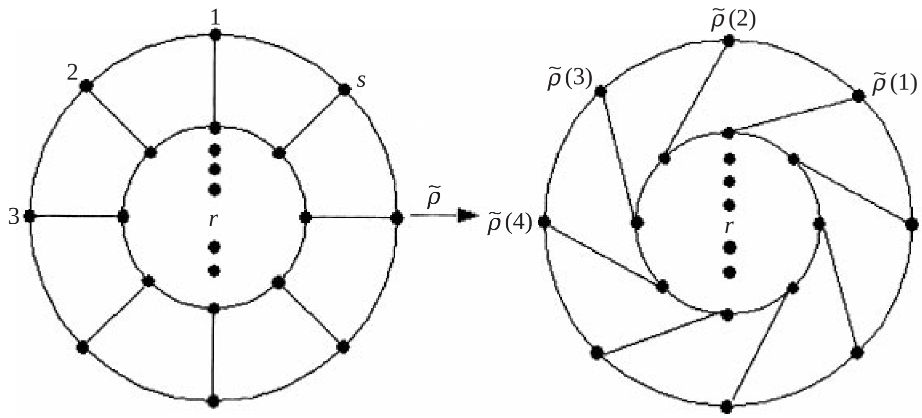

Figure 1. The automorphism $\tilde{\rho}$.

a set of canonical generators of Γ', where each x_i is elliptic of order 2, e is hyperbolic, each c_j is a reflection and

$$(c_j c_{j+1})^2 = 1;\ x_1 \dots x_r.e = 1;\ e^{-1} c_0 e c_s = 1.$$

We fix this notation throughout the paper.

Let us consider the automorphisms of Γ' induced by the assignments:

$$\left.\begin{array}{c} \hat{\rho} : x_i \mapsto x_i; \quad e \mapsto e; \quad c_0 \mapsto e c_{s-1} e^{-1}; \quad c_j \mapsto c_{j-1}, \quad 1 \leqslant j \leqslant s; \\ \hat{\sigma} : x_1 \mapsto x_1; \quad x_i \mapsto y_i x_i y_i^{-1}, \quad 2 \leqslant i \leqslant r; \quad e \mapsto e^{-1}; \quad c_0 \mapsto e^{-1} c_1 e; \\ c_j \mapsto c_{s-j+1}, \quad 1 \leqslant j \leqslant s, \end{array}\right\} \tag{4.1}$$

where $y_i = x_1 \dots x_{i-1}$ for $2 \leqslant i \leqslant r$.

Given the point $b_{j+1} \in A_2$ as the intersection of the axes of the reflections c_j and c_{j+1}, then $\pi(\hat{\rho})(b_{j+1})$ is the intersection of the axes of $\hat{\rho}(c_j) = c_{j-1}$ and $\hat{\rho}(c_{j+1}) = c_j$, i.e

$$\pi([\hat{\rho}])(b_{j+1}) = b_j = \rho(b_j),$$

and so $\pi([\hat{\rho}]) = \rho$. Analogously $\pi([\hat{\sigma}]) = \sigma$, and we say that $[\hat{\rho}]$ and $[\hat{\sigma}]$ represent ρ and σ, respectively. Note that $[\hat{\rho}]^s = [\hat{\sigma}]^2 = 1$.

The classes $[\hat{\rho}]$, $[\hat{\sigma}]$ can be represented as the automorphisms $\tilde{\rho}$, $\tilde{\sigma}$ of the orbifold $X/\langle\phi\rangle$ shown in Figures 1 and 2. The automorphisms given by $\tilde{\rho}$ and $\tilde{\sigma}$ in the orbifold fundamental group of $X/\langle\phi\rangle$ are exactly $\hat{\rho}$ and $\hat{\sigma}$. □

We shall now obtain a presentation by generators and relations of $\mathrm{mod}_{r,s}(D)$. For this purpose we compute $\ker \pi$ and we study first the case $s > 2$. Then, the elements in $\ker \pi$ are represented by homeomorphisms of D (which leave the boundary ∂D invariant), and fix $s \geqslant 4$ points in ∂D, and so all of them preserve the orientation. Let $\Delta = D \setminus A_1$ and F_r be the free group generated by $X_1, \dots, X_r$. The *braid group* B_r of the plane E^2 is the subgroup of the automorphism group $\mathrm{aut}(F_r)$ of F_r consisting of those

