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FROBENIUS INDUCTION FOR HIGHER WHITEHEAD GROUPS

ANDREW J. NICAS

0. Introduction. The theory of induced representations has served as a powerful tool in the computations of algebraic K-theory and L-theory ([**2**], [**7**], [**4, 5**], [**9**], [**10, 11, 12, 13**], [**14**], [**17**], [**18**]). In this paper we show how to apply this theory to obtain induction theorems for the higher Whitehead groups of Waldhausen. The same technique applies to the analogs of Whitehead groups in unitary K-theory and in L-theory.

For any ring A with unit, let $\mathbf{K}(A)$ be the spectrum of the algebraic K-theory of A ([**8**, p. 343]). Given a discrete group Γ and a subring R of the rational numbers, Loday defines a map of spectra:

$$(*) \qquad (B\Gamma) \wedge \mathbf{K}(R) \rightarrow \mathbf{K}(R\Gamma)$$

where $(B\Gamma)$ is the classifying space of Γ union with a disjoint base point and $R\Gamma$ is the group-ring of Γ over R. The map of spectra (*) induces a homomorphism:

$$h_j(B\Gamma; \mathbf{K}(R)) \rightarrow K_j(R\Gamma)$$

where $h_j(B\Gamma; \mathbf{K}(R))$ is the generalized homology theory corresponding to $\mathbf{K}(R)$. The fiber of the map of spectra (*) is a spectrum whose $(j - 1)$st homotopy group is called the jth Whitehead group of Γ over R and is denoted by $\mathrm{Wh}_j^R(\Gamma)$.

For a finite group π define $\bar{\pi}$ to be the category whose objects are subgroups of π and whose morphisms are group homomorphisms given by conjugation by some element of π. Let RINGS be the category of rings with unit. Recall that a Frobenius functor is a contravariant functor $F:\bar{\pi} \rightarrow$ RINGS together with a functorial induction, i.e., given a morphism $f:H \rightarrow K$ in $\bar{\pi}$ there is a homomorphism of abelian groups

$$f_*:F(H) \rightarrow F(K).$$

Furthermore, Frobenius reciprocity is valid:

$$f_*(f^*a \cdot b) = a \cdot f_*b$$

$$f_*(b \cdot f^*a) = f_*b \cdot a$$

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for all $f:H \to K$, $a \in F(K)$, and $b \in F(H)$ (see [7]). The ring homomorphism f^* is called the restriction map. An example of a Frobenius functor, important in K-theory, arises from the Swan ring, $G_0^R(\pi)$, of a group π and a subring R of the rational numbers ([18]). As an abelian group $G_0^R(\pi)$ has a presentation with generators $\langle M \rangle$ where M is a finitely generated right $R\pi$ module which is free as an R-module, and relations $\langle M \rangle = \langle M_1 \rangle + \langle M_2 \rangle$ whenever there is an exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

of $R\pi$ modules. Now assume π is finite. The tensor product over R gives $G_0^R(\pi)$ the structure of a commutative ring. The association $H \to G_0^R(H)$ where H is a subgroup of π defines a Frobenius functor where for a morphism $f:H \to K$ in $\bar{\pi}$ the restriction map is given by restriction of rings and the induction map is given by the tensor product with RK over RH.

Let Ab be the category of abelian groups and $F:\bar{\pi} \to$ RINGS a given Frobenius functor. A contravariant functor $V:\bar{\pi} \to Ab$ is said to be a Frobenius module over F if V has a functorial induction and for each subgroup H of π $V(H)$ is a $F(H)$ module. Furthermore for any morphism $f:H \to K$

$$f^*(x \cdot z) = f^*(x) \cdot f^*(z)$$

$$f_*(f^*(x) \cdot y) = x \cdot f_*(y)$$

$$f_*(v \cdot f^*(z)) = f^*(v) \cdot z$$

for all $v \in F(H)$, $x \in F(K)$, $y \in V(H)$, $z \in V(K)$.

Now suppose Γ is a discrete group and $p:\Gamma \to \pi$ is a surjection to a finite group. If H is a subgroup of π define $\Gamma_H = p^{-1}(H)$. According to [5], the functor $H \to K_j(R\Gamma_H)$ is a Frobenius module over $H \to G_0^R(H)$. We prove the following:

MAIN THEOREM. *The functor $H \to h_j(B\Gamma_H; \mathbf{K}(R))$ is a Frobenius module over $H \to G_0^R(H)$ and the Loday homomorphism*

$$L_j:h_j(B\Gamma_H; \mathbf{K}(R)) \to K_j(R\Gamma_H)$$

is a morphism of Frobenius modules (i.e., L_j is a $G_0^R(H)$ module morphism and is natural with respect to restriction and induction).

