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THE SPECTRUM OF A FINITE LATTICE: BREADTH AND LENGTH TECHNIQUES

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Efforts to determine the orders of the sublattices of an arbitrary finite lattice date back at least to the early 1930's, and notably, in the work of Fritz Klein-Barmen [3], [4]. Nevertheless, very little that is new has appeared in the literature since that time.

The *spectrum* of a lattice L, denoted by $\mathbf{sp}(L)$, is the set of all integers n such that L has an n-element sublattice. We say that the spectrum of a finite lattice L is *complete* provided that $\mathbf{sp}(L)=\{n \mid 0\le n\le |L|\}$. While Klein-Barmen [3] was the first to make the observation, it is a simple computation to verify that every lattice with at most seven elements has a complete spectrum. On the other hand, the lattice $\mathbf{2}^n$ of all subsets of an n-element set does not have a complete spectrum in case $n\ge 3$. A lattice may, however, have a complete spectrum even though sublattices of it do not. The lattice illustrated in Figure 1 is such an example; it is also an instance of our first main result.

Let $l(L)$ denote the *length* of a lattice L, that is, the order of a maximum-sized chain of L minus one.

THEOREM. *Every finite modular lattice L satisfying*

$$|L|\le 2l(L)+1$$

has a complete spectrum. Moreover, this is a best possible estimate.

The proof of this theorem involves a further arithmetical invariant of a lattice L, namely, its *breadth*, $b(L)$, that is, the least integer b such that every join $\bigvee_{i=1}^{n} x_i$, $n>b$, is a join of b of the x_i's. In fact, as the following result indicates the breadth and length together provide a great deal of information concerning the spectrum of a modular lattice.

A lattice L, is *linearly decomposable* if it contains nonempty sublattices A and B, $A\ne B$, such that $L=A\cup B$ and, for each $a\in A$ and for each $b\in B$, $a\ge b$; otherwise, L is said to be *linearly nondecomposable.*

THEOREM. *Let L be a finite, linearly nondecomposable, modular lattice. Then*

$$2^i+j\in \mathbf{sp}(L)$$

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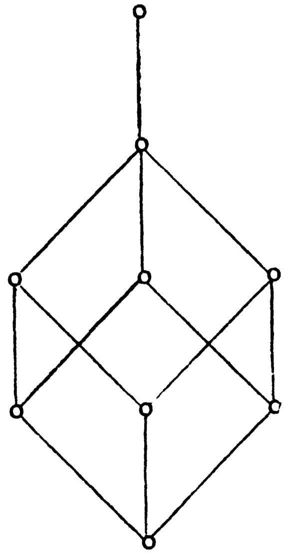

Figure 1

for all integers i and j satisfying

$$1 \le i \le b(L) \quad \textit{and} \quad 0 \le j \le 2(l(L)-i).$$

There is another approach to the problem of determining the spectrum of an arbitrary finite lattice and it, too, finds its roots in the work of Klein-Barmen. In particular, he showed in [4] that, for every integer $n \le 6$ and any modular lattice L, $n \in \mathbf{sp}(L)$ whenever $|L| \ge 6$. Again, the lattice $\mathbf{2}^3$ shows that this is no longer true for $n = 7$.

THEOREM. *Let L be a finite distributive lattice and let n be any positive integer. If*

$$|L| \ge n2^{n/4}$$

then $n \in \mathbf{sp}(L)$.

Interestingly enough, the proof of this theorem will rely on yet another relationship between breadth and length.

THEOREM. *Let L be a finite distributive lattice. Then*

$$|L| \le \left(\frac{l(L)}{b(L)}+1\right)^{b(L)}.$$

Moreover, this inequality is best possible.

Modular lattices with complete spectrum. The purpose of this section is to prove the first two theorems announced above. To this end, we first dispense with certain preliminary considerations.

