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Representing N – semigroups

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An N-semigroup is a commutative, cancellative, archimedean semigroup with no idempotent element. This paper obtains a representation of finitely generated N-semigroups as the subdirect product of an abelian group and a subsemigroup of the additive positive integers.

1. Introduction

The term N-semigroup was first used by Petrich in [3] to name a commutative, cancellative, nonpotent, archimedean semigroup. T. Tamura [5] characterized N-semigroups as the direct product of the nonnegative integers and an abelian group G, with the operation:

$$(n, g) \cdot (m, h) = (n + m + I(g, h), gh),$$

where n, m are nonnegative integers and $g, h \in G$. $I(g, h)$ is a non-negative integer-valued function, (called an index function), defined on $G \times G$ and satisfying the following four conditions for all $g, h, k \in G$:

(i) $I(g, h) = I(h, g)$,

(ii) $I(g, h) + I(gh, k) = I(g, hk) + I(h, k)$,

(iii) for any $g \in G$ there is a positive integer m, depending on g, such that $I(g^m, g) > 0$,

(iv) $I(e, e) = 1$, where e is the identity of G.

In [3] Petrich obtained a characterization of N-semigroups with two generators in terms of pairs of non-negative integers with a certain operation.

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In this paper, a representation of finitely generated N-semigroups in terms of a subdirect product of a finite abelian group and a subsemigroup of the additive positive integers is given. This representation is essentially different from that obtained by Tamura in [5]. A mapping is introduced from a finitely generated N-semigroup S into the additive positive integers, called an $\underline{\underline{I}}$ function, which mapping is a homomorphism.

I have been informed that Mr Sasaki has obtained an as yet unpublished result which extends my main representation theorem to power joined N-semigroup. The results of this paper constitute a portion of my dissertation for the Ph.D. degree in mathematics from the University of California at Davis under the direction of Professor T. Tamura. I would also like to express my most sincere appreciation to the referee of this paper for his many valuable suggestions.

2. Preliminaries

In what follows S will stand for an N-semigroup. For $a \in S$ we define a relation on S, called $\sim_a$, by:

$$\text{if } x, y \in S \text{ then } x \sim_a y \text{ iff } x = a^n y \text{ or } y = a^m x \text{ or } y = x ,$$

(m, n are positive integers).

(Note: it is convenient to define $x = a^0 x$ where we use the convention that a^0 is the empty symbol.) It is shown in [5] that $\sim_a$ is a congruence on S and that S^*_a, the homomorphic image of S under the homomorphism implied by $\sim_a$, is an abelian group. S^*_a is called the structure group of S with respect to a. We may also use a to obtain a partial ordering of S, called $<_a$, and defined by:

$$\text{for } x, y \in S, \ x <_a y \text{ iff } y = a^n x , \ (n \text{ a positive integer}).$$

It is also shown in [5] that $<_a$ on S satisfies the ascending chain condition and that every congruence class of S under $\sim_a$ contains one and only one element maximal with respect to the $<_a$ ordering. This allows us to associate in a rather natural way the elements of S^*_a with the elements of S which are maximal in the $<_a$ ordering. Elements maximal in the $<_a$ ordering, hereafter called $<_a$-maximal elements, are said to be *prime* to a ; a is called the *standard element* for determining S^*_a .

We denote by (x) the congruence class of S under $\sim_a$ which has x as its maximal element. We then define:

$$I((x)\ ,\ (y)) = n\ , \text{ where } xy = a^n z \text{ and } z \text{ is prime to } a\ .$$

It is shown in [5] that the function $I((x)\ ,\ (y))$ thus defined on the a-maximal elements of S , and thus by extension on the elements of S^*_a , satisfies properties (i) through (iv) of the Introduction and is an index function. Thus, we may represent S as outlined in the Introduction, where the group G is S^*_a and the index function is $I((x)\ ,\ (y))$.

The following Lemma is essential.

