

BOUNDEDNESS OF ROUGH SINGULAR INTEGRAL OPERATORS ON HOMOGENEOUS HERZ SPACES

GUOEN HU, SHANZHEN LU and DACHUN YANG

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Abstract

The authors establish the boundedness on the Herz spaces and the weak Herz spaces for a large class of rough singular integral operators and their corresponding fractional versions. Applications are given to Fefferman's rough singular integral operators, their fractional versions, their commutators with $BMO(\mathbb{R}^n)$ functions and Ricci-Stein oscillatory singular integral operators. Some new results are obtained.

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1. Introduction

A well-known result of Stein [23] states that if

$$(1.1) \quad Tf(x) = \text{p. v.} \int_{\mathbb{R}^n} H(x-y)f(y) dy$$

is bounded on $L^q(\mathbb{R}^n)$ for some $q \in (1, \infty)$, and if the kernel H satisfies the 'standard size' condition

$$(1.2) \quad |H(x)| \leq \frac{c}{|x|^n} \text{ for all } x \in \mathbb{R}^n \setminus \{0\},$$

then T is also bounded on $L^q_{|x|^\alpha}(\mathbb{R}^n)$ for $\alpha \in (-n, n(q-1))$. Here, and in what follows, $f \in L^q_\omega(\mathbb{R}^n)$ for any non-negative function ω on \mathbb{R}^n means that f is a measurable

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function and

$$\|f\|_{L^q_\omega(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^q \omega(x) dx \right)^{1/q} < \infty.$$

Let Ω be a homogeneous function of degree zero on \mathbb{R}^n and $\Omega \in L^r(S^{n-1})$ for some $r \in [1, \infty]$. If T in (1.1) is bounded on $L^q(\mathbb{R}^n)$ for some $q \in (1, \infty)$, and H satisfies, instead of (1.2),

$$(1.3) \quad |H(x)| \leq c \frac{|\Omega(x)|}{|x|^n} \text{ for all } x \in \mathbb{R}^n \setminus \{0\},$$

what then can be said on the boundedness of T on $L^q_{|x|^\alpha}(\mathbb{R}^n)$? The main purpose of this paper is to answer this question. Note that the homogeneous Herz spaces are some kind of generalizations of the weighted Lebesgue spaces. We establish our results in the setting of the Herz spaces.

Let $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for any $k \in \mathbb{Z}$. Denote χ_{A_k} by χ_k for any $k \in \mathbb{Z}$, where χ_E is the characteristic function of the set E .

DEFINITION 1.1. Let $\alpha \in \mathbb{R}, 0 < p \leq \infty$ and $0 < q \leq \infty$. The homogeneous Herz spaces $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ are defined by

$$\dot{K}^{\alpha,p}_q(\mathbb{R}^n) = \{f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p},$$

and the usual modifications are made when $p = \infty$ or $q = \infty$.

Obviously, $\dot{K}^{0,q}_q(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ and $\dot{K}^{\alpha/q,q}_q(\mathbb{R}^n) = L^q_{|x|^\alpha}(\mathbb{R}^n)$. We remark that studies involving these spaces have a long history; and we recommend the references [9, 14, 17] and [12].

Soria and Weiss [22] give some beautiful generalizations of Stein’s results in [23]. In particular, they obtain the result of Stein in the case $q = 1$. We generalize the results of Soria and Weiss in the case $q = 1$ to weak Herz spaces which are introduced by the authors in [12].

In what follows, for any $k \in \mathbb{Z}$ and any $\sigma > 0$, let $m_k(\sigma, f) = |\{x \in A_k : |f(x)| > \sigma\}|$.

DEFINITION 1.2. ([12]) Let $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leq \infty$. A measurable function f on \mathbb{R}^n is said to belong to the homogeneous weak Herz space $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, if

$$\|f\|_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda > 0} \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} m_k(\lambda, f)^{p/q} \right\}^{1/p} < \infty,$$

where the usual modifications are made when $p = \infty$.

Clearly, $W\dot{K}_q^{0,q}(\mathbb{R}^n) = WL^q(\mathbb{R}^n)$ and $W\dot{K}_q^{\alpha/q,q}(\mathbb{R}^n) = WL_{|x|^\alpha}^q(\mathbb{R}^n)$. Here $f \in WL_\omega^q(\mathbb{R}^n)$ means that f is a measurable function on \mathbb{R}^n and

$$(1.4) \quad \sup_{\lambda > 0} \lambda [\omega(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})]^{1/q} \leq c < \infty.$$

The smallest constant c satisfying (1.4) is denoted by $\|f\|_{WL_\omega^q(\mathbb{R}^n)}$.

Let $0 < l < n$. Instead of (1.3), if H satisfies

$$(1.5) \quad |H(x)| \leq c \frac{|\Omega(x)|}{|x|^{n-l}} \text{ for all } x \in \mathbb{R}^n \setminus \{0\},$$

what will happen for the operator T defined by (1.1)? We obtain some similar results to that operator T with a kernel H satisfying (1.5).

Our main theorems are stated in Section 2; and their proofs are also given in this section. In Section 3, we give some applications. Based on the results of Fan-Pan [6], Seeger [21], Ding [4], Chanillo-Christ [1], Christ-Rubio de Francia [2], Jiang-Lu [13] and Pan [19], we obtain some new results on rough singular integrals, fractional rough singular integrals and oscillatory singular integral operators.

We should point out that some special cases of our theorems have been obtained in [16, 10], and [15]. Some of the techniques used here are borrowed from Soria-Weiss [22]; see also [14].

Throughout this paper, c always means a constant independent of the main parameters involved, but may be different from line to line. For any power exponent p with $1 \leq p \leq \infty$, we denote the conjugate exponent $p/(p - 1)$ by p' .

2. Main theorems and their proofs

To begin with, the following theorem indicates that the boundedness of operator T on Lebesgue spaces with power weights implies its boundedness on homogeneous Herz spaces.