As the $G_0^R(H)$-module action is constructed at the level of spectra we obtain an action of $G_0^R(H)$ on $\mathrm{Wh}_j^R(\Gamma_H)$. The main theorem yields as a consequence the following induction theorem for higher Whitehead groups:

PROPOSITION A. *Let $p:\Gamma \to \pi$ be as above, R a subring of the rational numbers, and B a commutative ring which is flat as a $\mathbf{Z}$-module. Suppose C is a collection of subgroups of π such that the sum of the induction maps*

$$\bigoplus_{H \in C} G_0^R(H) \otimes B \to G_0^R(\pi) \otimes B$$

is surjective. Then

1. *The sum of the induction maps*

$$\bigoplus_{H \in C} \mathrm{Wh}_j^R(\Gamma_H) \otimes B \to \mathrm{Wh}_j^R(\pi) \otimes B$$

is surjective $j \geqq 0$.

2. *The product of the restriction maps*

$$\mathrm{Wh}_j^R(\Gamma) \otimes B \to \prod_{H \in C} \mathrm{Wh}_j^R(\Gamma_H) \otimes B$$

is injective $j \geqq 0$.

In particular the hypothesis is satisfied in the cases

1. $R = B = \mathbf{Z}$ and C is the class of hyperelementary subgroups of π.
2. $R = \mathbf{Z}$, $B = \mathbf{Q}$, and C is the class of cyclic subgroups of π.

Analogous results hold in Hermitian K-theory and in L-theory if one replaces $G_0^R(\ )$ with the equivariant Witt ring of **[2]** and the Whitehead groups by their unitary and L-theoretic analogs.

1. Notation and conventions. For any associative ring A with unit, $GL(n, A)$ will be the general linear group of invertible n by n matrices over A. There is a natural inclusion $GL(n, A) \to GL(n + 1, A)$ and the direct limit lim $GL(n, A)$ will be denoted by $GL(A)$. Whenever C and D are groups $C \times D$ will be the product group of C and D and Hom(C, D) will be the set of group homomorphisms $C \to D$. If A and A' are algebras over a commutative ring R, typically in our application a subring of the rationals, then the tensor product of A and A' over R denoted by $A \otimes_R A'$ (and more briefly by $A \otimes A'$ when R is understood from the context) is also an R-algebra.

The tensor product of matrices defines a group homomorphism:

$$GL(n, A) \times GL(m, A') \to GL(nm, A \otimes_R A').$$

Explicitly let $X = (x_{ij}) \in GL(n, A)$ and $Y = (y_{ij}) \in GL(m, A')$. Order the set

$$I = \{ (i, j) \quad 1 \leqq i \leqq n, 1 \leqq j \leqq m \}$$

lexicographically so that it makes sense to speak of the "(k, l), (i, j)th" entry of a nm by nm matrix. The $(k, 1)$, (i, j)th entry of the matrix $X \otimes Y$ is $x_{ki} \otimes x_{1j}$. In the special case $A = R$ we identify $R \otimes_R A'$ with A' via the isomorphism $r \otimes a \to ra$ and similarly when $A' = R$.

The tensor product induces a continuous map (see **[8]**)

$$\gamma : BGL(n, A)^+ \wedge BGL(m, A')^+ \to BGL(nm, A \otimes_R A')^+$$

where for any group G, BG is the classifying space G and where $BGL(n, A)^+$ is the plus construction relative to the perfect subgroup

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$$E(n, A) = [GL(n, A), GL(n, A)]$$

of $GL(n, A)$.

The cone on a ring A, denoted CA, is the ring of infinite matrices with entries in A such that each row and each column have only finitely many nonzero entries. Let $\widetilde{A}$ be the two-sided ideal of CA consisting of those matrices in CA with only finitely many nonzero entries. The suspension of A, denoted SA, is the ring $SA = CA/\widetilde{A}$. If A is an R-algebra then SA inherits an R-algebra structure from the R-algebra of infinite matrices over A and there is a natural isomorphism $SA = SR \otimes_R A$. The infinite matrix $(u_{ij})_{i,j \geqq 1}$ where

$$u_{ij} = \begin{cases} 1 \text{ if } j = i - 1 \text{ and } i \geqq 2 \\ 0 \text{ otherwise} \end{cases}$$

represents an invertible element of SA which will be denoted by τ.

2. The action of the Swan ring on homology. Let A be a ring. According to [**8**, p. 343] A determines a spectrum, denoted $\mathbf{K}(A)$, which is defined by a sequence of spaces

$$(2.1) \quad \mathbf{K}(A)_k = K_0(S^kA) \times BGL(S^kA)^+$$

where $k \geqq 0$ and S^kA is the k-fold suspension of the ring A and where the suspension maps

$$\epsilon_k : \mathbf{K}(A)_k \wedge S^1 \rightarrow \mathbf{K}(A)_{k+1}$$

are given as follows: consider the group homomorphism

$$j : \mathbf{Z} \rightarrow GL(1, S\mathbf{Z})$$

which sends $1 \in \mathbf{Z}$ to $\tau \in GL(1, S\mathbf{Z})$. For $n \geqq 1$ there is a sequence of group homomorphisms:

$$(2.2) \quad GL(n, S^kA) \times \mathbf{Z} \xrightarrow{1 \times j} GL(n, S^kA) \times GL(1, S\mathbf{Z})$$
$$\overset{\otimes}{\rightarrow} GL(n, S^{k+1}A).$$

