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On properties of countable character

Gabriel Sabbagh

It is proved that if a class X of algebras of countable similarity type is closed under isomorphism and ultrapower, then the class of subalgebras of direct products of elements of X is of countable character.

1. Introduction

This short paper is composed of variations on a theme of B.H. Neumann. In a recent talk in Nice, he introduced the notion of property *of countable character* and showed that several properties are of countable character. Various persons, including W.W. Boone, A. Robinson and the author, suggested the possibility of using a kind of Löwenheim-Skolem Theorem for deriving such results. Although the most obvious tool seems to be the downward Löwenheim-Skolem Theorem for $L_{\omega_1\omega}$ (*cf.* [6]) and it is possible to describe in an infinitary language universal properties of countable character, the main purpose of this note is to show how to use the ordinary Löwenheim-Skolem-Tarski Theorem [13] for unifying and improving some of the results of [9].

2. Preliminaries

For simplicity we will only deal with *algebras*, namely with sets endowed with an arbitrary number of finitary operations (functions), some of which may be of arity 0 . A being an algebra, we denote by $\alpha_n(A)$ the cardinal of the set of operations of arity n of the algebra A .

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The sequence $\langle \alpha_n(A) \rangle_{n \in \omega}$ is called the similarity type of A. We denote by X a nonvoid fixed class closed under isomorphism of algebras, all of which have the same similarity type. We denote by $\alpha = \alpha(X)$ the cardinal $\sum_{n \in \omega} \alpha_n(A)$ where A is an element of X. *To* X *is associated in the usual way a first-order language, the cardinal of which will be denoted by* γ *and coincides with* $\sup(\alpha, \aleph_0)$. Except if otherwise stated, all the logical concepts are considered with respect to this language.

As usual, we denote by SX (respectively PX, respectively RX) the class of algebras isomorphic to subalgebras (respectively cartesian products, respectively subcartesian products) of elements of X ([4], [8]). If X coincides with SX, X is said to be *universal*. If X coincides with the class of finite algebras, RX is said to be the class of *residually finite algebras*. In general one has

(1) $$RX \subseteq SRX = RSX = SPX ,$$

(2) $$PSX \subseteq SPX \ ; \quad SSX = SX .$$

For every infinite cardinal β, we denote by $L_\beta(X)$ the class of algebras all of whose subalgebras generated by strictly less than β elements belong to X. We may then introduce the following

DEFINITION. X is said to be of β-character if $L_\beta(X)$ is included in X.

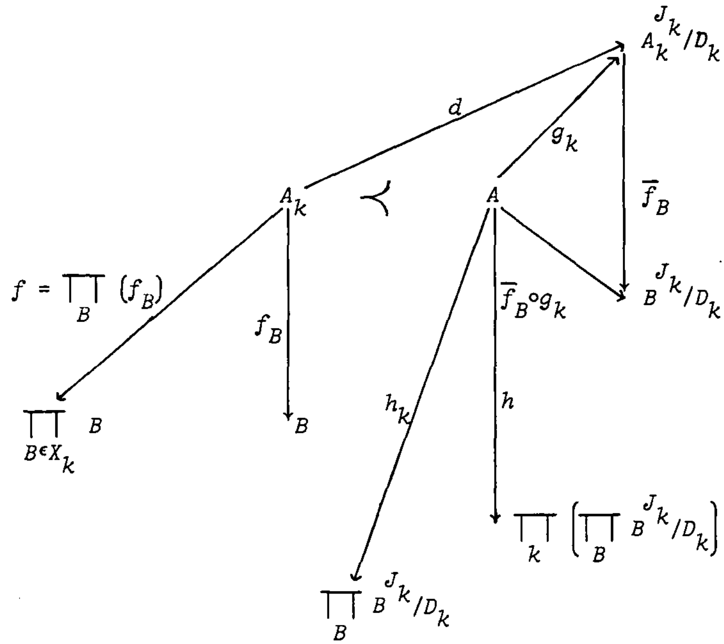
We will adapt the terminology of [9] in saying that X is of *local character* (respectively *countable character*) if X is of $\aleph_0$-character (respectively $\aleph_1$-character). The definitions of [9] are different from ours but coincide with them if X is universal and if α is strictly less than $\aleph_1$.