LEMMA 2.1 *If an N-semigroup S is finitely generated then every structure group of S , S^*_a , has finite order.*

Proof. Let $b_1 , \ldots , b_n$ be a generating set for S . For any $a \in S$ we have:

$$a = b_1^{k_1} \ . \ . \ . \ b_n^{k_n}\ .$$

In [3] p. 149 it is shown that for any pair of elements of a finitely generated N-semigroup, say $x , y \in S$ there are positive integers m , p such that $x^m = y^p$. (Note: a semigroup satisfying such property is called power joined.) Thus for any b_i we have m_i and p_i such that $a^{m_i} = b^{p_i}$. Thus, $c = b_1^{j_1} \ldots b_n^{j_n}$ could be prime to a only if $j_i < p_i$ for $i = 1,2,\ldots,n$. Clearly the number of such c is finite.

Using Lemma 2.1 we may now define a mapping $\underline{I}$ from S to the positive integers by:

$$\text{for } a \in S\ ,\ \underline{I}(a) = |S^*_a|\ ,$$

where $|S^*_a|$ denotes the order of the group S^*_a. We then obtain:

LEMMA 2.2 *Let a finitely generated* S *be represented by some structure group* S^*_a *and its associated* $\underline{\underline{I}}$*-function. Then, for* $x \in S$, *where* $x = (n, g)$ *in terms of this representation,*

$$\underline{\underline{I}}(x) = n\,|S^*_a| + \underline{\underline{I}}\,((0, g)) .$$

Proof. If $y = (m,h)$ in terms of this representation and $m < n$ then y is prime to x since $y = (n,g)\cdot(m,h') = (n + m' + I(g,h'), gh')$ but $I(g, h') \geqq 0$ and $n + m' + I(g,h') \leqq m$ is clearly impossible for $n, m' \geqq 0$ and $m < n$. There are $n\,|S^*_a|$ elements of this type. If $y = (n, h)$ then $y = (n, g)\cdot(m', g')$ if and only if $n + m' + I(g, g') = n$, which implies $m' = I(g, g') = 0$, and $gg' = h$. Thus $y = (n, h)$ is prime to x if and only if $I(g, g^{-1}h) > 0$. But, if $I(g, g^{-1}h) > 0$ then $(0, h) \nmid (0, g)$, $(0, g^{-1}h) = (I(g, g^{-1}h), h)$ and $(0, h)$ is prime to $(0, g)$. This shows that the number of $y = (n, h)$ prime to x is at least as great as $\underline{\underline{I}}((0, g))$. But if $(0, h)$ is not prime to $(0, g)$ then $y = (n, h)$ is not prime to x and the number of such y is exactly $\underline{\underline{I}}((0, g))$.

For finitely generated S, $x \in S$ is called a normal standard element if $\underline{\underline{I}}(x)$ is minimal.

3. Subsemigroups of the additive positive integers

In this section J represents the additive positive integers. Clearly J is an N-semigroup. Portions of the following may be found in [4] and [7].

LEMMA 3.1 *Let* L *be the subsemigroup of* J *generated by the integers* $\{a_1, a_2, \ldots, a_j\}$, $j > 1$. *If all the* a_i *have no common divisor then* L *contains all integers greater than some fixed positive integer* k.

Proof. (I am indebted to the referee for the following proof.) Let $k = 2a_1a_2 \ldots a_j$. Since $\{a_1, a_2, \ldots, a_j\}$ has no common divisor, for $b > k$ we may find integers $x_1, x_2, \ldots, x_j$ such that

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$x_1a_1 + \ldots + x_ja_j = b$. We may now find integers q_i and r_i such that $x_i = q_ia_1 \ldots a_{i-1}\, a_{i+1} \ldots a_j + r_i$ where $0 < r_i \leqq a_1 \ldots a_{i-1}a_{i+1} \ldots a_j$ $(i = 2, 3, \ldots, j)$. Now put $y_1 = x_1 + (q_2 + \ldots + q_j)\, a_2a_3 \ldots a_j$, $y_i = r_i$, $(i = 2, 3, \ldots, j)$. We now have $b = y_1a_1 + y_2a_2 + \ldots + y_ja_j$. We have chosen $y_i > 0$ for $i = 2, 3, \ldots, j$. But since $y_2a_2 + \ldots + y_ja_j = r_2a_2 + \ldots + r_ja_j \leqq a_1a_2 \ldots a_j < b$, clearly $r_1 > 0$.