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For elements a and b of a lattice L, we write $a \succ b$ or $b \prec a$ (*a covers b* or *b is covered by a*) if, for every element c of L $a \geq c > b$ implies $a = c$.

LEMMA 1. *Let a and b be noncomparable elements of a finite modular lattice L. Then there exists an element c of L such that* $a \vee c \succ a$ *and* $a \vee c \succ c$.

Proof. Choose elements x and y in L such that $a \prec x \leq a \vee b$ and $b \leq y \prec a \vee b$ and set $c = x \wedge y$.

Let $J(L)$, $M(L)$ and $D(L)$ denote, respectively, the set of all join irreducible, meet irreducible and doubly irreducible elements of a lattice L. Recall that a lattice L with n elements is *dismantlable* if there is a chain $L = L_0 \supset L_1 \supset \cdots \supset L_n = \varnothing$ of sublattices of L such that $|L_{i-1} - L_i| = 1$ for each $i = 1, 2, \ldots, n$. Evidently, $L_{i-1} - L_i \subseteq D(L)$ for each i; *a fortiori*, L has a complete spectrum.

LEMMA 2. *Let C be a maximal chain of a finite modular lattice L. Then there is a sublattice S of L containing C and satisfying:*
(i) *S is dismantlable;*
(ii) $l(S) = l(L)$;
(iii) $|S| = |C| + |C - J(L)|$.

Proof. Let $\{x_1 > x_2 > \cdots > x_n\} = C - J(L)$ and note that each x_i covers an element of $L - C$. Choose $y_1 \in L - C$ such that $x_1 \succ y_1$. For $i > 1$, choose $y_i \in L - C$ satisfying $x_i \succ y_i$ provided that $x_i \wedge y_{i-1} \in C$ while, if $x_i \wedge y_{i-1} \notin C$ choose $y_i = x_i \wedge y_{i-1}$. In view of this construction it suffices to show that $S = C \cup \{y_1, y_2, \cdots, y_n\}$ is a sublattice of L and that it is dismantlable.

First, we observe that, by virtue of modularity, $x_i \succ y_i$ for each $i = 1, 2, \ldots, n$. Let a and b be noncomparable elements of S. If $a = y_i$, say, then $b \in S - \{y_i, y_2, \ldots, y_n\}$, whence, $a \vee b = x_i \in S$. That $a \wedge b \in S$ is an immediate consequence of modularity; hence, S is a sublattice of L. Finally, as $y_1 \in D(S)$ and $y_i \in D(S - \{y_1, y_2, \ldots, y_{i-1}\})$, for each $i = 2, 3, \ldots, n$, it follows that S is dismantlable.

LEMMA 3. *Let L be a finite, linearly nondecomposable, modular lattice. Then L contains a dismantlable sublattice S such that* $l(S) = l(L)$ *and satisfying* $|S| = 2l(L)$.

Proof. In view of Lemma 2 it is enough to construct a maximal chain C of L such that $|C - J(L)| = l(L) - 1$. In fact, we construct a maximal chain $C = \{1 = c_1 \succ c_2 \succ \cdots \succ c_n = 0\}$ of L such that $C \cap J(L) = \{c_{n-1}, c_n\}$. As L is linearly nondecomposable $c_1 = 1 \notin J(L)$. Let us suppose that $c_1 \succ c_2 \succ \cdots \succ c_i$ have been chosen such that each of $c_1, c_2, \ldots, c_i$ is join reducible and let us suppose that $c_i \succ x \succ x_* > 0$ where $x \in J(L)$. Since L is linearly nondecomposable there exists $y \in L$ noncomparable to x_* and, in view of Lemma 1, we may suppose that $x_* \vee y \succ x_*$ and $x_* \vee y \succ y$. Now pick j maximum such that $y < c_j$.