Note that $S^kA \otimes_{\mathbf{Z}} S\mathbf{Z}$ is identified with $S^{k+1}A$ via a natural isomorphism. Explicitly, the composite (2.2) is given by

$$(A, s) \rightarrow A \otimes_{\mathbf{Z}} \tau^s.$$

Stabilizing, taking classifying spaces, and performing the plus construction we obtain a map:

$$\epsilon'_k : BGL(S^kA)^+ \wedge S^1 \rightarrow BGL(S^{k+1}A)^+.$$

An element

$$\alpha \in K_0(S^kA) = \pi_1 BGL(S^{k+1}A)^+$$

is represented by a continuous map

$$S^1 \to BGL(S^{k+1}, A)^+$$

which will also be denoted by alpha. Define

$$\epsilon_k(\alpha, x, t) = (0, \epsilon'_k(x, t) + \alpha(t)).$$

We need a more algebraic definition of ϵ_k. Define

$$V(k, n, A) = \mathrm{Hom}(\mathbf{Z}, GL(n, S^{k+1}A)),$$

$$V(k, A) = \mathrm{Hom}(\mathbf{Z}, GL(S^{k+1}A)).$$

Observe that $V(k, A)$ bijects to the set of homotopy classes $[B\mathbf{Z}, BGL(S^{k+1}A)]$, which surjects to

$$\pi_1 BGL(S^{k+1}A)^+ = K_0(S^kA).$$

Hence elements of $K_0(S^kA)$ are represented by elements of $V(k, p, A)$ for sufficiently large p. Define

$$\bar{\epsilon}_k: V(k, p, A) \times GL(n, S^kA) \times \mathbf{Z} \to V(k, p, A) \times GL(n + p, S^{k+1}A)$$

by

$$\bar{\epsilon}_k(f, A, s) = (0, A \otimes_{\mathbf{Z}} \tau^s \oplus f(s))$$

where $\oplus$ denotes the direct sum of matrices. Then after stabilization, passage to classifying spaces, and performing the plus construction, $\bar{\epsilon}_k$ induces ϵ_k.

Let $p:\Gamma \to \pi$ be an epimorphism of a discrete group to a finite group. If H is a subgroup of π define $\Gamma_H = p^{-1}(H)$. Also let R be a subring of the rational numbers. Suppose

$$\phi: H \to GL(m, R)$$

is a homomorphism representing an element of $G_0^R(H)$. Composition with p defines a homomorphism $\Gamma_H \to GL(m, R)$. All tensor products will be over R unless otherwise indicated. Note that the tensor product defines a map

$$GL(n, R) \times V(k, p, R) \to V(k, pm, R)$$

by $(M \otimes f)(s) = M \otimes f(s)$ where $s \in \mathbf{Z}$, $M \in GL(m, R)$, and $f \in V(k, p, R)$. Define a map

$$\bar{\phi}_k: \Gamma_H \times V(k, p, R) \times GL(n, S^kR) \to \Gamma_H \times V(k, pm, R) \times GL(nm, S^kR)$$

by

$$\bar{\phi}_k(g, f, A) = (g, \phi p(g) \otimes f, \phi p(g) \otimes A).$$

After stabilization, passage to classifying spaces, and performing the plus construction $\bar{\phi}_k$ induces a continuous map

$$\phi_k:(B\Gamma_H) \wedge \mathbf{K}(R)_k \to \mathbf{K}(R)_k.$$

Observe that the following diagram is commutative:

$$\begin{array}{ccc} \Gamma_H \times V(k,p,R) \times GL(n, S^kR) \times \mathbf{Z} & \xrightarrow{\bar{\phi}_k \times 1} & \Gamma_H \times V(k,pm,R) \times GL(nm, S^kR) \times \mathbf{Z} \\ \downarrow & & \downarrow \\ \Gamma_H \times V(k+1,p,R) \times GL(n+p, S^{k+1}R) & \xrightarrow{\bar{\phi}_{k+1}} & \Gamma_H \times V(k+1,pm,R) \times GL(nm+pm, S^{k+1}R) \end{array}$$

The vertical maps are $1 \times \bar{\epsilon}_k$. This diagram induces a commutative diagram (actually homotopy commutative via a canonical homotopy)

$$\begin{array}{ccc} (B\Gamma_H) \wedge \mathbf{K}(R)_k \wedge S^1 & \xrightarrow{\phi_k} & (B\Gamma_H) \wedge \mathbf{K}(R)_k \wedge S^1 \\ \downarrow & & \downarrow \\ (B\Gamma_H) \wedge \mathbf{K}(R)_{k+1} & \xrightarrow{\phi_{k+1}} & (B\Gamma_H) \wedge \mathbf{K}(R)_{k+1} \end{array}$$

Thus ϕ induces a map of spectra:

(2.3) $\quad \phi_*:(B\Gamma_H) \wedge \mathbf{K}(R) \to (B\Gamma_H) \wedge \mathbf{K}(R).$

In order to see that (2.3) induces an action of $G_0^R(H)$ on $h_*(B\Gamma_H; \mathbf{K}(R))$ making the functor $H \to h_*(B\Gamma_H; \mathbf{K}(R))$ a Frobenius module over the Frobenius functor $H \to G_0^R(H)$, we use the following equivalent description of the action defined by (2.3).