THEOREM 2.1. *If a sublinear operator T is bounded on $L_{|x|^\beta}^q(\mathbb{R}^n)$ for all $\beta \in (\beta_1, \beta_2)$ and some $q \in (1, \infty)$, where $\beta_1, \beta_2 \in \mathbb{R}$, then T is also bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ provided that $\alpha \in (\beta_1/q, \beta_2/q)$ and $0 < p \leq \infty$.*

PROOF. We choose $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\beta_1/q < \alpha_1/q < \alpha < \alpha_2/q < \beta_2/q$. Write

$$f(x) = \sum_{k=-\infty}^{\infty} f(x)\chi_k(x) \equiv \sum_{k=-\infty}^{\infty} f_k(x).$$

By the condition of the theorem, we then have

$$\begin{aligned} \|Tf\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|\chi_k Tf\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \\ &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k+1} \|\chi_k Tf_j\|_{L^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\quad + c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k+2}^{\infty} \|\chi_k Tf_j\|_{L^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_2/q)p} \left(\sum_{j=-\infty}^{k+1} \|\chi_k Tf_j\|_{L_{|\lambda|^{\alpha_2}}^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\quad + c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_1/q)p} \left(\sum_{j=k+2}^{\infty} \|\chi_k Tf_j\|_{L_{|\lambda|^{\alpha_1}}^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_2/q)p} \left(\sum_{j=-\infty}^{k+1} \|f_j\|_{L_{|\lambda|^{\alpha_2}}^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\quad + c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_1/q)p} \left(\sum_{j=k+2}^{\infty} \|f_j\|_{L_{|\lambda|^{\alpha_1}}^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\leq c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k+1} 2^{j\alpha} \|f_j\|_{L^q(\mathbb{R}^n)} 2^{(k-j)(\alpha-\alpha_2/q)} \right)^p \right\}^{1/p} \\ &\quad + c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha} \|f_j\|_{L^q(\mathbb{R}^n)} 2^{(k-j)(\alpha-\alpha_1/q)} \right)^p \right\}^{1/p}. \end{aligned}$$

If $0 < p \leq 1$, the above quantity is dominated by

$$\begin{aligned} &c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k+1} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p 2^{(k-j)(\alpha-\alpha_2/q)p} \right) \right\}^{1/p} \\ &\quad + c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p 2^{(k-j)(\alpha-\alpha_1/q)p} \right) \right\}^{1/p} \end{aligned}$$

$$\begin{aligned}
 &= c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \left(\sum_{k=j-1}^{\infty} 2^{(k-j)(\alpha-\alpha_2/q)p} \right) \right\}^{1/p} \\
 &\quad + c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(\alpha-\alpha_1/q)p} \right) \right\}^{1/p} \\
 &\leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \\
 &= c \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},
 \end{aligned}$$

since $\alpha - \alpha_1/q > 0$ and $\alpha - \alpha_2/q < 0$.

Now, let $1 < p \leq \infty$. By Hölder’s inequality, we then have

$$\begin{aligned}
 \|Tf\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &\leq c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k+1} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p 2^{(k-j)(\alpha-\alpha_2/q)p/2} \right) \right. \\
 &\quad \left. \times \left(\sum_{j=-\infty}^{k+1} 2^{(k-j)(\alpha-\alpha_2/q)p'/2} \right)^{p/p'} \right\}^{1/p} \\
 &\quad + c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p 2^{(k-j)(\alpha-\alpha_2/q)p/2} \right) \right. \\
 &\quad \left. \times \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\alpha_1/q)p'/2} \right)^{p/p'} \right\}^{1/p} \\
 &\leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \left(\sum_{k=j-1}^{\infty} 2^{(k-j)(\alpha-\alpha_2/q)p/2} \right) \right\}^{1/p} \\
 &\quad + c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(\alpha-\alpha_1/q)p/2} \right) \right\}^{1/p} \\
 &\leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \\
 &= c \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.
 \end{aligned}$$

This finishes the proof of Theorem 2.1. □

Let Ω be a homogeneous function of degree zero on \mathbb{R}^n and $\Omega \in L^r(S^{n-1})$ for certain $r \in [1, \infty]$. The rough Hardy-Littlewood maximal function M_Ω is defined by

$$M_\Omega f(x) = \sup_{t>0} \frac{1}{t^n} \int_{|y|<t} |\Omega(y')f(x-y)| dy,$$

where $y' = y|y|^{-1}$.

The following result for the boundedness of M_Ω on the Lebesgue spaces with power weights can be found in [5, p. 874].

LEMMA 2.1 ([5]). *If $\Omega \in L^r(S^{n-1})$, $1 \leq r \leq \infty$, $1 < q < \infty$ and*

$$\max(-n, -1 - (n - 1)q/r') < \beta < \min(n(q - 1), q - 1 + (n - 1)q/r'),$$

then M_Ω is bounded on $L^q_{|x|^\beta}(\mathbb{R}^n)$.

As a simple corollary of Theorem 2.1 and Lemma 2.1, we have the following corollary, which will be used in the proof of our main theorem.

COROLLARY 2.1. *Let $\Omega \in L^r(S^{n-1})$, $1 \leq r \leq \infty$, $1 < q < \infty$, $0 < p \leq \infty$ and*

$$\max\left(-\frac{n}{q}, -\frac{1}{q} - \frac{n-1}{r'}\right) < \alpha < \min\left(n\left(1 - \frac{1}{q}\right), 1 - \frac{1}{q} + \frac{n-1}{r'}\right),$$

then M_Ω is bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

To prove one of our main theorems, we also need the following two lemmas. The first one can be found in [18, p. 251] and the second one in [8].

LEMMA 2.2 ([18]). *Let $\Omega \in L^r(S^{n-1})$ and $r \in [1, \infty]$. If $a > 0$, $0 < d \leq r$ and $-n + (n - 1)d/r < \beta < \infty$, then*

$$\left(\int_{|x| \leq a|y|} |\Omega(x - y)|^d |x|^\beta dx\right)^{1/d} \leq c|y|^{(\beta+n)/d} \|\Omega\|_{L^r(\mathbb{R}^n)}.$$

In fact, when $1 < r \leq \infty$, this lemma is just a special case of [18, Lemma 1]. The case $r = 1$ is easy to prove using polar coordinates. We omit the details.

LEMMA 2.3 ([8]). *Let $\alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$. Then $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if and only if*

$$\left| \int_{\mathbb{R}^n} f(x)g(x) dx \right| < \infty$$

for every $g \in \dot{K}_q^{-\alpha,p'}(\mathbb{R}^n)$ and, in this case,

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sup \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x) dx \right| : \|g\|_{\dot{K}_q^{-\alpha,p'}(\mathbb{R}^n)} \leq 1 \right\}.$$

Now, we are in a position to state one of our main theorems.