3. Main section

We can now state our result.

THEOREM 1. *Let* δ *be the successor cardinal of* γ. *If* X *is closed under ultrapower, then* SPX *is of* δ*-character.*

Proof. The proof is best summarized by the following diagram:



Let A be an arbitrary element of $L_\delta(SPX)$. We wish to prove that A is an element of SPX. We can clearly assume that A is of cardinal $\geq \gamma$. By the Löwenheim-Skolem-Tarski Theorem, every subset k of cardinal 2 of A is contained in an elementary substructure A_k of A of cardinal γ. By assumption, there exist a family X_k of elements of X and for each element B of X_k a homomorphism f_B of A_k into B such that the "product" homomorphism $f = \prod_B (f_B)$ of A_k into $\prod_{B \in X_k} B$ is one-one. By Scott's Lemma ([2], p. 163), since A_k is an elementary substructure of A, there exists a one-one homomorphism g_k of A into an ultrapower $A_k^{J_k}/D_k$ of A_k whose restriction to A_k coincides with the canonical embedding d of A_k into A^{J_k}/D_k. For each element B of

X_k , f_B induces a homomorphism $\overline{f}_B$ of $A_k^{J_k}/D_k$ into B^{J_k}/D_k . The family $(\overline{f}_B \circ g_k)_{B \in X_k}$ allows us to define a homomorphism $h_k = \prod_{B \in X_k} (\overline{f}_B \circ g_k)$ of A into $\prod_{B \in X_k} \left(B^{J_k}/D_k\right)$.

Let K now denote the set of all two-element subsets k of A . The family $(h_k)_{k \in K}$ allows us to define a homomorphism $h = \prod_k (h_k)$ of A into $M = \prod_{k \in K} \left(\prod_{B \in X_k} \left(B^{J_k}/D_k\right)\right)$. Since by assumption X is closed under ultrapower, M is an element of PX . For proving that A is an element of SPX , it now suffices to show that h is one-one.

Let a and b be two distinct elements of A . Let q denote the subset $\{a, b\}$ of A . There exists an element B of X_q such that $f_B(a) \neq f_B(b)$. It easily follows that $\overline{f}_B(d(a))$ and $\overline{f}_B(d(b))$ are distinct and hence that $\overline{f}_B \circ g_q(a)$ and $\overline{f}_B \circ g_q(b)$ are distinct. We then obtain $h_q(a) \neq h_q(b)$, which implies $h(a) \neq h(b)$. The proof is finished.

COROLLARY 1. *Let* δ *be the successor cardinal of* γ . *If* X *is closed under ultrapower and is universal, then* RX *is of* δ*-character.*

Proof. Since X is universal, (1) implies that RX is equal to SPX . One then applies the theorem.

Corollary 1 yields under weaker assumptions Theorem 3 and Theorem 4 of [9]. Theorem 3 essentially states that if X is the union of a family of quasivarieties, then RX is of δ-character. A quasivariety is just a universal Horn class of algebras ([4], p. 235). It is now plain that Theorem 3 remains true if one only assumes that X is the union of a family of universal elementary (in the wider sense) classes of algebras. It is perhaps worthwhile to state formally our version of Theorem 4; its only advantage is that α need not be finite.

COROLLARY 2. *If* α *is countable, the class of residually finite algebras is of countable character.*

Proof. Indeed, an ultrapower of a finite set is finite.