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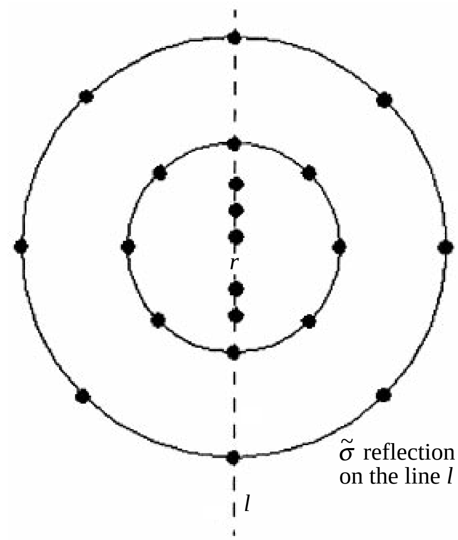

Figure 2. The automorphism $\tilde{\sigma}$.

automorphisms of F_r which fix the product $X_1 \dots X_r$ and map each generator X_i to a conjugate of some generator X_j. If we denote by $I(B_r) = B_r \cap \operatorname{inn}(F_r)$ the intersection of B_r with the subgroup of inner automorphisms of F_r, Birman [**2**] proved the existence of an epimorphism

$$\theta : \operatorname{homeo}(\varDelta; \partial\varDelta) \longrightarrow \frac{B_r}{I(B_r)}$$

from the group $\operatorname{homeo}(\varDelta; \partial\varDelta)$ of homeomorphisms of $\varDelta$ leaving $\partial\varDelta$ invariant, onto the quotient $B_r/I(B_r)$, whose kernel consists of those homeomorphisms homotopic to the identity. Thus, since

$$\ker\pi \approx \frac{\operatorname{homeo}(\varDelta; \partial\varDelta)}{\ker\theta} \approx \frac{B_r}{I(B_r)},$$

it is enough to describe this last quotient. With the notation in Magnus [**11**]:

$$B_r = \langle \delta_1, \dots, \delta_{r-1} : \delta_i\delta_j = \delta_j\delta_i \text{ if } |i-j| \geqslant 2;\ \delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1} \text{ and } 1 \leqslant i \leqslant r-2\rangle;$$

$$A_r^* = \frac{B_r}{I(B_r)} = \langle \delta_1, \dots, \delta_{r-1} : \delta_i\delta_j = \delta_j\delta_i \text{ if } |i-j| \geqslant 2;$$
$$\delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1},\ 1 \leqslant i \leqslant r-2;\ (\delta_1, \dots, \delta_{r-1})^r = 1\rangle,$$

where $A_0^* = A_1^*$ is the trivial group. Note also that $A_2^* = \boldsymbol{Z}/2\boldsymbol{Z}$.

By (4.1) we know how to represent the generators of D_s as elements in $\operatorname{mod}(\varGamma')$; to understand the meaning of the semidirect product $D_s \propto A_r^*$ we must provide an explicit presentation of the generators δ_j of A_r^* as elements in $\operatorname{mod}(\varGamma')$. Let $r \geqslant 2$, and for every $1 \leqslant j \leqslant r-1$, let $\alpha_j : \varGamma' \longrightarrow \varGamma'$ be the automorphism induced by the assignment

$$\left.\begin{array}{l} x_j \mapsto x_{j+1}; \quad x_{j+1} \mapsto x_{j+1}x_jx_{j+1}; \quad x_i \longmapsto x_i \quad \text{if } j \neq i \neq j+1; \\ \qquad e \longmapsto e; \quad c_k \longmapsto c_k \quad \text{for } 0 \leqslant k \leqslant s. \end{array}\right\} \tag{4.2}$$

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It is easy to check that $\alpha_j \hat{\rho} = \hat{\rho} \alpha_j$ and $\alpha_j \hat{\sigma} \alpha_j = \hat{\sigma}$. Summarizing, and since A_r^* is trivial if $r < 2$, we can now give a presentation for $\operatorname{mod}_{r,s}(D)$ if $s > 2$.

Theorem 4.3.