THEOREM 2.2. *Let $\Omega \in L^r(S^{n-1})$ for $r \in [1, \infty]$ and let $0 < p \leq \infty$, $1 < q < \infty$ and $\max(-n/q, -1/q - (n-1)/r') < \alpha < \min(n(1-1/q), 1-1/q + (n-1)/r')$. If a sublinear operator T is bounded on $L^q(\mathbb{R}^n)$ and there is a constant c independent of f such that*

$$(2.1) \quad |T(f)(x)| \leq c \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy$$

for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp } f$, then T is also bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

PROOF. Write

$$f(x) = \sum_{k=-\infty}^{\infty} f(x)\chi_k(x) \equiv \sum_{k=-\infty}^{\infty} f_k(x)$$

and

$$\begin{aligned} \|Tf\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|\chi_k Tf\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \\ &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| \chi_k T \left(\sum_{j=-\infty}^{k-2} f_j \right) \right\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \\ &\quad + c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| \chi_k T \left(\sum_{j=k-1}^{k+1} f_j \right) \right\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \\ &\quad + c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| \chi_k T \left(\sum_{j=k+2}^{\infty} f_j \right) \right\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \\ &\equiv D_1 + D_2 + D_3. \end{aligned}$$

For D_2 , by the $L^q(\mathbb{R}^n)$ -boundedness of T , we have

$$D_2 \leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| \sum_{j=k-1}^{k+1} f_j \right\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \leq c \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

For D_1 , note that when $x \in A_k$, $j \leq k-2$, and $y \in A_j$, then $2|y| \leq |x|$. Therefore, for $x \in A_k$,

$$\left| T \left(\sum_{j=-\infty}^{k-2} f_j \right) (x) \right| \leq c \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} \left| \sum_{j=-\infty}^{k-2} f_j(y) \right| dy$$

$$\begin{aligned} &\leq \frac{c}{|x|^n} \int_{|x-y| \leq 3|x|/2} |\Omega(x-y)| \left| \sum_{j=-\infty}^{k-2} f_j(y) \right| dy \\ &\leq c M_\Omega \left(\sum_{j=-\infty}^{k-2} f_j \right) (x). \end{aligned}$$

Thus, from Corollary 2.1, it follows that

$$\begin{aligned} D_1 &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{kap} \left\| \chi_k M_\Omega \left(\sum_{j=-\infty}^{k-2} f_j \right) \right\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \\ &\leq c \left\| M_\Omega \left(\sum_{j=-\infty}^{k-2} f_j \right) \right\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \\ &\leq c \left\| \sum_{j=-\infty}^{k-2} f_j \right\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \\ &\leq c \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}. \end{aligned}$$

Now, let us turn to the estimate for D_3 . We first assume $1 < p < \infty$. Note that when $j \geq k + 2, x \in A_k$, and $y \in A_j$, then $2|x| \leq |y|$ and

$$\begin{aligned} \left| T \left(\sum_{j=k+2}^{\infty} f_j \right) (x) \right| &\leq c \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} \left| \sum_{j=k+2}^{\infty} f_j(y) \right| dy \\ &\leq c \int_{2|x| \leq |y|} \frac{|\Omega(x-y)|}{|y|^n} \left| \sum_{j=k+2}^{\infty} f_j(y) \right| dy \\ &\equiv c T_0 \left(\left| \sum_{j=k+2}^{\infty} f_j \right| \right) (x). \end{aligned}$$

Set

$$T_1(g)(x) = \frac{1}{|x|^n} \int_{|y| \leq 2|x|} |\Omega(x-y)| g(y) dy.$$

Then by Lemma 2.3 and Corollary 2.1, when $1 < p < \infty$, we obtain

$$\begin{aligned} D_3 &\leq c \left\| T_0 \left(\sum_{j=k+2}^{\infty} f_j \right) \right\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \\ &= c \sup_{\|g\|_{\dot{K}_q^{-\alpha,p'}(\mathbb{R}^n)} \leq 1} \left| \left\langle T_0 \left(\sum_{j=k+2}^{\infty} f_j \right), g \right\rangle \right| \end{aligned}$$

$$\begin{aligned}
 &= c \sup_{\|g\|_{\dot{K}_q^{-\alpha,p'}(\mathbb{R}^n)} \leq 1} \left| \left\langle \left| \sum_{j=k+2}^{\infty} f_j \right|, T_1(g) \right\rangle \right| \\
 &\leq c \left\| \sum_{j=k+2}^{\infty} f_j \right\|_{\dot{K}_q^{-\alpha,p}(\mathbb{R}^n)} \sup_{\|g\|_{\dot{K}_q^{-\alpha,p'}(\mathbb{R}^n)} \leq 1} \|M_\Omega(g)\|_{\dot{K}_q^{-\alpha,p'}(\mathbb{R}^n)} \\
 &\leq c \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \sup_{\|g\|_{\dot{K}_q^{-\alpha,p'}(\mathbb{R}^n)} \leq 1} \|M_\Omega(g)\|_{\dot{K}_q^{\alpha,p'}(\mathbb{R}^n)} \\
 &\leq c \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \sup_{\|g\|_{\dot{K}_q^{-\alpha,p'}(\mathbb{R}^n)} \leq 1} \|g\|_{\dot{K}_q^{-\alpha,p'}(\mathbb{R}^n)} \\
 &\leq c \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},
 \end{aligned}$$

since $\max(-n/q', -1/q' - (n - 1)/r') < -\alpha < \min(n/q, 1/q + (n - 1)/r')$.

Now, let $0 < p \leq 1$. In this case, we consider two sub-cases. First, suppose $1 \leq r' \leq q$ and, therefore, $\alpha > -n/q$. When $x \in A_k$ and $j \geq k + 2$, by Hölder’s inequality, we have

$$\begin{aligned}
 \left| T \left(\sum_{j=k+2}^{\infty} f_j \right) (x) \right| &\leq c \sum_{j=k+2}^{\infty} \int_{\mathbb{R}^n} \frac{|\Omega(x - y)|}{|x - y|^n} |f_j(y)| dy \\
 &\leq c \sum_{j=k+2}^{\infty} 2^{-jn/q} \|\Omega\|_{L^{q'}(S^{n-1})} \|f_j\|_{L^q(\mathbb{R}^n)} \\
 &\leq c \|\Omega\|_{L^{q'}(S^{n-1})} \sum_{j=k+2}^{\infty} 2^{-jn/q} \|f_j\|_{L^q(\mathbb{R}^n)}.
 \end{aligned}$$

Thus we have

$$\begin{aligned}
 D_3 &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha+n/q)p} \left(\sum_{j=k+2}^{\infty} 2^{-jn/q} \|f_j\|_{L^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
 &\leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{-jnp/q} \|f_j\|_{L^q(\mathbb{R}^n)}^p \left(\sum_{k=-\infty}^{j-2} 2^{k(\alpha+n/q)p} \right) \right\}^{1/p} \\
 &= c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \\
 &\leq c \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.
 \end{aligned}$$

