A NOTE ON PSEUDO-UMBILICAL SURFACES

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1. Preliminaries

We follow the notations and basic equations of Chen (2). Let M be a surface immersed in an m-dimensional space form $R^m(c)$ of curvature $c = 1, 0$ or -1. We choose a local field of orthonormal frames $e_1, \ldots, e_m$ in $R^m(c)$ such that, restricted to M, the vectors e_1, e_2 are tangent to M. Let $\omega^1, \ldots, \omega^m$ be the field of dual frames. Then the structure equations of $R^m(c)$ are given by

$$d\omega^A = \Sigma\omega_A^B \wedge \omega^B, \quad \omega_B^A + \omega_A^B = 0$$
$$d\omega_B^A = \Sigma\omega_C^A \wedge \omega_C^B + c\omega^A \wedge \omega^B, \quad A, B, C = 1, \ldots, m. \tag{1}$$

Restricting these forms to M we have $\omega^r = 0$, where $r, s, t = 3, \ldots, m$. Since $0 = d\omega^r = \omega_r^1 \wedge \omega^1 + \omega_r^2 \wedge \omega^2$, by Cartan's Lemma we may write

$$\omega_i^r = \Sigma h_{ij}^r \omega^j, \quad h_{ij}^r = h_{ji}^r, \quad i, j = 1, 2. \tag{2}$$

From these we obtain

$$d\omega^i = \Sigma\omega_i^j \wedge \omega^j, \tag{3}$$

$$d\omega_2^1 = \{c + \sum_r \det(h_{ij}^r)\}\omega^1 \wedge \omega^2 = K\omega^1 \wedge \omega^2, \tag{4}$$

$$d\omega_i^r = \Sigma\omega_j^r \wedge \omega_j^i + \Sigma\omega_s^r \wedge \omega_s^i. \tag{5}$$

The second fundamental form h and the mean curvature vector H are given respectively by

$$h = \Sigma h_{ij}^r \omega^i \otimes \omega^j e_r,$$
$$H = \tfrac{1}{2}\Sigma h_{ii}^r e_r. \tag{6}$$

If there exists a function α on M such that $\langle h(X, Y), H\rangle = \alpha\langle X, Y\rangle$ for all tangent vectors X, Y, then M is called a pseudo-umbilical surface of $R^m(c)$. For points at which $H \neq 0$ we choose e_3 to be $H/|H|$ then

$$h_{11}^3 = h_{22}^3 = \alpha, \quad h_{12}^3 = 0. \tag{7}$$

The normal curvature K_N of M is given by

$$K_N = \sum_{r,s} [\sum_i (h_{1i}^r h_{2i}^s - h_{2i}^r h_{1i}^s)]^2. \tag{8}$$

We denote the square of the length of the second fundamental form by S, that is

$$S = \sum_r \sum_{i,j} h_{ij}^r h_{ij}^r. \tag{9}$$

In this paper we will consider pseudo-umbilical surfaces in $R^m(c)$ with $K_N = 0$ and S a constant. $S^1 \times S^1 \subset R^4$ $(= R^4(0))$ is one such surface. Another is the following example.

Let M be a product of two circular helices in R^6:

$$x = (\cos t,\ \sin t,\ t,\ \cos s,\ \sin s,\ s).$$

At each point of M we choose the following frame in R^6:

$$e_1 = \frac{1}{\sqrt{2}}(-\sin t,\ \cos t,\ 1,\ 0,\ 0,\ 0),\ e_2 = \frac{1}{\sqrt{2}}(0,\ 0,\ 0,\ -\sin s,\ \cos s,\ 1),$$

$$e_3 = (\cos t,\ \sin t,\ 0,\ 0,\ 0,\ 0), \qquad e_4 = (0,\ 0,\ 0,\ -\cos s,\ -\sin s,\ 0),$$