COROLLARY 3.1.1 *Every subsemigroup of* J *is finitely generated.*

Proof. Let L be a subsemigroup of J . If all of L has no common divisor then L contains all integers greater than some integer k . Then $L \cap \{1, 2, \ldots, 2k\}$ generates L , since for $m > 2k$ we have $m = qk + r$, but $q \geqq 2$, and $m = (q-1)k + (k+r)$ but k , $k + r \in \{L \quad \{1,2,\ldots,2k\}\}$. The case where all L have a common divisor is easily reduced to the case above.

It is clear from the proof of Corollary 2.1.1 that there are two types of subsemigroups of J . Those which contain all integers greater than some fixed integer will be designated *relatively prime semigroups*.

Let K , L be subsemigroups of J . We then have:

THEOREM 3.2 *A homomorphism of* K *into* L *is an isomorphism of the type:* $a \in K$ *is mapped onto* $r \cdot a \in L$ *where* r *is a fixed rational number which depends on* K *and* L .

Proof. From Corollary 2.1.1 both K and L are finitely generated. Let $\{a_1 , a_2 , \ldots , a_j\}$ be the generators of K . Let $\{b_1 , b_2 , \ldots , b_j\}$ be the images of the a_i in L under the homomorphism. If we apply the homomorphism to $a_ia_1 = a_1a_i$ we have $a_ib_1 = a_1b_i$ and $b_i = (b_1/a_1)a_i$.

Clearly, given a generating set $\{a_1 , a_2 , \ldots , a_j\}$, not any rational number $r = q/p$ defines a homomorphism on the $\{a_i\}$. Indeed, $(a_iq)/p$ must be an integer and since p , q may be chosen relatively prime p must divide a_i . But a mapping of this type is just a mapping:

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$$b_i \to n\, b_i\ ,$$

where b_i is a generating element of a relatively prime subsemigroup of J . Thus, we have obtained:

THEOREM 3.3 *For* K , L , *subsemigroups of* J , *if* L *is a homomorphic image of* K *then both* K *and* L *are integral multiples of some subsemigroup* K' *of* J , *where* K' *is a relatively prime subsemigroup.*

THEOREM 3.4 *In a subsemigroup of* J *the congruence* $\sim_a$ *as defined in 2, is just the congruence modulo* (a) *as usually defined for integers.*

Proof. By definition $x \sim_a y$ iff $y = a^m x$ or $x = a^n y$. But for subsemigroups of J this is just the condition $x \equiv y (\text{mod}\ a)$.

COROLLARY 3.4.1 *In a subsemigroup of* K , *say* L , *there is a unique normal standard element. This element is the least integer in the subsemigroup.*

Proof. If L is a relatively prime subsemigroup, the order of L^*_n is the number of congruence classes of L modulo (n), but L contains all integers greater than some fixed integer k and thus $|L^*_n| = n$. If L is not relatively prime, factor out the greatest common divisor of the elements of L , say j , and proceed as above. Clearly, the elements $0, 1, 2, \ldots, n-1$ are prime to n and also $0, j, 2j, \ldots, (n-1)j$ and only these are prime to nj .

4. The $\underline{\underline{I}}$-function homomorphism

As defined in Section 2 the $\underline{\underline{I}}$-function is a mapping from any finitely generated N-semigroup into the additive positive integers. We now show:

THEOREM 4.1 *Let* S *be a finitely generated* N*-semigroup. Then the* $\underline{\underline{I}}$*-function on* S *is a homomorphism from* S *into the additive positive integers.*

Proof. Take a representation for S in terms of some structure group S^*_a and its associated $\underline{\underline{I}}$-function. Let $(m\ , g)$ and $(n\ , h)$ be two elements of S thus represented. From the definition of the $\underline{\underline{I}}$-function we have:

(1) $$\underline{\underline{I}}((m, g)(n, h)) = \underline{\underline{I}}((m + n + I(g, h), gh)) = (m + n + I(g, h)) |S^*_a| + \underline{\underline{I}}((0, gh)) .$$

From property (ii) of I-functions and summing over S^*_a we have:

$$\sum I(g, h) + \sum I(gh, i) = \sum I(g, hi) + \sum I(h, i) ,$$

as i ranges over S^*_a .