Loday defines a homomorphism

$$\theta^0:G_0^R(\Gamma_H) \to h^0(B\Gamma_H; \mathbf{K}(R)).$$

(See Proposition 5.1.8 of **[8]** and the remark which follows it.) If

$$\psi:\Gamma_H \to GL(n, R)$$

is a representative of an element of $G_0^R(\Gamma_H)$ then $\theta^0(\psi)$ is represented by the homotopy class of the composite:

$$B\Gamma_H \xrightarrow{B\psi} BGL(R) \to BGL(R)^+ \to \mathbf{K}(R)_0.$$

Composition with $p:\Gamma_H \to H$ defines a homomorphism

$$p^*:G_0^R(H) \to G_0^R(\Gamma_H).$$

Let $\bar{\theta}^0$ be the composite of θ^0 with p^*. The tensor product of matrices

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makes $\mathbf{K}(R)$ a ring spectrum ([**8**, p. 346]) and thus there is a cap product:

$$\cap : h^p(X; \mathbf{K}(R)) \times h_q(X; \mathbf{K}(R)) \rightarrow h_{q-p}(X; \mathbf{K}(R)).$$

The action of $G_0^R(H)$ is given by the composite:

(2.4) $$G_0^R(H) \times h_*(B\Gamma_H; \mathbf{K}(R)) \xrightarrow{\bar{\theta}^0 \times 1} h^0(B\Gamma_H; \mathbf{K}(R)) \times h_*(B\Gamma_H; \mathbf{K}(R)) \xrightarrow{\cap} h_*(B\Gamma_H; \mathbf{K}(R)).$$

A comparison of the definitions immediately shows that this action is induced by (2.3). It is also clear that

$$\bar{\theta}^0 : G_0^R(H) \rightarrow h^0(B\Gamma_H; \mathbf{K}(R))$$

is natural with respect to restriction, i.e., given a homomorphism $f:H \rightarrow K$ of subgroups of π there is a commutative diagram

$$\begin{array}{ccc} G_0^R(K) & \xrightarrow{\bar{\theta}^0} & h^0(B\Gamma_K; \mathbf{K}(R)) \\ \downarrow f^* & & \downarrow f^* \\ G_0^R(H) & \xrightarrow{\bar{\theta}^0} & h^0(B\Gamma_H; \mathbf{K}(R)) \end{array}$$

$\bar{\theta}^0$ is a ring homomorphism by Proposition 4.1.4 of [**8**] where $h^0(\ )$ is a ring via the cup product. Now for any ring spectrum $\mathbf{E}$, in particular $\mathbf{K}(R)$, $H \rightarrow h^0(B\Gamma_H; \mathbf{E})$ is a Frobenius functor and $H \rightarrow h_*(B\Gamma_H; \mathbf{E})$ is a Frobenius module over $H \rightarrow h^0(B\Gamma_H; \mathbf{E})$ via the cap product (compare [**9**]). In the next section we will show that $\bar{\theta}^0$ is natural with respect to induction. Assuming this, $\bar{\theta}^0$ is thus a morphism of Frobenius functors and we have, as a consequence,

PROPOSITION 2.5. *$H \rightarrow h_*(B\Gamma_H; \mathbf{K}(R))$ is a Frobenius module over $H \rightarrow G_0^R(H)$ where the action of $G_0^R(H)$ is defined by* (2.3) *or equivalently by* (2.4).

3. The action of the Swan ring on K-theory. Let $p:\Gamma \rightarrow \pi$, $H \leqq \pi$, $\Gamma_H = p^{-1}(H)$ and $\phi:H \rightarrow GL(m, R)$ be as in Section 2. In this section all tensor products will be taken over R. Give R^m the right $R\Gamma_H$ module structure defined by

$$vh = \phi p(h^{-1})v \quad h \in \Gamma_H, v \in R^m.$$

Let $k \geqq 0$ and $\Lambda = S^k(R\Gamma_H)$. Recall that there is a natural ring isomorphism

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$$i{:}S^kR \otimes R\Gamma_H \to \Lambda.$$

Give $\Lambda^n \otimes R^m$ the diagonal right $S^kR \otimes R\Gamma_H$-module structure defined by

$$(x \otimes y)(\lambda \otimes h) = xi(\lambda \otimes h) \otimes yh$$

where $x \in \Lambda^n, y \in R^m, \lambda \in S^kR, h \in R\Gamma_H$. Then i^{-1} makes $\Lambda^n \otimes R^m$ into a free right Λ-module. A basis for $\Lambda^n \otimes R^m$ is given by $\{e_i \otimes f_j\}$ where $\{e_i\}$ is the standard basis for Λ^n and $\{f_j\}$ is the standard basis for R^m. An element $A = (a_{ij}) \in GL(n, \Lambda)$ determines a Λ-linear map of right Λ-modules:

$$L_A{:}\Lambda^n \to \Lambda^n, \quad L_A(e_j) = \sum_{i=1}^{n} e_i a_{ij}.$$

Tensoring with the identity map of R^m gives a Λ-linear map

$$L_A \otimes 1{:}\Lambda^n \otimes R^m \to \Lambda^n \otimes R^m$$

where $\Lambda^n \otimes R^m$ has the right Λ-module described above. Let $[L_A \otimes 1]$ be the matrix of $L_A \otimes 1$ with respect to the basis $\{e_i \otimes f_j\}$. There is a group homomorphism