Now suppose $r' > q$ and, therefore, $\alpha > -1/q - (n - 1)/r'$. Let $s = 1 - r + r/q = 1 - r/q'$. Then $(1 - s)q' = r$ and $sq = (1 - r)q + r \leq r$. When $x \in A_k$ and $j \geq k + 2$,

by Hölder’s inequality, we have

$$\begin{aligned} \left| T \left(\sum_{j=k+2}^{\infty} f_j \right) (x) \right| &\leq c \sum_{j=k+2}^{\infty} 2^{-jn/q} \|\Omega\|_{L^{(1-s)q'}(S^{n-1})}^{1-s} \\ &\quad \times \left(\int_{\mathbb{R}^n} |f_j(y)|^q |\Omega(x-y)|^{sq} dy \right)^{1/q} \\ &\leq c \sum_{j=k+2}^{\infty} 2^{-jn/q} \left(\int_{\mathbb{R}^n} |f_j(y)|^q |\Omega(x-y)|^{(1-r)q+r} dy \right)^{1/q}. \end{aligned}$$

We choose $\beta \in \mathbb{R}$ and $0 < \epsilon < 1$ such that $\alpha > \beta/q > -1/q - (n-1)/r'$ and $\alpha - \beta/q > (1-\epsilon)n/q$. Then, by Hölder’s inequality and Lemma 2.2, we obtain

$$\begin{aligned} &\left\| \chi_k T \left(\sum_{j=k+2}^{\infty} f_j \right) \right\|_{L^q(\mathbb{R}^n)} \\ &\leq c \left(\sum_{j=k+2}^{\infty} 2^{-jn(1-\epsilon)q'/q} \right)^{1/q'} \\ &\quad \times \left\| \chi_k \left[\sum_{j=k+2}^{\infty} 2^{-jn\epsilon} \left(\int_{\mathbb{R}^n} |f_j(y)|^q |\Omega(x-y)|^{(1-r)q+r} dy \right) \right] \right\|_{L^q(\mathbb{R}^n)}^{1/q} \\ &\leq c 2^{-kn(1-\epsilon)/q} \left\{ \sum_{j=k+2}^{\infty} 2^{-jn\epsilon} 2^{-k\beta} \int_{\mathbb{R}^n} |f_j(y)|^q \right. \\ &\quad \times \left. \left(\int_{|x|\leq|y|/2} |\Omega(x-y)|^{(1-r)q+r} |x|^\beta dx \right) dy \right\}^{1/q} \\ &\leq c 2^{-k(n(1-\epsilon)+\beta)/q} \left\{ \sum_{j=k+2}^{\infty} 2^{-jn\epsilon} \int_{\mathbb{R}^n} |f_j(y)|^q |y|^{\beta+n} dy \right\}^{1/q} \\ &\leq c 2^{-k(n(1-\epsilon)+\beta)/q} \left\{ \sum_{j=k+2}^{\infty} 2^{-j(n\epsilon-\beta-n)/q} \|f_j\|_{L^q(\mathbb{R}^n)} \right\}. \end{aligned}$$

Thus, in this case, we have

$$\begin{aligned} D_3 &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-(n(1-\epsilon)+\beta)/q)p} \left(\sum_{j=k+2}^{\infty} 2^{-j(n\epsilon-\beta-n)p/q} \|f_j\|_{L^q(\mathbb{R}^n)}^p \right) \right\}^{1/p} \\ &= c \left\{ \sum_{j=-\infty}^{\infty} 2^{-j(n\epsilon+\beta-n)p/q} \|f_j\|_{L^q(\mathbb{R}^n)}^p \left(\sum_{k=-\infty}^{j-2} 2^{k(\alpha-(n(1-\epsilon)+\beta)/q)p} \right) \right\}^{1/p} \end{aligned}$$

$$\begin{aligned}
 &= c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \\
 &= c \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.
 \end{aligned}$$

We leave the estimate for D_3 with $p = \infty$ to interested readers. This finishes the proof of Theorem 2.2. □

Note that $L_{|x|^\alpha}^q(\mathbb{R}^n) = \dot{K}_q^{\alpha/q,q}(\mathbb{R}^n)$. The following corollary is a consequence of Theorem 2.2.

COROLLARY 2.2. *Let $\Omega \in L^r(S^{n-1})$ and $r \in [1, \infty]$ and let $1 < q < \infty$ and*

$$\max \left(-n, -1 - \frac{(n-1)q}{r'} \right) < \beta < \min \left(n(q-1), q-1 + \frac{(n-1)q}{r'} \right).$$

If a sublinear operator T satisfies (2.1) and is bounded on $L^q(\mathbb{R}^n)$, then T is also bounded on $L_{|x|^\beta}^q(\mathbb{R}^n)$.

By examining the proof of Theorem 2.2, one can easily prove the following extension.

THEOREM 2.3. *Let $0 < p \leq \infty$, $1 \leq r \leq \infty$, and*

$$\max \left(-\frac{n}{q}, -\frac{1}{q} - \frac{n-1}{r'} \right) < \alpha < \min \left(n \left(1 - \frac{1}{q} \right), 1 - \frac{1}{q} + \frac{n-1}{r'} \right).$$

Suppose that $\Omega_j \in L^r(S^{n-1})$ for $j \in I$ (I is some index set) and

$$(2.2) \quad \|\Omega_j\|_{L^r(S^{n-1})} \leq c_0 < \infty,$$

where c_0 is independent of j . For sublinear operators T_j 's, we define

$$T^*f(x) = \sup_{j \in I} |T_j(f)(x)|.$$

If T^ is bounded on $L^q(\mathbb{R}^n)$ and T_j satisfies*

$$(2.3) \quad |T_j(f)(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{|\Omega_j(x-y)|}{|x-y|^n} |f(y)| dy$$

for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp } f$, then T^ is also bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.*

COROLLARY 2.3. *Let $1 < q < \infty, 1 \leq r \leq \infty$, and*

$$\max(-n, -1 - (n - 1)q/r') < \beta < \min(n(q - 1), q - 1 + (n - 1)q/r').$$

Suppose $\{\Omega_j\}_{j \in I}$ and $\{T_j\}_{j \in I}$ satisfy the conditions of Theorem 2.3. If T^ is bounded on $L^q(\mathbb{R}^n)$, then T^* is also bounded on $L^q_{|\cdot|^\beta}(\mathbb{R}^n)$.*

Now, let us turn to the fractional rough singular integrals. We first have the following theorem similar to Theorem 2.1.