$$e_5 = \tfrac{1}{2}(\sin t,\ -\cos t,\ 1,\ \sin s,\ -\cos s,\ 1),$$

$$e_6 = \tfrac{1}{2}(\sin t,\ -\cos t,\ 1,\ -\sin s,\ \cos s,\ -1).$$

Then we have

$$(h^3_{ij}) = \begin{pmatrix} -\dfrac{1}{2\sqrt{2}} & 0 \\ 0 & -\dfrac{1}{2\sqrt{2}} \end{pmatrix},\quad (h^4_{ij}) = \begin{pmatrix} \dfrac{1}{2\sqrt{2}} & 0 \\ 0 & -\dfrac{1}{2\sqrt{2}} \end{pmatrix},\quad (h^5_{ij}) = 0,\ (h^6_{ij}) = 0.$$

Hence M is pseudo-umbilical, $K_N = 0$, $S = \frac{1}{2}$ = constant.

We are going to prove the following theorems.

Theorem 1. *Let M be a pseudo-umbilical surface in $R^m(c)$ satisfying $K_N = 0$ and S is constant. Then M is either flat or totally umbilical in $R^m(c)$. Furthermore, if the interior of the set $\{x \in M \mid H = 0$ at $x\}$ is not empty then M is either flat and $c \geqq 0$ or totally geodesic.*

Theorem 2. *Let M be a simply-connected flat pseudo-umbilical surface in $R^m = R^m(0)$ satisfying $K_N = 0$ and S is constant. Then M is a product of two curves C_1 and C_2, $C_1 \subset R^l$, $C_2 \subset R^{m-l}$ so that the absolute values of the first curvatures of C_1 and C_2 are equal.*

2. Proof of Theorem 1

Since $K_N = 0$ on M, the second fundamental tensors of M in $R^m(c)$ are simultaneously diagonalisable. (For instance, see Chen (1), p. 101.) Let $U = \{x \in M \mid H \neq 0$ at $x\}$. Then U is an open set of M. The set

$$\{x \in M \mid H = 0 \text{ at } x\}$$

is closed. Let V be the interior of $\{x \in M \mid H = 0$ at $x\}$.

At each point of U we may choose a frame field $e_1, e_2, \ldots, e_m$ in $R^m(c)$ so that e_1, e_2 are tangent to M and e_3 is the direction of the mean curvature vector to M. Since M is pseudo-umbilical by (7) we have that at each point of U the second fundamental tensors are

$$(h_{ij}^3) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad (h_{ij}^r) = \begin{pmatrix} h_{11}^r & 0 \\ 0 & -h_{11}^r \end{pmatrix}, \quad 4 \leqq r \leqq m, \tag{10}$$

with respect to the frame field $e_1, e_2, \ldots, e_m$.

Hence we have the differential forms:

$$\omega_i^3 = \alpha\omega^i, \quad 1 \leqq i \leqq 2, \tag{11}$$

$$\omega_i^r = h_{ii}^r \omega^i, \; r \geqq 4, \; 1 \leqq i \leqq 2. \tag{12}$$

Exterior differentiation of (11) yields

$$\sum_{r=3}^{m} (h_{ii}^r \omega_r^3 + d\alpha) \wedge \omega^i = 0, \quad (i = 1, 2). \tag{13}$$

Exterior differentiation of (12) yields

$$dh_{ii}^r \wedge \omega^i + 2h_{ii}^r d\omega^i + \alpha\omega_3^r \wedge \omega^i = \sum_{s=4}^{m} h_{ii}^s \omega_r^s \wedge \omega^i, \; r \geqq 4, \quad (i = 1, 2). \tag{14}$$

Multiplying (14) by h_{ii}^r and summing for r from 4 to m we have by (1) and (13)

$$\sum_{r=4}^{m} h_{ii}^r dh_{ii}^r \wedge \omega^i + 2 \sum_{r=4}^{m} (h_{ii}^r)^2 d\omega^i + \alpha d\alpha \wedge \omega^i = 0, \quad (i = 1, 2). \tag{15}$$