(3.1) $\quad \bar{\phi}_k{:}GL(n, \Lambda) \to GL(nm, \Lambda)$

defined by $\bar{\phi}_k(A) = [L_A \otimes 1]$. An explicit formula for $\bar{\phi}_k$ is given as follows. Write

$$a_{ij} = \sum_{h \in \Gamma_H} r(i, j, h) \otimes h \quad \text{where } r(i, j, h) \in S^kR.$$

Then the $(k, l), (i, j)$th component of $\bar{\phi}_k(A)$ is

(3.2) $$\sum_{h \in \Gamma_H} (r(k, i, h)(\phi p)_{lj}(h)) \otimes h,$$

where $(\phi p)_{lj}$ is the (l, j)th component of the matrix $\phi p(h)$. This can be expressed more succinctly. If $B = (b_{ij})$ is a matrix over S^kR, and $h \in \Gamma_H$, we write $B \otimes h$ for the matrix $(b_{ij} \otimes h)$ over the ring $S^kR \otimes R\Gamma_H$ which is naturally identified with Λ. Writing

$$A = \sum_{h \in \Gamma_H} A(h) \otimes h$$

where all but finitely many of the matrices $A(h)$ are zero, we have

(3.3) $$\bar{\phi}_k(A) = \sum_{h \in \Gamma_H} (A(h) \otimes \phi p(h)) \otimes h,$$

where $A(h) \otimes \phi p(h)$ is a matrix over $S^kR \otimes R$, which we identify with S^kR. Define a map

(3.4) $\tilde{\phi}_k : V(k, p, R) \times GL(n, \Lambda) \to V(k, pm, R) \times GL(nm, \Lambda)$

by

$$\tilde{\phi}_k(f, x) = (\phi_{k+1} f, \phi_k(x))$$

where

$$f \in V(k, p, R) = \mathrm{Hom}(\mathbf{Z}, GL(p, S^{k+1}R))$$

and $x \in GL(n, \Lambda)$.

After stabilization, taking classifying spaces, and performing the plus construction, we have that (3.4) induces a continuous map

$$\phi_k : \mathbf{K}(R\Gamma_H)_k \to \mathbf{K}(R\Gamma_H)_k.$$

It is easy to verify that the ϕ_k's are compatible with the suspension maps ϵ_k of Section 2 and thus define a map of spectra:

(3.5) $\phi : \mathbf{K}(R\Gamma_H) \to \mathbf{K}(R\Gamma_H)$.

At the level of homotopy groups, (3.5) gives an action of $G_0^R(H)$ on $K_j(R\Gamma_H)$ making $H \to K_j(R\Gamma_H)$ a Frobenius module over $H \to G_0^R(H)$ (compare [**5**] and also see the remark preceding Proposition 5.1.6 of [**8**] for the Frobenius reciprocity formula in K-theory). The fact that the action determined by (3.5) is compatible with the relations of $G_0^R(H)$ is a consequence of Quillen's additivity theorem (see Theorem 2, chapter 3 of [**15**]).

Loday defines a map of spectra ([**8**])

(3.6) $L : (B\Gamma_H) \wedge \mathbf{K}(R) \to \mathbf{K}(R\Gamma_H)$

which arises as follows:

Consider

(3.7) $L_k : \Gamma_H \times V(k, p, R) \times GL(n, S^k R)$
$\qquad \to V(k, p, R\Gamma_H) \times GL(n, S^k(R\Gamma_H))$

given by

$$L_k(h, f, A) = (f \otimes h, A \otimes h).$$

Note that $f \otimes h$ is defined by

$$(f \otimes h)(s) = f(s) \otimes h \quad \text{for } s \in \mathbf{Z}.$$

After stabilization, passage to classifying spaces, and performing the plus construction, one has continuous maps

$$L_k : (B\Gamma_H) \wedge \mathbf{K}(R)_k \to \mathbf{K}(R\Gamma_H)_k$$

for $k \geqq 0$ which together give the map of spectra (3.6). Using the explicitly

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algebraic description of the Loday map (3.7) and of the action of ϕ, it is then an easy exercise to verify that the following diagram is commutative:

$$\begin{array}{ccc} \Gamma_H \times V(k, p, R) \times GL(n, S^kR) & \xrightarrow{L_k} & V(k, p, R\Gamma_H) \times GL(n, S^kR\Gamma_H) \\ \downarrow 1 \times \tilde{\varphi}_k & & \downarrow \tilde{\varphi}_k \\ \Gamma_H \times V(k, pm, R) \times GL(nm, S^kR) & \xrightarrow{L_k} & V(k, pm, R\Gamma_H) \times GL(nm, S^kR\Gamma_H). \end{array}$$