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Then $y \vee c_{j+1} = c_j$. Moreover, by modularity, $c_j = y \vee c_{j+1} \succ y \vee c_{j+2} \succ \cdots \succ y \vee c_i \succ y \vee x \succ y \vee x_*$. Then each of the $i+1$ elements in the chain

$$c_1 \succ c_2 \succ \cdots \succ c_j \succ y \vee c_{j+2} \succ \cdots \succ y \vee c_i \succ y \vee x \succ y \vee x_*$$

is join reducible. Proceeding in this way we can construct a maximal chain of L with the desired properties.

We are now in a position to prove

THEOREM 4. *Let L be a finite, linearly nondecomposable, modular lattice. Then*

$$2^i + j \in \mathbf{sp}(L)$$

for all integers i and j satisfying

$$1 \le i \le b(L) \quad \textit{and} \quad 0 \le j \le 2(l(L) - i).$$

Proof. The first step in the proof is to construct a sublattice S of L such that $l(S) = l(L)$ and $|S| = 2^{b(L)} + 2(l(L) - b(L))$.

It is well known that every finite modular lattice L contains a sublattice $T \cong \mathbf{2}^{b(L)}$ such that, for $x, y \in T$, $x \succ y$ in T whenever $x \succ y$ in L. If the least element 0_T of T coincides with the least element 0_L of L and the greatest element 1_T of T coincides with the greatest element 1_L of Lthen we set $S = T$. Since L is modular it follows that $l(L) = b(L)$ so that S satisfies the required properties.

Let $0_T > 0_L$. Note that every element a which covers 0_T belongs to $[0_T, 1_T]$ since otherwise L contains a sublattice isomorphic to $T \times \{0_T, a\} \cong \mathbf{2}^{b(L)+1}$ which, however, would imply that L has breadth at least $b(L)+1$. For each $a \succ 0_T$ choose $x_a \in L$ distinct from 0_L and minimal with respect to the condition $[0_L, a] = [0_L, x_a] \cup [x_a, a]$ and let x_0 be a minimal element of $\{x_a \mid a \succ 0_T\}$.

Since L is linearly nondecomposable, there exists $y \in L$ noncomparable to x_0. By Lemma 1 we may choose y such that $x_0 \vee y \succ x_0$ and $x_0 \vee y \succ y$ whence, by modularity, $0_T \vee y \succ 0_T$. Let $x_0 \prec x_1 \prec \cdots \prec x_m = 0_T$. Then $x_i \vee y$ is noncomparable to x_{i+1} for each $i = 1, 2, \ldots, m-1$ so that $x_b < x_0$, where $b = 0_T \vee y$, contradicting the minimality of x_0. It follows that $[0_L, a]$ is linearly nondecomposable for some $a \succ 0_T$. By Lemma 3 there exists a dismantlable sublattice A of $[0_L, a]$ such that $l(A) = l([0_L, a])$ and $|A| = 2l(A)$. By duality there exists a dismantlable sublattice B of $[b, 1_L]$ such that $l(B) = l([b, 1_L])$ and $|B| = 2l(B)$, where $1_T \succ b$.

We now select a sublattice T' *of* $[0_T, 1_T]$ containing a and b and isomorphic to $T \cong \mathbf{2}^{b(L)}$. If a is noncomparable to b choose elements $a_1, a_2, \ldots, a_{b(L)-1}$ from $[0_T, 1_T]$, each covering 0_T, such that $a_1 \vee a_2 \vee \cdots \vee a_{b(L)-1} = b$ and take T' to be the sublattice of L generated by $\{a_1, a_2, \ldots, a_{b(L)-1}, a\}$. If $b \ge a$ choose

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elements $a_1, a_2, \ldots, a_{b(L)-2}$ from $[0_T, 1_T]$, each covering 0_T, such that $a_1 \vee a_2 \vee \cdots \vee a_{b(L)-2} \vee a = b$. Then choose an element $a' > 0_T$ noncomparable to b and take T' to be the sublattice of L generated by $\{a_1, a_2, \ldots, a_{b(L)-2}, a, a'\}$.