(i) *If $s > 2$ and $r \geqslant 2$, then $\operatorname{mod}_{r,s}(D) = D_s \propto A_r^*$, where*

$$D_s \propto A_r^* = \langle \delta_1, \dots, \delta_{r-1}, \rho, \sigma : \rho^s = \sigma^2 = 1;\ \delta_i \delta_j = \delta_j \delta_i \text{ if } |i - j| \geqslant 2; \\ \delta_i \delta_{i+1} \delta_i = \delta_{i+1} \delta_i \delta_{i+1},\ 1 \leqslant i \leqslant r - 2;\ (\delta_1 \dots \delta_{r-1})^r = 1; \\ \delta_j \rho = \rho \delta_j;\ \delta_j \sigma \delta_j = \sigma,\ 1 \leqslant i \leqslant r - 1 \rangle.$$

(ii) *If $s > 2$ and $r < 2$, then $\operatorname{mod}_{r,s}(D) = D_s$.*

Moreover, formulae (4.1) and (4.2) show the way the generators of $\operatorname{mod}(\Gamma')$ can be expressed as (classes of) automorphisms of Γ'.

The cases $s = 2$ and $s = 0$ follow similar arguments as the previous one. We just sketch the proof, pointing out the specific facts for these cases.

Assume now that $s = 2$, and let us denote by R the reflection of the disc D whose axis is the line joining the two points in A_2. Each homeomorphism of D whose restriction to A_2 is the identity is homotopic to either R or the identity, and so we get an epimorphism,

$$\eta : \ker \pi \longrightarrow \boldsymbol{Z}/2\boldsymbol{Z} : [\varphi] \longmapsto \begin{cases} 1, & \text{if } \varphi \text{ is homotopic to } R, \\ 0, & \text{otherwise,} \end{cases}$$

whose kernel is A_r^*. Hence, $\ker \pi$ is a semidirect product $\ker \pi = \boldsymbol{Z}/2\boldsymbol{Z} \propto A_r^*$. Moreover, if $r \geqslant 2$, let $\hat{\mu} : \Gamma' \longrightarrow \Gamma'$ be the automorphism of Γ' induced by the assignment

$$\left.\begin{array}{r} \hat{\mu} : x_1 \mapsto x_1; \quad x_i \mapsto y_i x_i y_i^{-1}, \quad 2 \leqslant i; \quad e \mapsto e^{-1}; \\ c_0 \mapsto e^{-1} c_1 e; \quad c_k \mapsto c_{3-k}, \quad k = 1, 2, \end{array}\right\} \tag{4.3}$$

where $y_i = x_1 \dots x_{i-1}$.

It is boring, but straightforward, to check that $\hat{\mu}$ is actually an automorphism of Γ' of order 2 such that $\alpha_j \hat{\mu} \alpha_j = \hat{\mu}$, $1 \leqslant j \leqslant r - 1$, where $\alpha_1, \dots, \alpha_{r-1}$ are the automorphisms of Γ' defined in (4.2). Moreover, $\eta([\widehat{\mu}]) = 1$.

Also, with the notation of (4.1), one can check that $(\hat{\sigma}\hat{\mu})^2$ is an inner automorphism of Γ' (conjugation by $x_r \cdot x_{r-1} \dots x_1$), and so, if μ is the class of $\hat{\mu}$, $\operatorname{mod} \operatorname{inn}(\Gamma')$, we obtain that if $s = 2$ and $r \geqslant 2$, then

$$\ker \pi = \boldsymbol{Z}/2\boldsymbol{Z} \propto A_r^* = \langle \delta_1, \dots, \delta_{r-1},\ \mu : \mu^2 = 1,\ \delta_i \delta_j = \delta_j \delta_i \text{ if } |i - j| \geqslant 2; \\ \delta_i \delta_{i+1} \delta_i = \delta_{i+1} \delta_i \delta_{i+1},\ \text{for } 1 \leqslant i \leqslant r - 2; \\ (\delta_1 \dots \delta_{r-1})^r = 1;\ \delta_i \mu \delta_i = \mu,\ 1 \leqslant i \leqslant r - 1 \rangle.$$

Consequently, if $r \geqslant 2$,

$$\operatorname{mod}_{r,2}(D) = G_s \propto \ker \pi = \boldsymbol{Z}/2\boldsymbol{Z} \propto \ker \pi = \boldsymbol{Z}/2\boldsymbol{Z} \propto (\boldsymbol{Z}/2\boldsymbol{Z} \propto A_r^*).$$

Theorem 4.4.