The above diagram induces a commutative diagram in the category of spectra:

$$\begin{array}{ccc} (B\Gamma_H) \wedge \mathbf{K}(R) & \xrightarrow{L} & \mathbf{K}(R\Gamma_H) \\ \downarrow \varphi & & \downarrow \varphi \\ (B\Gamma_H) \wedge \mathbf{K}(R) & \xrightarrow{L} & \mathbf{K}(R\Gamma_H) \end{array}$$

Hence the action of $G_0^R(H)$ is compatible with the Loday homomorphism at the level of spectra. As a consequence there is a natural action of $G_0^R(H)$ on $\mathrm{Wh}_j^R(\Gamma_H)$, the $(j - 1)$st homotopy group of the fibre of the Loday map.

In order to complete the proof of the main theorem we need to show that the Loday map is compatible with restriction and that the map

$$\theta^0 : G_0^R(\Gamma_H) \to h^0(B\Gamma_H; \mathbf{K}(R))$$

discussed in Section 2 is compatible with induction. Note that in cohomology theory the induction map is called the transfer and in homology theory the restriction map is called the transfer. First, we recall the construction of the transfer in generalized homology and cohomology theory. Let Σ_q be the symmetric group of permutations of q symbols. If G is a group then the wreath product $\Sigma_q \int G$ is the semidirect product

$$1 \to G^q \to \Sigma_q \int G \to \Sigma_q \to 1,$$

where Σ_q acts on G^q (the q-fold product group of G) by permuting the coordinates. Suppose F is a subgroup of finite index q in L. Let

$$L = \bigcup_{i=1}^{q} g_i F$$

be a decomposition of L into left cosets. There is a group monomorphism

$$u : L \to \Sigma_q \int F$$

(see [**3**]) given as follows: For $x \in L$ we have

$$xg_i = g_{\sigma(i)}f_i, \quad i = 1, \dots, q,$$

for some unique permutation $\sigma \in \Sigma_q$ and $f_i \in F$. Then

$$u(x) = ((f_1, \dots, f_q), \sigma) \in \Sigma_q \int F.$$

Let W be a contractible CW complex on which Σ_q acts freely. The induced map of classifying spaces

$$u{:}BL \to B(\Sigma_q \int F) = (BF)^q \times_{\Sigma_q} W$$

agrees with the map

$$\Phi{:}BL \to (BF)^q \times_{\Sigma_q} W$$

of [**6**] (Φ is what is called the "pretransfer" in [**6**]).

For an unpointed space X, let $\Sigma^\infty X$ be the suspension spectrum associated to X union with a disjoint base point. There is a commutative triangle in the category of spectra:

$$(3.8) \qquad \begin{array}{ccc} \Sigma^\infty BL & \xrightarrow{\ u_*\ } & \Sigma^\infty B(\Sigma_q \int F) \\ & {\scriptstyle t}\searrow \quad \swarrow & \\ & \Sigma^\infty BF & \end{array}$$

The map of spectra u_* is induced from the map

$$u{:}BL \to B(\Sigma_q \int F).$$

The map of spectra $t{:}\Sigma^\infty BL \to \Sigma^\infty BF$ induces the transfer

$$t{:}h_*(BL) \to h_*(BF)$$

in any generalized homology theory h_* and the transfer

$$t{:}h^*(BF) \to h^*(BL)$$

in any generalized cohomology theory h^* (see [**6**]). By viewing Σ_q as the group of q by q permutation matrices, we have for any group E a natural inclusion homomorphism

$$(3.9) \qquad \begin{array}{c} \Sigma_q \int E \to GL(q, \mathbf{Z}E) \\ ((e_1, \dots, e_q), \sigma) \to (\delta_{i\sigma(j)}e_i) \end{array}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

Consider the following generalization of the Loday homomorphism. Let E be a group and fix a positive integer q. As in (3.7) define for $k \geqq 0$

(3.10) $$L_k : GL(q, \mathbf{Z}E) \times V(k, p, R) \times GL(n, S^kR) \to V(k, pq, RE) \times GL(nq, S^k(RE))$$

by the formula

$$L_k(h, f, A) = (f \otimes h, A \otimes h)$$

where $\otimes$ denotes the tensor product of matrices.