Under any circumstances we obtain a sublattice $S = A \cup T' \cup B$ such that $l(S) = l(L)$ and $|S| = 2^{b(L)} + 2(l(L) - b(L))$.

The next step of the proof is to construct a sublattice S' of S such that $l(S') = l(S)$ and $|S'| = 2^i + (l(S) - i)$, where $1 \le i \le b(L)$. If a is noncomparable to b choose $a_1, a_2, \ldots, a_{i-1}$ from T' each covering $0_{T'}$ and each beneath b and choose $a_1 \vee a_2 \vee \cdots \vee a_{i-1} \prec x_i \prec x_{i+1} \prec \cdots \prec x_{b(L)-2} \prec b \prec 1_{T'}$. Set $B' = B \cup \{a_1 \vee a_2 \vee \cdots \vee a_{i-1}, x_i, x_{i+1}, \ldots, x_{b(L)-2}, a_1 \vee a_2 \vee \cdots \vee a_{i-1} \vee a, a \vee x_i, a \vee x_{i+1}, \ldots, a \vee x_{b(L)-2}\}$ and let T'' be the sublattice of T' generated by $a_1, a_2, \ldots, a_{i-1}, a$. Then $S' = A \cup T'' \cup B'$ is the required sublattice. If $a \le b$ replace a_1 by a and a by an element $a' > 0_{T'}$, $a' \nleq b$, in the preceding argument.

Finally, $T'' \cong \mathbf{2}^i$, while both A and B' are dismantlable sublattices of L.

The next theorem is motivated by the following elementary result established in [6] by I. Rival.

Lemma 5. *Every finite lattice L satisfies*

$$|L| \ge 2(l(L)+1) - |D(L)|.$$

In particular, a finite lattice L, with no doubly irreducible elements, must have at least $2(l(L)+1)$ elements. For example, any finite linearly decomposable lattice consisting of disjoint linearly nondecomposable lattices each isomorphic to $\mathbf{2}^3$ is of this type (see Figure 2). In other words, there are finite (modular) lattices L with precisely $2(l(L)+1)$ elements but without a complete spectrum. This, in turn, shows that the estimate of the next theorem is best possible.

Theorem 6 (cf. I. Rival [5]). *Every finite modular lattice L satisfying*

$$|L| \le 2l(L) + 1$$

has a complete spectrum.

Proof. We proceed by induction on $|L|$.

In view of Lemma 5, $D(L) \ne \emptyset$. Moreover, if $D(L)$ is not a chain in L then there exists $a \in D(L)$ such that $l(L - \{a\}) = l(L)$ and $|L - \{a\}| \le 2l(L - \{a\}) + 1$ so that by the induction hypothesis $L - \{a\}$, and hence L, both have a complete spectrum. We shall assume, then, that $D(L)$ is a chain in L. If $D(L) = L$, that is, L is a chain, then we are obviously done.

Otherwise, let $L_1, L_2 \ldots, L_m$ be the maximal, linearly nondecomposable lattices of which L is composed; evidently, $m > 1$. Moreover, L contains a nontrivial linearly nondecomposable sublattice L_1, say, such that $D(L_1) = \emptyset$. Let $L' = L - L_1$ and $L'' = L' - D(L)$.