(i) *If* $s = 2$ *and* $r \geqslant 2$, $\mathrm{mod}_{r,2}(D) = D_2 \propto A_r^*$, *where*

$$\begin{aligned} D_2 \propto A_r^* = \langle \sigma, \mu, \delta_1, \dots, \delta_{r-1} : \sigma^2 = \mu^2 = (\sigma\mu)^2 = 1,\ \delta_i\delta_j = \delta_j\delta_i \text{ if } |i-j| \geqslant 2; \\ \delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1},\ \text{for } 1 \leqslant i \leqslant r-2;\ (\delta_1 \dots \delta_{r-1})^r = 1; \\ \delta_i\mu\delta_i = \mu,\ \delta_i\sigma\delta_i = \sigma,\ \text{for } 1 \leqslant i \leqslant r-1\rangle. \end{aligned}$$

(ii) *If* $s = 2$ *and* $r \leqslant 1$, $\mathrm{mod}_{r,2}(D) = D_2$.

To finish this section we compute $\mathrm{mod}_{r,0}(D)$. Each homotopy class in $\mathrm{mod}_{r,0}(D)$ contains a representative φ with a fixed point $p \in \partial D = S^1$. Hence, if φ^* denotes the automorphism of the fundamental group $\pi_1(S^1) = \boldsymbol{Z}$ with base point p induced by the restriction $\varphi \mid S^1$, one gets an epimorphism

$$\mathrm{mod}_{r,0}(D) \to \boldsymbol{Z}/2\boldsymbol{Z} = \mathrm{aut}(\pi_1(S^1)) : [\varphi] \mapsto \varphi^*,$$

with kernel A_r^* (this epimorphism can be described in the same way as η in the case $s = 2$). Hence $\mathrm{mod}_{r,0}(D) = \boldsymbol{Z}/2\boldsymbol{Z} \propto A_r^*$, where the generator ς of $\boldsymbol{Z}/2\boldsymbol{Z}$ can be seen as the class, $\mathrm{mod}\,\mathrm{inn}(\Gamma')$, of the automorphism $\hat{\varsigma} : \Gamma' \to \Gamma'$ induced by the assignment

$$x_1 \mapsto x_1, \quad x_i \mapsto y_i x_i y_i^{-1}, \quad \text{for } 2 \leqslant i \leqslant r; \qquad e \mapsto e^{-1}; \qquad c_0 \mapsto e^{-1} c_0 e. \tag{4.4}$$

It is easy to check that $\delta_j\varsigma\delta_j = \varsigma$, and so we have the following theorem.

Theorem 4.5.

(i) *If* $s = 0$ *and* $r \geqslant 2$, *then*

$$\begin{aligned} \mathrm{mod}_{r,0}(D) &= \boldsymbol{Z}/2\boldsymbol{Z} \propto A_r^* \\ &= \langle \varsigma, \delta_1, \dots, \delta_{r-1} : \varsigma^2 = 1,\ \delta_i\delta_j = \delta_j\delta_i \text{ if } |i-j| \geqslant 2; \\ &\qquad \delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1},\ \text{for all } 1 \leqslant i \leqslant r-2; \\ &\qquad (\delta_1 \dots \delta_{r-1})^r = 1;\ \delta_i\varsigma\delta_i = \varsigma,\ \text{for } 1 \leqslant i \leqslant r-1\rangle. \end{aligned}$$

(ii) *If* $s = 0$ *and* $r < 2$, *then* $\mathrm{mod}_{r,0}(D) = \boldsymbol{Z}/2\boldsymbol{Z}$.

5. The hyperelliptic mapping class group $\mathrm{map}^h(g, k, \varepsilon)$

We have introduced the mapping class group $\mathrm{map}^h(g, k, \varepsilon)$ of hyperelliptic Klein surfaces of fixed topological type (g, k, ε). The group $\mathrm{map}^h(g, k, \varepsilon)$ is isomorphic to the relative modular group

$$\mathrm{mod}(\Gamma', \Gamma) = \frac{\mathrm{aut}(\Gamma', \Gamma)}{\mathrm{inn}(\Gamma')},$$

which is a subgroup of the modular group

$$\mathrm{mod}(\Gamma') = \frac{\mathrm{aut}(\Gamma')}{\mathrm{inn}(\Gamma')}$$

that we computed in the previous section.