Then (3.10) gives a map of spectra:

(3.11) $$L : BGL(q, \mathbf{Z}E) \wedge \mathbf{K}(R) \to \mathbf{K}(RE),$$

which yields the map of homotopy groups

$$L_j : h_j(BGL(q, \mathbf{Z}E); \mathbf{K}(R)) \to K_j(RE).$$

We now recall the construction of the restriction map in K-theory. Let $L \to GL(q, \mathbf{Z}F)$ be the composite of $u : L \to \Sigma_q \int F$ with the homomorphism of (3.9). This homomorphism determines a ring homomorphism

$$RL \to \mathrm{Mat}(q, RF)$$

where $\mathrm{Mat}(q, RF)$ is the R-algebra of q by q matrices over RF. Hence we obtain a group homomorphism

$$GL(n, RL) \to GL(n, \mathrm{Mat}(q, RF)) = GL(nq, RF).$$

Stabilizing, taking classifying spaces, and performing the plus construction yields a map

$$BGL(RL)^+ \to BGL(RF)^+$$

and it is clear how to extend this to a map of spectra

$$i^* : \mathbf{K}(RL) \to \mathbf{K}(RF)$$

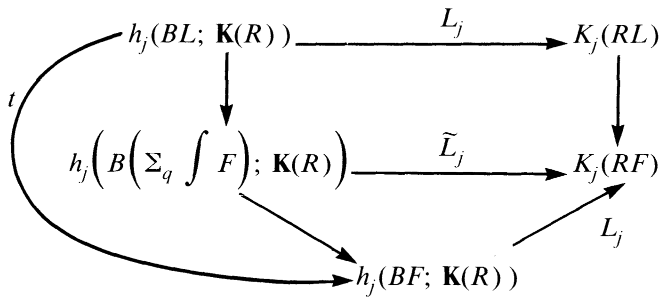
which in homotopy gives the restriction map

$$i^* : K_j(RL) \to K_j(RF).$$

Using (3.7) and (3.10) it is easy to verify that the following diagram of maps of spectra is commutative:

$$\begin{array}{ccc}
(BL) \wedge \mathbf{K}(R) & \xrightarrow{\ L\ } & \mathbf{K}(RL) \\
\Big\downarrow {\scriptstyle u \wedge 1} & & \Big\downarrow \\
(B(\Sigma_q \int F)) \wedge \mathbf{K}(R) & \xrightarrow{\ \widetilde{L}\ } & \mathbf{K}(RF)
\end{array}$$

where $\widetilde{L}$ is the composite of $B(\Sigma_q \int F) \to BGL(q, \mathbf{Z}F)$ smashed with identity and the generalized Loday map (3.11). Combining this square with (3.8) and taking homotopy groups we have a commutative diagram



It follows that the Loday map commutes with the restriction map.

The transfer in the cohomology theory associated with the spectrum $\mathbf{K}(R)$ can also be described by specifying "structure maps" as defined in chapter 4 of [**1**]. Let A be any ring with unit. Viewing Σ_q as the group of q by q permutation matrices, we have an embedding of groups

$$\Sigma_q \int GL(n, A) \to GL(nq, A).$$

After stabilization, taking classifying spaces, and performing the plus construction, this induces a map

$$s:(BGL(A)^+)^q \times_{\Sigma_q} W\Sigma_q \to BGL(A)^+.$$

By letting $A = S^kR$ we obtain structure maps for $\mathbf{K}(R)$,

$$s_k:\mathbf{K}(R)_k^+ \times_{\Sigma_q} W\Sigma_q \to \mathbf{K}(R)_k.$$

The transfer in the cohomology theory with coefficients in $\mathbf{K}(R)$ is given as follows. Let $f:BF \to \mathbf{K}(R)_k$ represent an element of $h^k(BF; \mathbf{K}(R))$. Then the transfer of f is represented by the composite:

$$BL \xrightarrow{u} B(\Sigma_q \int F) = (BF)^q \times_{\Sigma_q} W\Sigma_q$$

$$\downarrow (f)^q \times_{\Sigma_q} 1$$

$$\mathbf{K}(R)_k^q \times_{\Sigma_q} W\Sigma_q$$

$$\downarrow s_k$$

$$\mathbf{K}(R)_k$$

Using this description of the transfer it is readily verified that the map

$$\theta^0:G_0^R(E) \to h^0(BE; \mathbf{K}(R))$$

discussed in the previous section commutes with the induction map.

Remarks. The following statements are easy consequences of the main theorem.

1. The functors

$$H \to G_0^R(H), \quad H \to h_j(\Gamma_H, \mathbf{K}(R)), \quad \text{and } H \to K_j(R\Gamma_H)$$

all satisfy double coset formulas, so in the terminology of [**2**] the Loday map L_j is a morphism of Green modules over the Green functor $H \to G_0^R(H)$.

2. For j in the range $0 \leqq j \leqq 3$ the Whitehead groups are a quotient of K-theory, i.e.,

$$\mathrm{Wh}_j(\Gamma_H) = \mathrm{coker}(L_j).$$

It follows that $H \to \mathrm{Wh}_j(\Gamma_H)$ is a Frobenius module (a Green module by the above remark) over the Swan ring for $0 \leqq j \leqq 3$.