THEOREM 2.4. *Let $0 < l < n$ and T_l be a sublinear operator. If T_l is bounded from $L^q_{|\cdot|^{\alpha q_1}}(\mathbb{R}^n)$ into $L^q_{|\cdot|^{\alpha q_2}}(\mathbb{R}^n)$ for all $\alpha \in (\alpha_1, \alpha_2)$ and some $q_1, q_2 \in (1, \infty)$, where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $1/q_2 = 1/q_1 - l/n$, then T_l is also bounded from $\dot{K}^{\alpha, p_1}_{q_1}(\mathbb{R}^n)$ into $\dot{K}^{\alpha, p_2}_{q_2}(\mathbb{R}^n)$ provided that $\alpha \in (\alpha_1, \alpha_2)$ and $0 < p_1 \leq p_2 \leq \infty$.*

PROOF. Note that if $p_1 < p_2$, then $\dot{K}^{\alpha, p_1}_{q_2}(\mathbb{R}^n) \subset \dot{K}^{\alpha, p_2}_{q_2}(\mathbb{R}^n)$. We only need to prove the theorem in the case $p_1 = p_2$. Choose $\beta_1, \beta_2 \in \mathbb{R}$ such that $\alpha_1 < \beta_1 < \alpha < \beta_2 < \alpha_2$. Write

$$f(x) = \sum_{k=-\infty}^{\infty} f(x)\chi_k(x) \equiv \sum_{k=-\infty}^{\infty} f_k(x),$$

it follows that

$$\begin{aligned} \|T_l f\|_{\dot{K}^{\alpha, p_1}_{q_2}(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \|\chi_k T_l f\|_{L^{q_2}(\mathbb{R}^n)}^{p_1} \right\}^{1/p_1} \\ &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k+1} \|\chi_k T_l f_j\|_{L^{q_2}(\mathbb{R}^n)} \right)^{p_1} \right\}^{1/p_1} \\ &\quad + c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k+2}^{\infty} \|\chi_k T_l f_j\|_{L^{q_2}(\mathbb{R}^n)} \right)^{p_1} \right\}^{1/p_1} \\ &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\beta_2)p_1} \left(\sum_{j=-\infty}^{k+1} \|\chi_k T_l f_j\|_{L^{q_2}_{|\cdot|^{\beta_2 q_2}}(\mathbb{R}^n)} \right)^{p_1} \right\}^{1/p_1} \\ &\quad + c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\beta_1)p_1} \left(\sum_{j=k+2}^{\infty} \|\chi_k T_l f_j\|_{L^{q_2}_{|\cdot|^{\beta_1 q_2}}(\mathbb{R}^n)} \right)^{p_1} \right\}^{1/p_1} \\ &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\beta_2)p_1} \left(\sum_{j=-\infty}^{k+1} \|f_j\|_{L^{q_1}_{|\cdot|^{\beta_2 q_1}}(\mathbb{R}^n)} \right)^{p_1} \right\}^{1/p_1} \\ &\quad + c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\beta_1)p_1} \left(\sum_{j=k+2}^{\infty} \|f_j\|_{L^{q_1}_{|\cdot|^{\beta_1 q_1}}(\mathbb{R}^n)} \right)^{p_1} \right\}^{1/p_1} \end{aligned}$$

$$\begin{aligned}
 &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\beta_2)p_1} \left(\sum_{j=-\infty}^{k+1} \|f_j\|_{L^{q_1}(\mathbb{R}^n)} 2^{j\beta_2} \right)^{p_1} \right\}^{1/p_1} \\
 &\quad + c \left\{ \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\beta_1)p_1} \left(\sum_{j=k+2}^{\infty} \|f_j\|_{L^{q_1}(\mathbb{R}^n)} 2^{j\beta_1} \right)^{p_1} \right\}^{1/p_1} \\
 &\leq c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k+1} 2^{j\alpha} \|f_j\|_{L^{q_1}(\mathbb{R}^n)} 2^{j(\beta_2-\alpha)} \right)^{p_1} \right\}^{1/p_1} \\
 &\quad + c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha} \|f_j\|_{L^{q_1}(\mathbb{R}^n)} 2^{j(\beta_1-\alpha)} \right)^{p_1} \right\}^{1/p_1} \\
 &\leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p_1} \|f_j\|_{L^{q_1}(\mathbb{R}^n)}^{p_1} \right\}^{1/p_1} \\
 &= c \|f\|_{\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)},
 \end{aligned}$$

where in the second-to-last inequality, we have employed the facts $\beta_2 - \alpha > 0$ and $\beta_1 - \alpha < 0$. This finishes the proof of Theorem 2.4. \square

The following lemma can be found in [18, p. 250].

LEMMA 2.4. *Let $0 < l < n$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, $n/(n-l) \leq r \leq \infty$ and $\Omega \in L^r(S^{n-1})$. If there is a constant c independent of f such that the sublinear operator T_l satisfies that*

$$(2.4) \quad |T_l(f)(x)| \leq c \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-l}} |f(y)| dy$$

for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp } f$, then T_l is also bounded from $L_{|x|^{q_1\beta}}^{q_1}(\mathbb{R}^n)$ into $L_{|x|^{q_2\beta}}^{q_2}(\mathbb{R}^n)$ provided that

$$l + \max\left(-\frac{n}{q_1}, -\frac{1}{q_1} - \frac{n-1}{r'}\right) < \beta < -l + \min\left(\frac{n}{q_2}, \frac{1}{q_2} + \frac{n-1}{r'}\right).$$

As a simple corollary of Theorem 2.4 and Lemma 2.4, we have

COROLLARY 2.4. *Assume that $0 < l < n$, $0 < p_1 \leq p_2 \leq \infty$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, $n/(n-l) \leq r \leq \infty$ and $\Omega \in L^r(S^{n-1})$. If a sublinear operator T_l satisfies (2.4), then T_l is also bounded from $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ into $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$, provided that*

$$l + \max\left(-\frac{n}{q_1}, -\frac{1}{q_1} - \frac{n-1}{r'}\right) < \beta < -l + \min\left(\frac{n}{q_2}, \frac{1}{q_2} + \frac{n-1}{r'}\right).$$

Now we come to consider the end-point cases of Theorems 2.1, 2.2 and 2.4. First, we will establish the boundedness of M_Ω on weak Herz spaces.

THEOREM 2.5. *Let $\Omega \in L'(S^{n-1})$ and $r \in [1, \infty]$ and let $0 < p \leq \infty$ and*

$$-n + (n - 1)/r < \alpha < 0.$$

If $r > 1$, or $r = 1$ and M_Ω is of weak type $(1, 1)$, then M_Ω is type $(\dot{K}_1^{\alpha,p}(\mathbb{R}^n), W\dot{K}_1^{\alpha,p}(\mathbb{R}^n))$.