We consider only the case that the greatest element of L_1 does not belong to

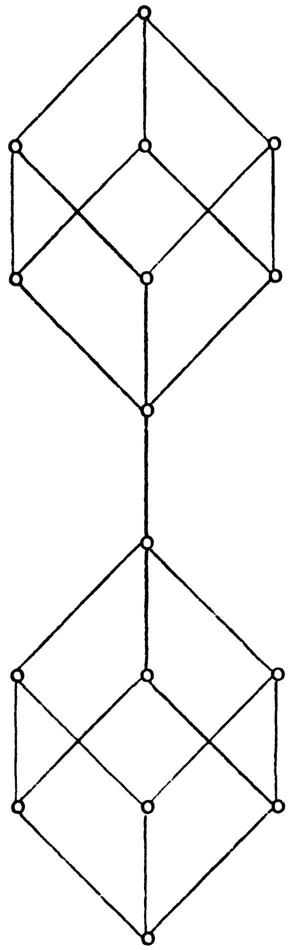

Figure 2

$\bigcup_{i=2}^{m} L_i$; the other case is similar. First, we observe that

$$\begin{aligned}|L| &= |L_1|+|L''|+|D(L)|\\ &\leq 2(l(L_1)+l(L'')+|D(L)|+1)+1\end{aligned}$$

whence,

$$|D(L)\geq|L''|-(2l(L'')+1)+|L_1|-2(l(L_1)+1).$$

Since $D(L'')=\varnothing$ we have, by Lemma 5, that $|L''|\geq 2(l(L'')+1)$ so that

(*) $$|D(L)|\geq|L_1|-(2l(L_1)+1).$$

Similarly, $D(L_1)=\varnothing$ which, in view of Lemma 5 again, yields $|L_1|\geq 2(l(L_1)+1)$. But $|L|\leq 2l(L)+1$ and $l(L)=l(L_1)+l(L')+1$ from which it follows that

$$|L'| \leq 2l(L)+1-2(l(L_1)+1) \leq 2l(L')+1.$$

Applying the induction hypothesis to L' we have that L' has a complete spectrum, that is, $n \in \mathbf{sp}(L')$ for each integer n satisfying $1 \leq n \leq |L''|+|D(L)|$. In view of (*) we conclude that

$$\{1, 2, \ldots, |L''|+|L_1|-(2l(L_1)+1)\} \subseteq \mathbf{sp}(L').$$

By Theorem 4,

$$\{1, 2, \ldots, 2l(L_1)\} \subseteq \mathbf{sp}(L_1)$$

so that

$$\{1, 2, \ldots, |L''|+|L_1|-1\} \subseteq \mathbf{sp}(L_1 \cup L') = \mathbf{sp}(L).$$

Finally, by considering the sublattices of L obtained by removing doubly irreducible elements one at a time we have that

$$\{|L''|+|L_1|, |L''|+|L_1|+1, \ldots, |L|\} \subseteq \mathbf{sp}(L)$$

and L has a complete spectrum.

Distributive lattices with n-element sublattices. For a positive integer n let $\Delta(n)$ denote the smallest integer such that every finite distributive lattice with at least $\Delta(n)$ elements contains an n-element sublattice. Our purpose in this section is twofold: first, we determine $\Delta(n)$ for small n (actually for $1 \leq n \leq 14$); second, we establish an upper bound for $\Delta(n)$.

A finite distributive lattice has breadth at most two if and only if it is dismantlable (cf. D. Kelly and I. Rival [2]); in particular, every finite distributive lattice with breadth at most two has a complete spectrum. Hence, in order to show that a given finite distributive lattice L contains an n-element sublattice we may assume that L has breadth at least three. Moreover, as such a lattice must contain a sublattice isomorphic to $\mathbf{2}^3$ we conclude at once that $\Delta(n)=n$, for $1 \leq n \leq 6$, and $\Delta(8)=8$.

Let L be a finite distributive lattice. It is well known that $l(L) \geq m+1$ whenever $|L| > 2^m$, where m is any positive integer. Let $|L| \geq 9$. Then $l(L) \geq 4$ and since $b(L) \geq 3$, L contains a sublattice $S \cong \mathbf{2}^3$ in which, for $x, y \in S$, $x \succ y$ in S if $x \succ y$ in L. Obviously, $7 \in \mathbf{sp}(L)$. Since, however, $7 \notin \mathbf{sp}(\mathbf{2}^3)$ we conclude that $\Delta(7)=9$. This argument also shows that $\Delta(9)=9$. Let $|L| \geq 10$. If L is linearly decomposable then a simple application of the values of $\Delta(n)$ for $n \leq 9$ yields that $10 \in \mathbf{sp}(L)$. Otherwise, L is linearly nondecomposable and, since $b(L) \geq 3$, $l(L) \geq 4$. Applying Theorem 4 we obtain $\Delta(10)=10$. As the lattice $\mathbf{2}^4$ contains no 11-element *sublattice*, $\Delta(11) \geq 17$. Furthermore, the same technique which established $\Delta(10)=10$ above, now yields $\Delta(11)=17$..