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Let $p : \mathit{\Gamma}' \to \boldsymbol{Z}/2\boldsymbol{Z}$ be an epimorphism with $\ker p = \mathit{\Gamma}$. It is clear that an automorphism $f \in \operatorname{aut}(\mathit{\Gamma}')$ occurs in $\operatorname{aut}(\mathit{\Gamma}', \mathit{\Gamma})$ if and only if $p \circ f = p$. We use this to prove the following theorem.

Theorem 5.1.

(i) *If* $s = 0$, *then* $\operatorname{mod}(\mathit{\Gamma}', \mathit{\Gamma}) = \operatorname{mod}(\mathit{\Gamma}') = \boldsymbol{Z}/2\boldsymbol{Z} \propto A_r^*$.

(ii) *If* $s = 2$, *then* $\operatorname{mod}(\mathit{\Gamma}', \mathit{\Gamma}) = A_r^*$.

(iii) *If* $s > 2$, *then* $\operatorname{mod}(\mathit{\Gamma}', \mathit{\Gamma}) = D_{s/2} \propto A_r^*$.

Proof. (i) If $s = 0$, the epimorphism p is defined by

$$p : c_0 \mapsto 0; \quad x_i \mapsto 1; \quad 1 \leqslant i \leqslant r; \quad e \mapsto \begin{cases} 0, & \text{if } r \text{ is even,} \\ 1, & \text{if } r \text{ is odd.} \end{cases}$$

With the notation in (4.2) and (4.4) it is easy to check the equalities $p \circ \alpha_j = p$, $1 \leqslant j \leqslant r - 1$ and $p \circ \hat{\varsigma} = p$. Thus, from Theorem 4.5 it follows that $\operatorname{mod}(\mathit{\Gamma}', \mathit{\Gamma}) = \operatorname{mod}(\mathit{\Gamma}') = \boldsymbol{Z}/2\boldsymbol{Z} \propto A_r^*$.

(ii), (iii) Now let $s \neq 0$. Then $p : \mathit{\Gamma}' \to \boldsymbol{Z}/2\boldsymbol{Z}$ is defined by

$$p : c_j \mapsto \begin{cases} 1, & \text{if } j \text{ is even,} \\ 0, & \text{if } j \text{ is odd,} \end{cases} \qquad x_i \mapsto 1, \quad 1 \leqslant i \leqslant r;$$

$$e \mapsto \begin{cases} 0, & \text{if } r \text{ is even,} \\ 1, & \text{if } r \text{ is odd.} \end{cases}$$

With the notation of (4.1), (4.2) and (4.3), one easily checks that $p \circ \alpha_j = p$, $1 \leqslant j \leqslant r - 1$; $p \circ \hat{\sigma} = p$; but $p \circ \hat{\rho} \neq p$, if $s > 2$ and $p \circ \hat{\mu} \neq p$; $p \circ \hat{\sigma} \neq p$ if $s = 2$. Thus, if we write $\operatorname{mod}(\mathit{\Gamma}') = D_s \propto A_r^* = (\boldsymbol{Z}/s\boldsymbol{Z} \propto \boldsymbol{Z}/2\boldsymbol{Z}) \propto A_r^*$, then $\operatorname{mod}(\mathit{\Gamma}', \mathit{\Gamma}) = D_{s/2} \propto A_r^*$ if $s > 2$, and $\operatorname{mod}(\mathit{\Gamma}', \mathit{\Gamma}) = A_r^*$ if $s = 2$. □

Remark 5.2. Of course, from the proof above the semidirect product $D_{s/2} \propto A_r^*$ if $s > 2$ is easily presented by generators and relations using the presentations given in theorem 4.3.

6. Some consequences

(1) If $g = 0$, then $r = 0$ and $s = 2k > 2$, and so the hyperelliptic mapping class group $\operatorname{map}^h(0, k, 2) = D_k$.

(2) If $g \leqslant 2$ and $\varepsilon = 1$, then

$$\operatorname{map}^h(g, k, 1) = \begin{cases} D_k, & \text{if } g = 1, \quad k > 1, \\ \boldsymbol{Z}/2\boldsymbol{Z}, & \text{if } g = 2, \quad k = 1, \\ D_k \propto \boldsymbol{Z}/2\boldsymbol{Z}, & \text{if } g = 2, \quad k > 1. \end{cases}$$

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(3) Since A_r^* is not finite for $r \geqslant 3$ and we just consider Klein surfaces of algebraic genus $p \geqslant 2$, i.e. $\varepsilon g + k \geqslant 3$, the cases quoted in (1) and (2) are the only ones in which the hyperelliptic mapping class group $\mathrm{map}^h(g, k, \varepsilon)$ is finite.