On the other hand, by (10) S in (9) has the form

$$\tfrac{1}{2}S = \alpha^2 + \sum_{r=4}^{m} (h_{ii}^r)^2, \quad (i = 1, 2). \tag{16}$$

Differentiating this equality and using (15) we have

$$\tfrac{1}{4}dS \wedge \omega^i + 2 \sum_{r=4}^{m} (h_{ii}^r)^2 d\omega^i = 0, \quad (i = 1, 2). \tag{17}$$

Since S is assumed to be a constant we have

$$\left\{ \sum_{r=4}^{m} (h_{11}^r)^2 \right\} d\omega^1 = 0 \quad \text{and} \quad \left\{ \sum_{r=4}^{m} (h_{22}^r)^2 \right\} d\omega^2 = 0.$$

Noticing that $h_{22}^r = -h_{11}^r$ we then have either $h_{ii}^r = 0$ $(4 \leqq r \leqq m, \; 1 \leqq i \leqq 2)$ or $d\omega^i = 0$ $(1 \leqq i \leqq 2)$. U is thus either totally umbilical or flat.

By (4) the Gauss curvature K of U is given by

$$K = c + \alpha^2 - \sum_{r=4}^{m} (h_{ii}^r)^2 \; (i = 1 \text{ or } 2).$$

If U is flat, then $K = 0$. Otherwise $h_{ii}^r = 0$ $(4 \leqq r \leqq m, \; 1 \leqq i \leqq 2)$ on U, we then have from (16) that $\alpha^2 = \tfrac{1}{2}S$ = constant and $K = c + \alpha^2$ = constant. Hence for either case U has constant Gauss curvature.

Next we consider points in V. Since the mean curvature vector is zero on V, V is a minimal surface of $R^m(c)$. The second fundamental tensors of V in $R^m(c)$ are simultaneously diagonalisable on V. We may choose a local frame field on V in such a way that $h^r_{12} = 0$, $r = 3, \ldots, m$. Then

$$h^r_{11} = -h^r_{22} \; (3 \leqq r \leqq m),$$

since V is minimal. Now

$$S = \sum_{i,j,r} (h^r_{ij})^2 = \sum_{r=3}^{m} \sum_{i=1}^{2} (h^r_{ii})^2 = 2 \sum_{r=3}^{m} (h^r_{11})^2,$$

$$K = c + \sum_{r=3}^{m} (\det h^r_{ij}) = c - \sum_{r=3}^{m} (h^r_{11})^2 = c - \tfrac{1}{2}S.$$

The assumption that S = constant implies that V has constant Gauss curvature. V thus is a minimal surface of $R^m(c)$ with constant Gauss curvature and $K_N = 0$. By Lemma 2 of (3) V is either flat and $c \geqq 0$ or totally geodesic. This conclusion may also be reached by taking account that $\alpha = 0$, r runs from 3 to m in formulas (14) through (17).

Finally we consider the entire surface M. If $V = \varnothing$ then any point

$$p \in \{x \in M \mid H = 0 \text{ at } x\}$$

is a limit point of U. At every point of U we have proved that either

$$h^3_{11} = h^3_{22} = \alpha \text{ and } h^r_{ii} = 0 \; (r \geqq 4)$$

or $K = 0$. h^3_{ii}, h^4_{ii} and K are continuous on M, we have also $h^3_{11} = h^3_{22}$ and $h^r_{ii} = 0$ $(r \geqq 4)$ or $K = 0$ at p. Hence M is either totally umbilical or flat. If $V \neq \varnothing$ we have shown that V is minimal with constant Gauss curvature. So M has constant Gauss curvature K. If $K = 0$ then M is flat and $c \geqq 0$. If $K \neq 0$ then $h^r_{ii} = 0$ $(r \geqq 3, i = 1, 2)$ and hence $K = c \neq 0$. We have shown in U if $K \neq 0$ then $K = c + \alpha^2$. This means that U is empty. M thus is totally geodesic and Theorem 1 is proved.