Since S^*_a is a finite group, hi ranges over all S^*_a as i does; using this fact and Lemma 2.3 we may write the above as

$$I(g, h) |S^*_a| + I(0, gh) = \underline{\underline{I}}((0, g)) + \underline{\underline{I}}((0, h)) .$$

Substituting the above in (1) we have:

$$\underline{\underline{I}}((m, g)(n, h)) = m + n + \underline{\underline{I}}((0, g)) + \underline{\underline{I}}((0, h)) .$$

We then use Lemma 2.2 to obtain:

$$\underline{\underline{I}}((m, g)(n, h)) = \underline{\underline{I}}((m, g)) + \underline{\underline{I}}((n, h)) .$$

We next define what is meant by a semigroup having a greatest homomorphic image of type Γ . Let Ξ be a set of implications. Let Γ be the class of all semigroups satisfying all implications in Ξ . Then a semigroup T has a greatest homomorphic image of type Γ if:

(i) there is a homomorphism α from T onto $T_0 \in \Gamma$,

(ii) if β is a homomorphism from T onto $T_1 \in \Gamma$.

then there is a γ from T_0 to T_1 such that $\beta = \alpha\gamma$. The following is found in [6].

THEOREM 4.2 *Every semigroup,* T *, has a greatest homomorphic image of type* Γ .

A semigroup, T , is said to be *power cancellative* if for any $a, b \in T$, when $a^n = b^n$ then we have $a = b$. The following is found in [2] .

THEOREM 4.3 *Any power joined, power cancellative N-semigroup containing at least two elements can be embedded in the additive positive rationals.*

We now obtain

THEOREM 4.4 *Let* S *be a finitely generated N-semigroup. Then, there is a unique subsemigroup of the additive positive integers,* K_s *, such that* K_s *is a relatively prime subsemigroup and* K_s *is a homomorphic image of* S *.* K_s *is isomorphic to the* $\underline{\underline{I}}$*-function homomorphic image of* S *.*

Proof. The condition "power cancellative" is given by the set of implications:

$$(n) \quad \{a \, , \, b \in S \, , \quad a^n = b^n \to a = b\} \, .$$

Thus, by Theorem 4.2, S has a greatest power cancellative homomorphic image. It has been previously noted that all S are power joined and this condition is clearly preserved by homomorphisms. The property of being finitely generated is also preserved by homomorphisms. Thus S has a greatest power joined, power cancellative homomorphic image, T . This image is clearly finitely generated. From Theorem 4.3 T is isomorphic to a finitely generated subsemigroup of the additive positive rationals if T contains two or more elements. The $\underline{\underline{I}}$-function provides a power joined, power cancellative homomorphic image of S , say K'_s by Theorem 4.1. Thus, K'_s is a homomorphic image of T . But K'_s contains an infinite number of elements and thus T is a finitely generated subsemigroup of the additive positive rationals. Clearly any such semigroup is isomorphic to a subsemigroup of the positive integers under addition. From Theorem 3.3 we thus conclude that T and K'_s are isomorphic. Also from Theorem 3.3 we may find K_s isomorphic to T and K'_s such that K_s is a relatively prime subsemigroup. The uniqueness of K_s is guaranteed by Theorem 3.2 and 3.3.

LEMMA 4.5 *Let* S *be a finitely generated N-semigroup. Let* G *be a group homomorphic image of* S *, under the mapping* α *. Then* G *is the homorphic image of some structure group,* S^*_a *, of* S *.*

Proof. Let the set S_e be the pre-image of the identity of G under α . Since S_e is not empty select $a \in S_e$. Consider the relation $\sim_a$ as defined in the introduction, and the associated structure group S^*_a . Since

$a^n \in S_e$, if for $x , y \in S$ we have $x \sim_a y$ then either $x = a^n y$ or $y = a^m x$ and $(x)\alpha = (y)\alpha$. Thus, if for $(x) \in S^*_a$, where x is prime to a , we define $((x))a^* = (x)\alpha$, the mapping a^* is clearly a homomorphism from S^*_a onto G .

5. Subdirect products

We now use the results of the previous sections to obtain a new representation for finitely generated N semigroups.

DEFINITION 5.1 Let R and T be semigroups. A semigroup S is a subdirect product of $R \times T$ if and only if there exist homomorphisms α , β from S onto R and T respectively such that the pre-image of $r \in R$, in S , under α ; and the pre-image of $t \in T$, in S , under β ; intersect in at most one element.