We digress momentarily to prove the

THEOREM 7. *Let L be a finite distributive lattice. Then*

$$|L| \le \left(\frac{l(L)}{b(L)}+1\right)^{b(L)}.$$

Moreover, this inequality is best possible.

Proof. According to a well known result of R. P. Dilworth [1] L can be embedded in the direct product of chains $C_1, C_2, \ldots, C_{b(L)}$. Moreover, this embedding can be so carried out that the universal bounds of L correspond to the universal bounds of $C_1 \times C_2 \times \cdots \times C_{b(L)}$ and

$$l(L) = \sum_{i=1}^{b(L)} l(C_i).$$

It follows that

$$|L| \le \prod_{i=1}^{b(L)} (l(C_i)+1).$$

A simple argument shows that *dexter* is maximized when

$$\frac{l(L)}{b(L)} = l(C_i)$$

for each $i = 1, 2, \ldots, b(L)$.

Finally, the lattices $\mathbf{2}^n$ satisfy $l(L) = b(L)$, which shows that the inequality is best possible.

This inequality is handy. For instance, let L be a finite distributive lattice with at least twelve elements. If $b(L)=3$ and $l(L)=4$ then Theorem 7 implies that $|L|=12$. If $b(L)=4$ and $l(L)=4$ then $L \cong \mathbf{2}^4$ in which case $12 \in \mathbf{sp}(L)$. Otherwise, $l(L) \ge 5$. Finally, applying Theorem 4 and the values of $\Delta(n)$ for $n<12$ to the linearly nondecomposable lattices which constitute L yields $\Delta(12)=12$. Similar arguments show that $\Delta(13)=17$ and $\Delta(14)=18$. On the other hand, since the lattice illustrated in Figure 3 has no 15-element sublattice it follows that $\Delta(15) \ge 21$.

Theorems 4 and 7 provide the essential ingredients for our final result – an upper bound on $\Delta(n)$.

THEOREM 8. *For every positive integer n*

$$\Delta(n) \le n2^{n/4}.$$

Proof. We divide the proof into two parts.

First, for $n \ge 8$, we show that $\Delta(n) \le 2^{n/2-1}$. Indeed, we have already verified this inequality for $8 \le n \le 12$. Now, let L be a finite distributive lattice and let

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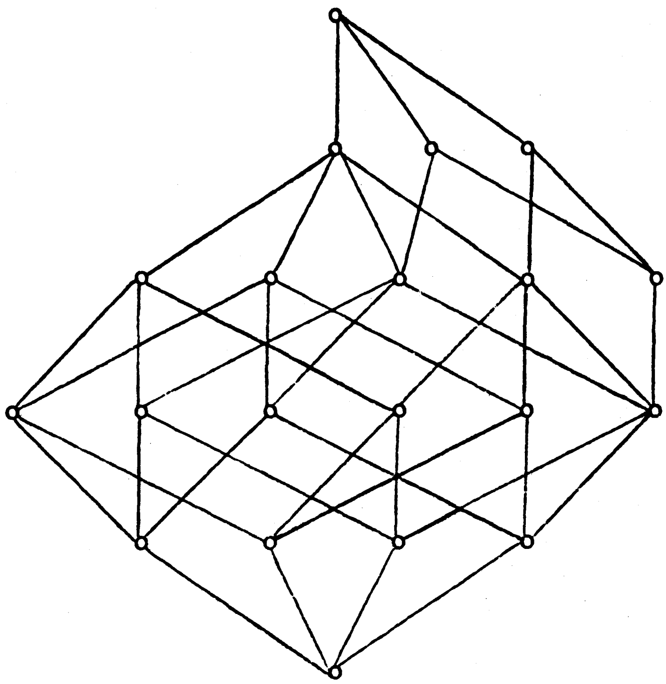