(4) In particular, since every Klein surface of algebraic genus $p = 2$ is hyperelliptic, the mapping class groups

$$\mathrm{map}(g, k, \varepsilon) \simeq \mathrm{map}^h(g, k, \varepsilon) \times \boldsymbol{Z}/2\boldsymbol{Z}$$

if $\varepsilon g + k = 3$, and since $k > 0$, from the precedent analysis, it follows that they are finite unless $g = k = 1$, $\varepsilon = 2$.

(5) Birman and Chillingworth [**3**] studied the homeotopy group $H(X)$ of a non-orientable surface X, i.e. the quotient $H(X) = G(X)/D(X)$ of the group $G(X)$ of all autohomeomorphisms of X under the subgroup $D(X)$ of homeomorphisms isotopic to the identity. In particular, they proved the following theorem (see [**3**, Theorem 3].

Theorem 6.1 (Birman–Chillingworth). *Let X be the sum of three projective planes. The homeotopy group $H(X)$ of X has the following presentation:*

$$H(X) = \langle A, B, Y : ABA = BAB;\ YAY^{-1} = A^{-1};\ YBY^{-1} = B^{-1};\ Y^2 = (ABA)^4 = 1\rangle.$$

To finish the paper we give another proof of this result with some extra information.

First of all, notice that $h = (ABA)^2$ has order 2 and it is easy to check that it is a central element of $H(X)$. Let a, b, y be the classes of A, B, Y mod $\langle h\rangle$. Then $(ab)^3 = ababab = (aba)^2 = 1$, and so

$$H(X) = \langle h\rangle \times G(X),$$

where

$$G(X) = \langle a, b, y : aba = bab;\ aya = y;\ byb = y;\ y^2 = 1,\ (ab)^3 = 1\rangle.$$

On the other hand, the NEC group Γ uniformizing X admits a set $\{z_1, z_2, z_3\}$ of canonical generations which are glide reflections satisfying the relation $z_1^2 z_2^2 z_3^2 = 1$, and

$$H(X) = \mathrm{mod}(\Gamma) = \frac{\mathrm{aut}(\Gamma)}{\mathrm{inn}(\Gamma)}.$$

In this case $g = 3$, $k = 0$, $\varepsilon = 1$, and so, by (2.2), $r = 3$, $s = 0$.

Hence, the group Γ' containing Γ as a subgroup of index 2 has a set of canonical generators $\{x_1, x_2, x_3, e, c\}$ and the restriction map $r : \mathrm{aut}(\Gamma', \Gamma) \to \mathrm{aut}(\Gamma) : f \mapsto f \mid \Gamma$ is a group isomorphism. However, if $x \in \Gamma' \setminus \Gamma$, the inner automorphism

$$c_x : \Gamma' \to \Gamma^{'} : \varphi \mapsto x\varphi x^{-1}$$

is a trivial element in $\mathrm{mod}(\Gamma', \Gamma)$, but $r(c_x)$ induces a non-zero element of $\mathrm{mod}(\Gamma)$. In other words, we have a group epimorphism

$$\mathrm{mod}(\Gamma) \to \mathrm{mod}(\Gamma', \Gamma) : [f] \mapsto [r^{-1}(f)],$$

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whose kernel is the group of order 2 generated by the class of $r(c_x)$. Consequently

$$H(X) = \mathrm{mod}(\Gamma) = \boldsymbol{Z}/2\boldsymbol{Z} \propto \mathrm{mod}(\Gamma', \Gamma)$$

and we are going to see that this is, in fact, a direct product.