3. Proof of Theorem 2

Let M be simply-connected and such that $d\omega^i = 0$, $i = 1, 2$. For this case, $\omega^1_2 = 0$. M is flat and both the distributions $T_i = \{\lambda e_i \mid \lambda \in R\}$, $i = 1, 2$ are parallel. By the de Rham decomposition theorem we have that $M = C_1 \times C_2$ where C_i is the maximal integral manifold of T_i.

From now on we consider that $M \subset R^m(0)$. Thus M is a simply-connected surface in a euclidean space R^m. Since the second fundamental forms given by (10) satisfy $h^r_{12} = 0$ $(r > 3)$, Moore in (4) has proved that there are euclidean spaces R^l and R^{m-l} so that $C_1 \subset R^l$, $C_2 \subset R^{m-l}$ and

$$M = C_1 \times C_2 \subset R^l \times R^{m-l} = R^m.$$

Let the curve C_1 in R^l be $x(s)$ and the curve C_2 in R^{m-l} be $y(t)$, here s, t are arc length for curves C_1 and C_2. Then M in R^m is given by $(x(s), y(t))$ and $e_1 = (x'(s), 0)$, $e_2 = (0, y'(t))$ are the tangent vectors of M. Let us write the Frenet formulas for C_1, C_2 as follows:

$$\frac{de_1}{ds} = k_1(s)e_3 \qquad \frac{de_2}{dt} = h_1(t)e_4$$

$$\vdots \qquad \vdots$$

$$\frac{de_{2i-1}}{ds} = -k_{i-1}(s)e_{2i-3}+k_i(s)e_{2i+1}, \qquad \frac{de_{2i}}{dt} = -h_{i-1}(t)e_{2i-2}+h_i(t)e_{2i+2},$$

$$\vdots \quad 2 \leqq i \leqq l-1, \qquad \vdots \quad 2 \leqq i \leqq m-l-1,$$

$$\frac{de_{2l-1}}{ds} = -k_{l-1}(s)e_{2l-3}; \qquad \frac{de_{2(m-l)}}{dt} = -h_{m-l-1}(t)e_{2(m-l-1)}.$$

Here k_i, h_i are the ith curvatures of C_1, C_2.

It is then easy to see that the basic forms and connection forms of M are

$$\omega^1 = ds, \quad \omega^2 = dt;$$

$$\omega_1^3 = -k_1\omega_1, \quad \omega_2^4 = -h_1\omega_2, \quad \omega_2^3 = \omega_1^4 = 0;$$

$$\omega_i^r = 0 \quad (i = 1, 2; \ t \geqq 5).$$

The second fundamental forms of M thus are

$$(h_{ij}^3) = \begin{pmatrix} -k_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & 0 \\ 0 & -h_1 \end{pmatrix}, \quad (h_{ij}^r) = 0 \ (r \geqq 5).$$

Hence the mean curvature of C_1 is $|k_1|$, the mean curvature of C_2 is $|h_1|$ and the mean curvature vector of M is $\frac{1}{2}(-k_1e_3-h_1e_4)$. That the length of the second fundamental form of M is constant implies that $h_1^2+k_1^2 = \text{constant}$. That M is pseudo-umbilical implies that $h_1^2 = k_1^2$. Hence we have that

$$|h_1| = |k_1| = \text{constant}.$$

Thus Theorem 2 is proved.

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REFERENCES

(1) B. Y. Chen, *Geometry of submanifolds* (Marcel-Dekker, New York, 1973).

(2) B. Y. Chen, Pseudo-umbilical surfaces with constant Gauss curvature, *Proc. Edinburgh Math. Soc.* **18** (1972), 143-148.

(3) B. Y. Chen, Minimal surfaces with constant Gauss curvature, *Proc. Amer. Math. Soc.* **34** (1972), 504-508.

(4) J. D. Moore, Isometric immersions of riemannian products. *J. Diff. Geometry* **5** (1971), 159-168.

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