Figure 3

$|L| \geq 2^{n/2-1}$. Then $l(L) \geq n/2-1$. We may suppose that $b(L) \geq 3$. If L is linearly nondecomposable, Theorem 4 implies that $m \in \mathbf{sp}(L)$ for every integer m satisfying

$$8 \leq m \leq n \leq 8+2(l(L)-3) = 2l(L)+2.$$

Now let L be linearly decomposable and let $L_1, L_2, \ldots, L_k$, $k \geq 2$, be the maximal, linearly nondecomposable lattices which constitute L. We may assume that $|L_1| \leq |L_i|$ for each $i \leq k$. If $|L_1| \leq 3$ then

$$|L-L_1| \geq 2^{n/2-1}-3 \geq 2^{(n-1)/2-1}.$$

Hence, by the induction hypothesis, $L-L_1$ contains an $(n-1)$-element sublattice. Adjoining a disjoint element from L_1 yields $n \in \mathbf{sp}(L)$. If $|L_1| \geq 4$ then

$$|L-L_1| \geq \frac{k-1}{k}|L| \geq \frac{k-1}{k} 2^{n/2-1} \geq 2^{(n-2)/2-1}$$

so that $L-L_1$ contains an $(n-2)$-element sublattice which together with two disjoint elements from L_1 produces an n-element sublattice.

In view of these remarks it is enough to show that for every finite distributive lattice L, $n \in \mathbf{sp}(L)$ whenever $|L| \geq n2^{n/4}$ and $n \geq 16$. To this end let m be a positive integer such that $2^m \leq n < 2^{m+1}$. Then

$$2(\log_2 |L| - m) + 2^m \geq n.$$

Under any circumstances, $|L| \leq 2^{l(L)}$ so that

$$2(l(L) - m) + 2^m \geq n \geq 2^m.$$

Let L be linearly nondecomposable. If $b(L) \geq m$ then Theorem 4 guarantees, in view of the last inequality, that $n \in \mathbf{sp}(L)$. If, on the other hand, $b(L) \leq m-1$ then

$$\left(\frac{l(L)}{b(L)} + 1\right)^{b(L)} \geq |L| \geq n2^{n/4}$$

implies that

$$l(L) \geq b(L)((n2^{n/4})^{1/b(L)} - 1).$$

As we may assume that $b(L) \geq 3$ we conclude that

$$l(L) \geq 3((n2^{n/4})^{1/(-1+\log_2 n)} - 1) \geq n/2$$

for $n \geq 16$. Again, Theorem 4 guarantees that L contains an n-element sublattice.

Finally, let us suppose that L is linearly decomposable and that L consists of the maximal, linearly nondecomposable lattices $L_1, L_2, \ldots, L_k$, such that $|L_1| \leq |L_i|$ for each $i \leq k$. Then

$$|L - L_1| \geq \frac{k-1}{k} n2^{n/4} > (n-4)2^{(n-4)/4}.$$

Hence, by the induction hypothesis, $L - L_1$ contains an $(n-4)$-element sublattice S. If $|L_1| \geq 5$ we may adjoin a 4-element sublattice of L_1, disjoint from $L - L_1$, to S. If $|L_1| \leq 4$ then

$$|L - L_1| \geq n2^{n/4} - 4 \geq (n-1)2^{(n-1)/4}$$

for $n \geq 7$. Hence, $L - L_1$ contains an $(n-1)$-element sublattice to which we may adjoin a disjoint element of L_1 and again $n \in \mathbf{sp}(L)$.

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