4. Other approaches

1. The obvious strengthening of Corollary 2 and of Theorem 1 fails: the class of residually finite commutative groups is not of local character; indeed every finitely generated commutative group is residually finite, while a non-trivial divisible group is never residually finite. However, if one assumes in the theorem that X is closed under ultraproduct, one may conclude that SPX is of local character. For proving that fact, it is enough by a standard embedding theorem (see for example [3] which has a nearly complete bibliography, [7] or [10]) to establish the following

LEMMA. *If* X *is closed under ultraproduct, then* SPX *is a universal elementary class.*

Proof. The shortest way is to derive the lemma from a similar, slightly weaker, result of Vaught [14]. According to that result, if Y is an elementary class (or even a PC_Δ class), then SPY is a universal elementary class. Let us denote by X' the elementary class generated by X, namely the class of models of all sentences valid in all elements of X. It is easy to see that X' is the class of the algebras which are elementarily embeddable in an element of X. (A more general result is given in [11].) One then has $SPX \subseteq SPX' \subseteq SPSX$; by (2) one obtains $SPSX \subseteq SPX$ and hence one has $SPX = SPX'$. Since X' is an elementary class, the proof is finished.

As an immediate application, one has

COROLLARY 3. *Let* n *be a positive integer and let* X_n *be the class of (finite) algebras of cardinal* $< n$. RX_n *is of local character.*

Corollary 3 is implicit in [9].

2. As mentioned in the introduction and expounded in [6], it is tempting to try to use some infinitary logic for proving that a given

class is of countable character. In some cases, it is enough to consider the language $L_{\omega_1\omega}$: for example, let N be the class of nilpotent groups. It is easy to find a sentence σ of the $L_{\omega_1\omega}$ theory of groups such that N is the class of models of σ . If N were not of countable character, there would exist a group G such that G is a model of $\neg\sigma$ and every countable subgroup of G is a model of σ , which would contradict the Löwenheim-Skolem theorem for $L_{\omega_1\omega}$.

On the other hand, as noticed in conversation with A. Macintyre, there are many classes which are of countable character and which are not definable in $L_{\omega_1\omega}$ (nor in $L_{\infty\omega}$) . Two simple examples are the class of commutative reduced p-groups ([1], Theorem 2.4) and the class of noetherian rings ([5], Theorem 11). It is not hard in fact to give a syntactical characterization of *universal* classes of countable character if one is willing to devise an ad hoc language:

THEOREM 2. *Let* β *be an infinite cardinal. Let* μ *denote the cardinal* $\sup(\beta, \gamma)$. *If* X *is universal, then the following assertions are equivalent:*

(i) X *is of* β*-character;*

(ii) there exists a set S *of sentences* ψ *of the form*

$$\psi = (\forall x_1) \ldots (\forall x_\lambda) \ldots {}_{\lambda<\rho<\beta}\varphi(x_\lambda)$$

where ρ *is a cardinal* $< \beta$ *(*ρ *depends upon* ψ*) and* $\varphi(x_\lambda)$ *is a quantifier-free formula of* $L_{\infty\infty}$ *, that is a quantifier-free formula of possibly infinite length, such that* X *is the class of models of* S *;*

(iii) there exists a set T *of sentences* ψ *of the form*

$$\psi = (\forall x_1) \ldots (\forall x_\lambda) \ldots {}_{\lambda<\rho<\beta}\varphi(x_\lambda)$$

where ρ *is a cardinal* $< \beta$ *and* $\varphi(x_\lambda)$ *is a disjunction of length at most equal to* μ *of atomic formulas and negations of atomic formulas, such that* X *is the class of models of* T .

Proof. *(iii)* ⇒ *(ii)* is obvious and *(ii)* ⇒ *(i)* is easy. For establishing the implication *(i)* ⇒ *(iii)* we will follow an argument due to Tarski ([12], Theorems 1.1 and 1.2).