THEOREM 5.2 *Every finitely generated N-semigroup,* S *, is the subdirect product of a finite abelian group and a subsemigroup of the additive positive integers and conversely.*

Proof. As a homomorphism from S to the additive positive integers use the $\underline{\underline{I}}$-mapping. Let Q be the mapping from S to S^*_a , some structure group of S , induced by the relation $\sim_a$ which defines S^*_a . Schematically, this may be represented as:

$$\begin{array}{lcl} S & \xrightarrow{\;Q\;} & S^*_a \\ \downarrow \underline{\underline{I}} & & \\ K' \subset K & & \end{array}$$

We associate with $\underline{\underline{I}}$ the congruence $\sim_{\underline{\underline{I}}}$ which $\underline{\underline{I}}$ induces on S . Let us use S^*_a and its associated $\underline{\underline{I}}$-function to represent S . If, under this representation, (m , g) and (n , h) are two elements of S and if (m , g) and (n , h) are in the same class under $\sim_{\underline{\underline{I}}}$ we have:

$$m \; |S^*_a| + \underline{\underline{I}}((0 , g)) = n \; |S^*_a| + \underline{\underline{I}}((0 , h)) \; .$$

If (m , g) and (n , h) are in the same class under $\sim_a$ we have, from definition of $S^*_a : g = h$. Thus $m = n$ and $(m , g) = (n , h)$.

Clearly any subdirect product of $G \times K$ where G is an abelian group and K a subsemigroup of the additive positive integers is an N-semigroup.

The following example shows that in some instances the representation outlined in Theorem 5.2 is properly a subdirect product. Let S^*_a be the cyclic group of order three with the following I-function:

	e	g	g^2
e	1	1	1
g	1	0	4
g^2	1	4	5

This N-semigroup is generated by $(0, e)$ and $(0, g)$, (i.e., $(0, g^2) = (0, g)(0, g) = 0 + 0 + I(g, g), g^2 = (0 + 0 + 0, a^2))$. $\underline{\underline{I}}((0, e)) = 3$, $\underline{\underline{I}}((0, a)) = 5$ and the image of this N-semigroup under the $\underline{\underline{I}}$-mapping is the sub-semigroup of the additive positive integers generated by 3 and 5. The intersection of the pre-image of 3 and pre-image of 5 is empty. We then obtain:

THEOREM 5.3 *A finitely generated N-semigroup S is the direct product of a subsemigroup of the positive integers and a structure group S^*_a if, using the representation for S given by S^*_a and its I-function, every element of the form $(0, g)$ is a normal standard element.*

Proof. Consider the pre-image of any $\underline{\underline{I}}$-class, say all (m, g) such that $\underline{\underline{I}}((m, g)) = n$. For $\underline{\underline{I}}$-mappings we have:

$$\underline{\underline{I}}((m, g)) = m\,|S^*_a| + \underline{\underline{I}}((0, g)) .$$

But $\underline{\underline{I}}((0, g))$ is the same for all $g \in S^*_a$. This $\underline{\underline{I}}$-class intersects the pre-image of any $h \in S^*_a$ in the element (m, h). Thus, $S = K_s \times S^*_a$.

The question of which other classes of N-semigroups may be represented as the direct product of an abelian group and a subsemigroup of the additive integers remains open.

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References

[1] John Clayborn Higgins, *Finitely generated commutative archimedean semigroups with idempotent,* doctoral dissertation, University of California at Davis, unpublished (1966).

[2] T. Tamura and R. Levin, "On locally cyclic semigroups", *Proc. Japan Acad.* 42 (1966), 377-379.

[3] Mario Petrich, "On the structure of a class of commutative semigroups", *Czechoslovak Math. J.,* 14 (1964), 147-153.

[4] L. Rédei, *Theorie der endlich erzeugbaren kommutativen Halbgruppen,* (Physica - Verlag, Würzburg, 1963).

[5] T. Tamura, "Commutative nonpotent archimedean semigroup with cancellation law 1", *J. Gakugei, Tokushima Univ.* 8 (1957), 5-11.

[6] T. Tamura, "The theory of operations on binary relations", *Trans. Amer. Math. Soc.* 120 (1965) 343-358.

[7] T. Tamura and T. Tetsyua, "On the base of the infinite quasipower semigroup" (Japanese) *Shikokie Shijo. Danwa* 13 (July 1963) 295-298.

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