PROOF. Write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x)$$

and

$$\begin{aligned} & \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} |\{x \in A_k : M_\Omega f(x) > \lambda\}|^p \right\}^{1/p} \\ & \leq c\lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left| \left\{ x \in A_k : \left| M_\Omega \left(\sum_{j=-\infty}^{k+1} f_j \right) (x) \right| > \lambda/2 \right\} \right|^p \right\}^{1/p} \\ & \quad + c\lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left| \left\{ x \in A_k : \left| M_\Omega \left(\sum_{j=k+2}^{\infty} f_j \right) (x) \right| > \lambda/2 \right\} \right|^p \right\}^{1/p} \\ & \equiv E_1 + E_2. \end{aligned}$$

In [2], Christ and Rubio de Francia have proved that when $r > 1$, M_Ω is of weak type $(1, 1)$. By this and the hypothesis in the case $r = 1$, for E_1 , we have

$$\begin{aligned} E_1 & \leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| \sum_{j=-\infty}^{k+1} f_j \right\|_{L^1(\mathbb{R}^n)}^p \right\}^{1/p} \\ & \leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k+1} \|f_j\|_{L^1(\mathbb{R}^n)} \right)^p \right\}^{1/p}. \end{aligned}$$

Note that $\alpha < 0$. If $0 < p \leq 1$, then

$$E_1 \leq c \left\{ \sum_{j=-\infty}^{\infty} \|f_j\|_{L^1(\mathbb{R}^n)}^p \left(\sum_{k=j-1}^{\infty} 2^{k\alpha p} \right) \right\}^{1/p}$$

$$\begin{aligned}
 &= c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^1(\mathbb{R}^n)}^p \right\}^{1/p} \\
 &= c \|f\|_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)}.
 \end{aligned}$$

Now, let $1 < p \leq \infty$. By Hölder’s inequality, we then deduce

$$\begin{aligned}
 E_1 &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p/2} \left(\sum_{j=-\infty}^{k+1} 2^{j\alpha p/2} \|f_j\|_{L^1(\mathbb{R}^n)}^p \right) \left(\sum_{j=-\infty}^{k+1} 2^{(k-j)\alpha p'/2} \right)^{p/p'} \right\}^{1/p} \\
 &\leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p/2} \|f_j\|_{L^1(\mathbb{R}^n)}^p \left(\sum_{j=k-1}^{\infty} 2^{k\alpha p/2} \right) \right\}^{1/p} \\
 &\leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^1(\mathbb{R}^n)}^p \right\}^{1/p} \\
 &= c \|f\|_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)}.
 \end{aligned}$$

To estimate E_2 , note that $-n + (n - 1)/r < \alpha < 0$. We can choose $\alpha_1 \in \mathbb{R}$ such that $0 < \alpha_1 < \alpha + n - (n - 1)/r$. When $j \geq k + 2$, by Lemma 2.2, we have

$$\begin{aligned}
 &\left\| \chi_k M_{\Omega} \left(\sum_{j=k+2}^{\infty} f_j \right) \right\|_{L^1(\mathbb{R}^n)} \\
 &\leq c \sum_{j=k+2}^{\infty} 2^{-jn} \int_{A_k} \left(\int_{A_j} |\Omega(x - y)| |f_j(y)| dy \right) dx \\
 &\leq c \sum_{j=k+2}^{\infty} 2^{-jn+k(\alpha_1-\alpha)} \int_{A_j} |f_j(y)| \left(\int_{|x|\leq|y|/2} |\Omega(x - y)| |x|^{\alpha_1-\alpha} dx \right) dy \\
 &\leq c \sum_{j=k+2}^{\infty} 2^{-jn+k(\alpha_1-\alpha)} \int_{A_j} |f_j(y)| |y|^{\alpha-\alpha_1+n} dy \\
 &\leq c \sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha_1-\alpha)} \|f_j\|_{L^1(\mathbb{R}^n)}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 E_2 &\leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| \chi_k M_{\Omega} \left(\sum_{j=k+2}^{\infty} f_j \right) \right\|_{L^1(\mathbb{R}^n)}^p \right\}^{1/p} \\
 &\leq c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha} \|f_j\|_{L^1(\mathbb{R}^n)} 2^{(k-j)\alpha_1} \right)^p \right\}^{1/p}
 \end{aligned}$$

Note that $\alpha_1 > 0$. If $0 < p \leq 1$, then

$$\begin{aligned} E_2 &\leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^1(\mathbb{R}^n)}^p \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)\alpha_1 p} \right) \right\}^{1/p} \\ &\leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^1(\mathbb{R}^n)}^p \right\}^{1/p} \\ &= c \|f\|_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)}. \end{aligned}$$

If $1 < p \leq \infty$, then by Hölder’s inequality, we obtain

$$\begin{aligned} E_2 &\leq c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha p} \|f_j\|_{L^1(\mathbb{R}^n)}^p 2^{(k-j)\alpha_1 p/2} \right) \left(\sum_{j=k+2}^{\infty} 2^{(k-j)\alpha_1 p'/2} \right)^{p/p'} \right\}^{1/p} \\ &\leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^1(\mathbb{R}^n)}^p \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)\alpha_1 p/2} \right) \right\}^{1/p} \\ &= c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^1(\mathbb{R}^n)}^p \right\}^{1/p} \\ &= c \|f\|_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)}. \end{aligned}$$

This finishes the proof of Theorem 2.5. □

By a proof similar to that of Theorem 2.5, one can prove the following

THEOREM 2.6. *Suppose that $\Omega \in L^r(S^{n-1})$ and $r \in [1, \infty]$ and let $0 < p \leq \infty$ and $-n + (n - 1)/r < \alpha < 0$. If a sublinear operator T is of weak type $(1, 1)$ and satisfies (2.1), then T is also bounded from $\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$ into $W\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$.*

Note that $\dot{K}_1^{\alpha,1}(\mathbb{R}^n) = L^1_{|x|^\alpha}(\mathbb{R}^n)$. Thus, Theorems 2.5 and 2.6 with the case $p = 1$ are the same as [22, Proposition 8] with the power weights. In addition, Theorem 2.6 is also true for the maximal singular integrals. That is,

THEOREM 2.7. *Let $1 \leq r \leq \infty$, $0 < p \leq \infty$ and $-n + (n - 1)/r < \alpha < 0$. Suppose $\{\Omega_j\}_{j \in I}$, $\{T_j\}_{j \in I}$ and T^* are the same as in Theorem 2.3 and satisfies (2.2) and (2.3). If T^* is of weak type $(1, 1)$, then T^* maps continuously $\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$ into $W\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$.*

For fractional singular integrals, we have some theorems similar to Theorems 2.6 and 2.7.

THEOREM 2.8. *Let $0 < l < n, 0 < p_1 \leq p_2 \leq \infty, q_1 = n/(n - l) \leq r \leq \infty,$*

$$l - n + (n - 1)/r < \alpha < 0,$$

and $\Omega \in L^r(S^{n-1})$. If a sublinear operator T_l satisfies (2.4) and is of weak type $(1, q_1)$, then T_l is also type $(\dot{K}_1^{\alpha, p_1}(\mathbb{R}^n), W\dot{K}_{q_2}^{\alpha, p_2}(\mathbb{R}^n))$.