We remark that the surface X has no boundary, but since it has genus 3, then the quotient of X by the hyperelliptic involution has boundary and then we can apply Theorem 4.5. Then

$$\mathrm{mod}(\Gamma', \Gamma) = \langle \tau, \delta_1, \delta_2 : \delta_1\delta_2\delta_1 = \delta_2\delta_1\delta_2;\ \delta_i\tau\delta_i = \tau,\ i = 1, 2;\ \tau^2 = (\delta_1\delta_2)^3 = 1\rangle.$$

In fact τ, δ_1 and δ_2 are the classes $\mathrm{mod}\,\mathrm{inn}(\Gamma')$ of the following automorphisms of Γ' (see (4.2) and (4.4)):

$$\alpha_1 : \begin{cases} x_1 \mapsto x_2, \\ x_2 \mapsto x_2x_1x_2, \\ x_3 \mapsto x_3, \\ e \mapsto e, \\ c \mapsto c; \end{cases} \qquad \alpha_2 : \begin{cases} x_1 \mapsto x_1, \\ x_2 \mapsto x_3, \\ x_3 \mapsto x_3x_2x_3, \\ e \mapsto e, \\ c \mapsto c; \end{cases} \qquad \widehat{\tau} : \begin{cases} x_1 \mapsto x_1, \\ x_2 \mapsto x_1x_2x_1, \\ x_3 \mapsto x_1x_2x_3x_2x_1, \\ e \mapsto e^{-1}, \\ c \mapsto e^{-1}ce. \end{cases}$$

To compute the restrictions $r(\alpha_1)$, $r(\alpha_2)$ and $r(\hat{\tau})$ we must write the generators z_i of Γ in terms of the ones of Γ'. It is not hard to see that

$$z_1 = cx_1; \quad z_2 = x_1cx_1x_2; \quad z_3 = x_2x_1cx_1x_2x_3.$$

From this it follows immediately that

$$z_2z_3 = x_2x_3; \quad z_1z_2 = x_1x_2; \quad z_1z_2^2z_3 = x_1x_3.$$

These relations are useful to compute the restrictions of α_1, α_2 and $\hat{\tau}$ which are also denoted by α_1, α_2 and $\hat{\tau}$:

$$\alpha_1 : \begin{cases} z_1 \mapsto z_1^2z_2, \\ z_2 \mapsto z_2^{-1}z_1^{-1}z_2, \\ z_3 \mapsto z_3; \end{cases} \qquad \alpha_2 : \begin{cases} z_1 \mapsto z_1, \\ z_2 \mapsto z_2^2z_3, \\ z_3 \mapsto z_3^{-1}z_2^{-1}z_3; \end{cases} \qquad \hat{\tau} : \begin{cases} z_1 \mapsto z_3^{-2}z_2^{-2}z_1^{-1}, \\ z_2 \mapsto z_1z_2^2z_3^2z_2^{-1}z_1^{-1}, \\ z_3 \mapsto z_1z_2z_3^{-1}z_2^{-1}z_1^{-1}. \end{cases}$$

Moreover, $x_1 \in \Gamma' \setminus \Gamma$ and so the automorphism $\hat{h}$ of Γ given by 'conjugation by x_1' is defined by

$$\hat{h} : \begin{cases} z_1 \mapsto z_1^{-1}, \\ z_2 \mapsto z_1z_2^{-1}z_1^{-1}, \\ z_3 \mapsto z_1z_2^2z_3^{-1}z_2^{-1}z_1^{-1}. \end{cases}$$

Now, $\hat{h}$ commutes with α_2 and $\hat{\tau}$. Moreover, if $w = z_1z_2$, then $\hat{h} \circ \alpha_1 = C_w \circ (\alpha_1 \circ \hat{h})$, where C_w denotes 'conjugation by w'. Thus, if h is the class of $\hat{h}$ mod $\mathrm{inn}(\Gamma)$, h commutes with δ_1, δ_2 and τ, and so

$$H(X) = (\boldsymbol{Z}/2\boldsymbol{Z} = \langle h \rangle) \times \mathrm{mod}(\Gamma', \Gamma).$$

Notice also that δ_1, τ and $\delta_2 h$ generate $\mathrm{mod}(\Gamma)$, because $(\delta_1 \cdot (\delta_2 h))^3 = h$. Hence, the assignment

$$\tau \mapsto y; \qquad \delta_1 \mapsto a; \qquad \delta_2 \mapsto b$$

induces an isomorphism between $G(X)$ and $\mathrm{mod}(\Gamma', \Gamma)$.

In this way we have reproved Theorem 6.1, with an explicit representation of a set of generators of $H(X)$, which are not those given by Birman and Chillingworth, as classes of automorphisms of the NEC group Γ that uniformizes X.

Acknowledgements. The authors are grateful to the referees for their helpful comments and corrections. E.B. and A.F.C. were partly supported by DGICYT PB 98-0017 and J.M.G. was partly supported by DGICYT PB 98-0756.

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