We will first prove a more precise version of a *consequence* of this implication. Let A be an algebra of the same similarity type as X and let Y be the class of algebras of the same similarity type as X into which A cannot be embedded. We assume that A admits a generating subset of cardinal $\rho < \beta$. We want to show that there exists a single sentence ψ_A of the form described in *(iii)* such that Y is the class of models of ψ_A.

Let $(a_\lambda)_{\lambda<\rho}$ be a non-repeating enumeration of a generating subset of A. Let F be the algebra of words of the same similarity type as X freely generated by a set $\{x_\lambda\}_{\lambda<\rho}$ of distinct elements. To each pair $C = \{P(x_\lambda), Q(x_\lambda)\}$ of words of F we associate the formula U_C defined as follows:

$$U_C = \begin{cases} P(x_\lambda) = Q(x_\lambda) & \text{if the elements } P(a_\lambda) \text{ and } Q(a_\lambda) \text{ of } A \text{ are distinct,} \\ P(x_\lambda) \neq Q(x_\lambda) & \text{if the elements } P(a_\lambda) \text{ and } Q(a_\lambda) \text{ of } A \text{ are equal.} \end{cases}$$

It is easy to check that one can take for ψ_A the formula

$$(\forall x_1) \ldots (\forall x_\lambda) \ldots_{\lambda<\rho} \left[\bigvee_C U_C\right] .$$

We now proceed to the proof of the general case. Let T be the set of all sentences of the form given in *(iii)* which are valid in all the elements of X. Assuming that X is universal and of β-character, we will show that X is the class of models of T. It clearly suffices to prove that an arbitrary model M of T is an element of $L_\beta(X)$. Let B be a subalgebra of M generated by strictly less than β elements. It is plain that the sentence ψ_B is not an element of T. It follows that B can be embedded in an element of X. Since X is universal, the proof is complete.

It is easy to derive from the previous theorem a straightforward generalization of Theorem 2 of [9].

COROLLARY 4. *Let* β *be an infinite cardinal and let* I *be a set of cardinal strictly less than the smallest cardinal co-final with* β *. The union of a family indexed by* I *of universal classes of* β*-character is a universal class of* β*-character.*

We have been unable to deduce Theorem 1 from Theorem 2. A more interesting question would be to know if there exists a $L_{\omega_1\omega}$ analogue of the result of Vaught previously mentioned. We do not even know if the class of residually finite groups is definable in $L_{\omega_1\omega}$; of course, the class of commutative residually finite groups is.

3. Theorem 1 constitutes a model-theoretic generalization of Corollary 2. There is a different generalization, which is due to J. Mycielski and is included here with his kind permission.

THEOREM 3. (Mycielski) *Let* δ *be the successor cardinal of* γ *. If* X *is a class of atomic compact algebras, then* SPX *is of* δ*-character.*

Proof. Let A be an arbitrary element of $L_\delta(SPX)$. We wish to prove that A is an element of SPX . We can clearly assume that A is of cardinal $\geq \gamma$. By the Löwenheim-Skolem-Tarski Theorem, every subset k of cardinal 2 of A is contained in an elementary substructure A_k of A of cardinal γ . By assumption, there exists an embedding of A_k into a product B_k of elements of X . By [15], p. 107 B_k is atomic compact. By a well-known theorem of Weglorz, a version of which may be found in [15], p. 105, the embedding of A_k into B_k can be extended to a homomorphism h_k of A into B_k . It is easy to see that the "product" homomorphism $\prod_k (h_k)$ of A into $\prod_k B_k$ is an embedding and makes A an element of SPX .

To derive Corollary 2 from Theorem 3, it is enough to use the fact that every finite algebra, and more generally every (Hausdorff) compact algebra, is atomic compact ([*15*], p. 75).

Note added in proof. A version of Theorem 2 appears in Tarski's paper, "Remarks on predicate logic with infinitely long expressions", *Colloq. Math.* 6 (1958), 171-176.

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1, square François Couperin,
92-Antony,
France.