PROOF. Note that if $p_1 < p_2$, then $W\dot{K}_{q_1}^{\alpha, p_1}(\mathbb{R}^n) \subset W\dot{K}_{q_1}^{\alpha, p_2}(\mathbb{R}^n)$. We only need to prove the theorem in the case $p_1 = p_2$. Write

$$f(x) = \sum_{j=-\infty}^{\infty} f_j(x) \chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x)$$

and

$$\begin{aligned} & \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} |\{x \in A_k : |T_l(f)(x)| > \lambda\}|^{p_1/q_1} \right\}^{1/p_1} \\ & \leq c\lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left| \left\{ x \in A_k : \left| T_l \left(\sum_{j=-\infty}^{k+1} f_j \right) (x) \right| > \lambda/2 \right\} \right|^{p_1/q_1} \right\}^{1/p_1} \\ & \quad + c\lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left| \left\{ x \in A_k : \left| T_l \left(\sum_{j=k+2}^{\infty} f_j \right) (x) \right| > \lambda/2 \right\} \right|^{p_1/q_1} \right\}^{1/p_1} \\ & \equiv F_1 + F_2. \end{aligned}$$

Since T_l is of weak type $(1, q_1)$, for F_1 , we then have

$$\begin{aligned} F_1 & \leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left\| \sum_{j=-\infty}^{k+1} f_j \right\|_{L^1(\mathbb{R}^n)}^{p_1} \right\}^{1/p_1} \\ & \leq c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k+1} 2^{j\alpha} \|f_j\|_{L^1(\mathbb{R}^n)} 2^{(k-j)\alpha} \right)^{p_1} \right\}^{1/p_1} \\ & \leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p_1} \|f_j\|_{L^1(\mathbb{R}^n)}^{p_1} \right\}^{1/p_1} \\ & = c \|f\|_{\dot{K}_1^{\alpha, p_1}(\mathbb{R}^n)}, \end{aligned}$$

where we have used that $\alpha < 0$ and some similar estimates used for the estimate of E_1 . For the estimate of F_2 , note that $l - n + (n - 1)/r < \alpha < 0$. We can choose $\beta_1 \in \mathbb{R}$ such that

$$0 < \beta_1 < \alpha + n - (n - 1)/r - l.$$

When $j \geq k + 1$, by Lemma 2.2, we have

$$\begin{aligned} & \left\| \chi_k T_l \left(\sum_{j=k+2}^{\infty} f_j \right) \right\|_{L^{q_1}(\mathbb{R}^n)} \\ & \leq c \sum_{j=k+2}^{\infty} 2^{j(n-l)} \left[\int_{A_k} \left(\int_{A_j} |\Omega(x-y)| |f_j(y)| dy \right)^{q_1} dx \right]^{1/q_1} \\ & \leq c \sum_{j=k+2}^{\infty} 2^{-j(n-l)-k\beta/q_1} \int_{A_j} |f_j(y)| \left(\int_{A_k} |\Omega(x-y)|^{q_1} |x|^\beta dx \right)^{1/q_1} dy \\ & \leq c \sum_{j=k+2}^{\infty} 2^{-j(n-l)-k\beta/q_1} \int_{A_j} |f_j(y)| |y|^{(\beta+n)/q_1} dy \\ & \leq c \sum_{j=k+2}^{\infty} 2^{(j-k)\beta/q_1} \|f_j\|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

where we choose $\beta \in \mathbb{R}$ such that $\alpha > \beta/q_1 > -n/q_1 + (n-1)/r = (n-1)/r - n + l$. Since $\alpha > (n-1)/r - n + l$, this is possible. Therefore,

$$\begin{aligned} F_2 & \leq c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left\| \chi_k T_l \left(\sum_{j=k+2}^{\infty} f_j \right) \right\|_{L^{q_2}(\mathbb{R}^n)}^{p_1} \right\}^{1/p_1} \\ & \leq c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha} \|f_j\|_{L^1(\mathbb{R}^n)} 2^{(j-k)(\beta q_2 - \alpha)} \right)^{p_1} \right\}^{1/p_1} \\ & \leq c \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p_1} \|f_j\|_{L^1(\mathbb{R}^n)}^{p_1} \right\}^{1/p_1} \\ & = c \|f\|_{\dot{K}_1^{\alpha, p_1}(\mathbb{R}^n)} \end{aligned}$$

by some similar estimates to those used for E_2 . This finishes the proof of Theorem 2.8. □

If we take $p_1 = 1$ and $p_2 = q_1$ in Theorem 2.8, we then obtain

COROLLARY 2.5. *Let $0 < l < n$, $q_1 = n/(n-l) \leq r \leq \infty$, $-n + (n-1)/r < \alpha < 0$ and $\Omega \in L^r(S^{n-1})$. If a sublinear operator T_l satisfies (2.4) and is of weak type $(1, q_1)$, then T_l is also of type $(L_{|x|^\alpha}(\mathbb{R}^n), WL_{|x|^{\alpha/q_1}}^{q_1}(\mathbb{R}^n))$.*

We remark that Corollary 2.5 is the end-point case of [18, Theorem 2].

3. Some applications

In this section, we will give some applications of the theorems in Section 2. Let us begin with the following singular integral operators of Fefferman [7]:

$$(3.1) \quad T_h f(x) = \text{p. v.} \int_{\mathbb{R}^n} h(x-y) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy,$$

where h is a bounded radial function, Ω is homogeneous of degree 0 and belongs to $L^1(S^{n-1})$. A result of Fan-Pan [6] states that if $\Omega \in H^1(S^{n-1})$, the Hardy space on S^{n-1} (see [3] or [6] for the definition), then this operator is bounded on $L^q(\mathbb{R}^n)$ for all $q \in (1, \infty)$. As a consequence of this result and Theorem 2.2, we have

COROLLARY 3.1. *Let $0 < p \leq \infty$, $1 < q < \infty$ and $-1/q < \alpha < 1 - 1/q$. Suppose T is defined as in (3.1). If h is a bounded radial function and $\Omega \in H^1(S^{n-1})$, then T_h is bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.*

From [18, Theorem 1] and Theorem 2.1 or [18, Theorem 1] with $\alpha = 0$ and Theorem 2.3, we deduce the following.