4. Proof of proposition A and concluding remarks. We now prove Proposition A of the introduction. Let $p:\Gamma \to \pi$ be an epimorphism to a finite group, R a subring of the rational numbers, and B a commutative ring which is flat as a $\mathbf{Z}$-module. Suppose C is a collection of subgroups of π such that the sum of the induction maps:

$$\bigoplus_{H \in C} G_0^R(H) \otimes B \to G_0^R(\pi) \otimes B,$$

is surjective. For a subgroup H of π there is a long exact sequence:

$$\to h_j(B\Gamma_H; \mathbf{K}(R)) \xrightarrow{L_j} K_j(R\Gamma_H)$$
$$\to \mathrm{Wh}_j^R(\Gamma_H) \to \ldots \to \mathrm{Wh}_0^R(\Gamma_H) \to 0.$$

This sequence remains exact when tensored with B. Write

$$W_j(H) = \mathrm{image}(L_j) \otimes B,$$
$$V_j(H) = \ker(L_{j-1}) \otimes B \quad (\text{set } V_0(H) = 0).$$

By the main theorem the Loday map is a morphism of Frobenius modules over the Frobenius functor $H \to G_0^R(H)$. Consequently, $H \to W_j(H)$ and $H \to V_j(H)$ are Frobenius modules over

$$H \to G_0^R(H) \otimes B$$

(compare chapter 6 of [**10**]). For any subgroup H of π there is an exact sequence:

$$(4.1) \qquad 0 \to W_j(H) \to \mathrm{Wh}_j^R(\Gamma_H) \otimes B \to V_j(H) \to 0.$$

Since the sum of the induction maps

$$\bigoplus_{H \in C} G_0^R(H) \otimes B \to G_0^R(\pi) \otimes B$$

is surjective it follows by Proposition 1.2 of [**2**] that the product of the restriction maps

$$W_j(\pi) \to \prod_{H \in C} W_j(H)$$

$$V_j(\pi) \to \prod_{H \in C} V_j(H)$$

are injective and the sum of the induction maps

$$\bigoplus_{H \in C} W_j(H) \to W_j(\pi)$$

$$\bigoplus_{H \in C} V_j(H) \to V_j(\pi)$$

are surjective. It follows from (4.1) and a simple diagram chase that the product of the restriction maps

$$\mathrm{Wh}_j^R(\Gamma) \otimes B \to \prod_{H \in C} \mathrm{Wh}_j^R(\Gamma_H) \otimes B$$

is injective and the sum of the induction maps

$$\bigoplus_{H \in C} \mathrm{Wh}_j^R(\Gamma_H) \otimes B \to \mathrm{Wh}_j^R(\Gamma) \otimes B$$

is surjective.

In particular the hypothesis is satisfied in the following two cases.

1. $R = B = \mathbf{Z}$ and C is the class of hyperelementary subgroups of π.
2. $R = \mathbf{Z}$, $B = \mathbf{Q}$, and C is the class of cyclic subgroups of π. (See [**18**].)

Proposition A generalizes a result of T.-Y. Lam to higher Whitehead groups (compare Proposition 1.1, chapter 4 of [**7**]).

In L-theory, Ranicki has defined for a ring with involution A the spectrum of the L-theory of A, denoted $\mathbf{L}(A)$, and for a discrete group Γ a map of spectra:

$$(4.2) \quad (B\Gamma) \wedge \mathbf{L}(\mathbf{Z}) \to \mathbf{L}(\mathbf{Z}\Gamma)$$

in analogy with the Loday map $(B\Gamma) \wedge \mathbf{K}(\mathbf{Z}) \to \mathbf{K}(\mathbf{Z}\Gamma)$ (see [**16**]). The groups

$$\pi_j \mathbf{L}(\mathbf{Z}\Gamma) = L_j^s(\mathbf{Z}\Gamma)$$

are the L-groups of Wall. The $(j - 1)$st homotopy group of the fibre of (4.2) is the L-theoretic analog of the jth Whitehead group and we will use the notation $UWh_j(\Gamma)$ for it.

Remark. In the special case $M = K(\Gamma, 1)$ is a closed orientable aspherical manifold

$$UWh_j(\Gamma) = S_{\mathrm{TOP}}(M \times I^{j+3}, \partial)$$

where S_{TOP} is the structure set of topological surgery.

If the Swan ring $G_0^Z(\ )$ is replaced with the equivariant Witt ring $GW_0(\ )$ of [2], then the same type of argument used to prove the main theorem shows that Ranicki's map

$$A_j:h_j(B\Gamma_H;\ \mathbf{L}(\mathbf{Z})\ ) \to L_j^s(\Gamma_H)$$

is a morphism of Frobenius modules over the Frobenius functor $H \to GW_0(H)$ where as before $p:\Gamma \to \pi$ is an epimorphism to a finite group and $\Gamma_H = p^{-1}(H)$. Thus as a consequence, if C is the collection of 2-hyperelementary subgroups and p-elementary subgroups for odd primes p of the group π then we have that the product of the restriction maps

$$UWh_j(\Gamma) \to \prod_{H\in C} UWh_j(\Gamma_H)$$

is injective and the sum of the restriction maps

$$\bigoplus_{H\in C} UWh_j(\Gamma_H) \to UWh_j(\Gamma)$$

is surjective (compare with Theorems 6.2.11-13 of [**10**]). These arguments also apply to Hermitian K-theory of type discussed in [**8**] where the equivariant Witt ring is used in place of the Swan ring.

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University of Toronto,
Toronto, Ontario