COROLLARY 3.2. *Suppose that when $r \in (1, \infty]$,*

$$\Omega \in L^r(S^{n-1});$$

or when $r = 1$, $\Omega \in L \log^+ L(S^{n-1})$. If $0 < p \leq \infty$, $1 < q < \infty$,

$$\max\left(-\frac{n}{q}, -\frac{1}{q} - \frac{n-1}{r'}\right) < \alpha < \min\left(n\left(1 - \frac{1}{q}\right), 1 - \frac{1}{q} + \frac{n-1}{r'}\right)$$

and Ω has mean value 0 on S^{n-1} , then

$$T_1^* f(x) = \sup_{\epsilon > 0} \left| \int_{|y|>\epsilon} \frac{\Omega(y)}{|y|^n} f(x-y) dy \right|$$

is bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

We point out that the conditions of [18, Theorem 1] require

$$\Omega \in L \log^+ L(S^{n-1})$$

when $r = 1$.

When $q = p$, Corollary 3.2 is just [18, Theorem 1] and is sharp in this case; see [18].

Now, let us consider the following commutators of high order defined by

$$(3.2) \quad T_{b,l}^m f(x) = \text{p.v.} \int_{\mathbb{R}^n} (b(x) - b(y))^m \frac{\Omega(x - y)}{|x - y|^{n-l}} f(y) dy,$$

where $0 \leq l < \infty, b \in \text{BMO}(\mathbb{R}^n)$ and $m \in \mathbb{N}$.

From [11, Theorem 1], [4, Theorem 4.3], Theorems 2.1 and 2.4, we deduce

COROLLARY 3.3. *Let $0 < p_1 \leq p_2 \leq \infty, 0 \leq l < n, 1 < q_1 < n/l, 1/q_2 = 1/q_1 - l/n, 1 - 1/r > 1/q_1 > l/n$ and $-n/q_1 + l < \alpha < n(1 - 1/r - 1/q_1)$. If $\Omega \in L^r(S^{n-1}), b \in \text{BMO}(\mathbb{R}^n)$ and $m \in \mathbb{N}$, then $T_{b,l}^m$ in (3.2) is bounded from $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ into $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$.*

We now turn to the boundedness of T_l in (3.1) on weak type Herz spaces. By the proposition in [21] and Theorem 2.6, we obtain

COROLLARY 3.4. *Let $0 < p \leq \infty$ and $-1 < \alpha < 0$. If $\Omega \in L \log^+ L(S^{n-1})$ and has mean value zero, then T_l in (3.1) is bounded from $\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$ into $W\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$.*

Similarly, for the following fractional singular integrals T^l defined by

$$(3.3) \quad T^l f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-l}} f(x - y) dy,$$

we have the following corollary.

COROLLARY 3.5. *Let $0 < p_1 \leq p_2 \leq \infty, 0 < l < n, q_1 = n/(n - l) < r \leq \infty$ and $l - n + (n - 1)/r < \alpha < 0$. If $\Omega \in L^r(S^{n-1})$, then the operator T^l in (3.3) maps continuously $\dot{K}_1^{\alpha,p_1}(\mathbb{R}^n)$ into $W\dot{K}_{q_1}^{\alpha,p_2}(\mathbb{R}^n)$.*

In [4], Ding proved that if $\Omega \in L^r(S^{n-1})$ and $q_1 = n/(n - l) < r \leq \infty$, then the operator T^l in (3.1) is of type $(1, q_1)$. This result and Theorem 2.8 imply Corollary 3.5.

Now let us turn to the boundedness of oscillatory singular integral operators defined by

$$(3.4) \quad Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} H(x - y) f(y) dy,$$

where $P(x, y)$ is a real polynomial on $\mathbb{R}^n \times \mathbb{R}^n$, and $H(x) = \Omega(x)|x|^{-n}$ for all $|x| \neq 0$, and Ω is homogeneous of degree zero. If H is a standard Calderón-Zygmund operator, the operator T in (3.4) was introduced by Ricci-Stein in [20]. Jiang and Lu in [13] proved that if $\Omega \in L \log^+ L(S^{n-1})$ and Ω has mean value 0 on S^{n-1} , then T in (3.4) is bounded on $L^q(\mathbb{R}^n)$ for $q \in (1, \infty)$. The following corollary is a simple application of Theorem 2.2.

COROLLARY 3.6. *Let $0 < p \leq \infty$, $1 < q < \infty$ and $-1/q < \alpha < 1 - 1/q$. If $H(x) = \Omega(x)|x|^{-n}$, $\Omega \in L \log^+ L(S^{n-1})$ and Ω has mean value 0 on S^{n-1} , then T in (3.4) is bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ with the operator norm independent of the coefficients of P .*

Chanillo and Christ in [1] showed that if H is a standard Calderón-Zygmund kernel, then the operator T in (3.4) is of weak type $(1, 1)$ with the operator norm independent of the coefficients of P . Combining this with Theorem 2.6, we have

COROLLARY 3.7. *Let $0 < p \leq \infty$ and $-n < \alpha < 0$. If H is a standard Calderón-Zygmund kernel, then the operator T in (3.4) is bounded from $\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$ into $W\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$ with the operator norm independent of the coefficients of P .*

Pan in [19] also studied the uniform weak $(1, 1)$ -boundedness of the following oscillatory singular integral operators defined by

$$(3.5) \quad T_\lambda f(x) = \text{p. v.} \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} H(x-y)\varphi(x,y)f(y) dy,$$

where $\lambda \in \mathbb{R}$, $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, Φ is a real-valued smooth function on $\mathbb{R}^n \times \mathbb{R}^n$ and H is a Calderón-Zygmund kernel. By Pan's Theorem 4 in [19] and Theorem 2.6, we obtain

COROLLARY 3.8. *Let $0 < p \leq \infty$, $-n < \alpha < 0$ and H be a Calderón-Zygmund kernel. If $\Phi(x, y) = \phi(x - y)$ with $\phi \in C^\infty(\mathbb{R}^n)$ and there exists an $\alpha \in \mathbb{N}^n$ with $|\alpha| > 1$ such that $\partial^\alpha \phi(0)/\partial x^\alpha \neq 0$, then the operator T_λ in (3.5) is uniformly bounded in $\lambda \in \mathbb{R}$ from $\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$ into $W\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$.*

Before we end this section, let us remark that more applications can be given to the (fractional) Hardy-Littlewood maximal operators related to T_1 in (3.1) and T^l in (3.3), the two-dimensional Marcinkiewicz integrals and the commutators of the oscillatory singular integral operators with $\text{BMO}(\mathbb{R}^n)$ functions; see [10, 21] and [4]. We omit the details.

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Department of Applied Mathematics
Institute of Information Engineering
Box 1001-47, Zhengzhou 450002
P. R. China

Department of Mathematics
Beijing Normal University
Beijing 100875
P. R. China
e-mail: lusz@bnu.edu.cn
e-mail: dcyang@bnu.edu.cn