Can. J. Math., Vol. XXXV, No. 2, 1983, pp. 274–299

UNITARILY INVARIANT OPERATOR NORMS

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1. Introduction.

1.1. Over the past 15 years there has grown up quite an extensive theory of operator norms related to the numerical radius

(1) $w(T) = \sup\{|(Th, h)| : \|h\| = 1\}$

of a Hilbert space operator T. Among the many interesting developments, we may mention:

(a) C. Berger's proof of the "power inequality"

(2) $w(T^n) \leqq (w(T))^n \quad (n = 1, 2, \ldots);$

(b) R. Bouldin's result that

(3) $w(VT) \leqq w(T)$

for any isometry V commuting with T;

(c) the unification by B. Sz.-Nagy and C. Foias, in their theory of ρ-dilations, of the Berger dilation for T with $w(T) \leqq 1$ and the earlier theory of strong unitary dilations (Nagy-dilations) for norm contractions;

(d) the result by T. Ando and K. Nishio that the operator radii $w_\rho(T)$ corresponding to the ρ-dilations of (c) are log-convex functions of ρ.

The following bibliographic notes will assist the reader who wishes more information on items (a–d). Berger's first announcement of the power inequality (originally a conjecture of P. R. Halmos) is in [**5**]; see also the papers [**6, 7**] by Berger and J. G. Stampfli, and the nice discussion of C. Pearcy [**15**]. Bouldin's theorem about isometries is in [**10**]. Sz.-Nagy and Foias provide in [**17**, Section 11 of Chapter I] a convenient account of their theory of ρ-dilations and the corresponding classes of operators; in [**18**], J. P. Williams also introduced such operator classes. The results on log-convexity of $w_\rho(T)$ are in [**3**].

Our main purpose in this paper is to formulate certain "structural" properties of such operator norms that will determine a general class of norms for which analogues of the results (a), (b), and (d), along with

Received November 4, 1981 and in revised form February 3, 1982. This work was supported in part by NSERC of Canada under operating grant A8745.

many others, can be established. In the course of this work we also obtained some more specialized results relating to the numerical radius itself (in the algebra of Hilbert space operators and in more general Banach algebras); these are described below in Sections 3 and 6.

In this context it is natural to ask what structural properties characterize the norms w_ρ. For example, when J. P. Williams, in [**18**], introduced functions equivalent to the w_ρ ($\rho \leqq 2$), he noted that they are "Schwarz norms":

(4) $\quad w_\rho(T) \leqq 1 \Rightarrow w_\rho(f(T)) \leqq w_\rho(T)$

for every holomorphic f mapping the unit disc into itself and such that $f(0) = 0$, and he wondered whether there are other Schwarz norms. There are (for example, $v(T) = \|C^{-1}T\|$, for any fixed invertible contraction C, defines a Schwarz norm), but it does not seem clear what additional properties are sufficient to single out the operator radii w_ρ. Here we emphasize other structural properties such as "unitary invariance":

(5) $\quad w_\rho(U^*TU) = w_\rho(T) \quad (U \text{ unitary}).$

In his classic study [**16**], R. Schatten has developed a theory of operator norms that may be seen as similar in spirit to the present work, though Schatten is able to characterize the norms in his class quite satisfactorily. His basic assumption is a "unitary invariance" that is much stronger than (5): he considers norms u such that

(6) $\quad u(VTU) = u(T) \quad (V, U \text{ unitary}).$

1.2. We shall need to refer now and then to properties of the operator radii $w_\rho(\cdot)$, so we collect some of the more basic of these properties here, along with some general remarks on notation and motivation. For those properties stated without further comment, proofs may be found in one or more of the following: [**17**, I, 11], [**12**] or [**2**].

All normed spaces in this paper have the complex numbers **C** as scalars, and $\mathscr{B}(H)$ denotes the algebra of all (bounded, linear) operators on a Hilbert space H. Given $T \in \mathscr{B}(H)$, we say that T belongs to the class $\mathscr{C}_\rho(\rho \in (0, \infty))$ if there is a unitary operator $U \in \mathscr{B}(K)$, where K contains H as a subspace, such that

(7) $\quad T^n = \rho P_H U^n|H \quad (n = 1, 2, \ldots).$

Such a U is called a ρ-dilation of T. The "operator radii" $w_\rho : \mathscr{B}(H) \to [0, \infty)$ are defined by homogeneity and the condition

(8) $\quad w_\rho(T) \leqq 1 \Leftrightarrow T \in \mathscr{C}_\rho.$

It turns out that $w_1(T) = \|T\|$, $w_2(T) = w(T)$, and $\lim_{\rho\to\infty} w_\rho(T)$ (= "$w_\infty(T)$") is the spectral radius $r(T)$. For fixed T, $w_\rho(T)$ is a con-

https://doi.org/10.4153/CJM-1983-015-3 Published online by Cambridge University Press

tinuous nonincreasing function of ρ. Each w_ρ, for $\rho \leqq 2$, is a norm on $\mathscr{B}(H)$ and $w_\rho(I) = 1$ when $\rho \geqq 1$. In this paper we shall be mainly interested in the case $\rho \in [1, 2]$.

The statements of (8) may be characterized intrinsically:

(9) $w_2(T) \leqq 1 \Leftrightarrow |(Th, h)| \leqq 1 \quad (\|h\| = 1),$

and, for $\rho \in [1, 2)$,

(10) $w_\rho(T) \leqq 1 \Leftrightarrow \|(2 - \rho)zT + (\rho - 1)I\| \leqq 1 \quad (|z| \leqq 1)$

(see, for example, [**17**, I, 11, Remark 2]).

From (7) (with $n = 1$), it is apparent that $T \in \mathscr{C}_\rho \Rightarrow \|T\| \leqq \rho$, so that

(11) $\|T\| \leqq \rho w_\rho(T).$

On the other hand, if $T^2 = 0$ any 1-dilation for T is also a ρ-dilation for ρT so that

(12) $T^2 = 0 \Rightarrow w_\rho(T) = \|T\|/\rho.$

It is clear from (9) and (10) that

$$w_\rho(T) \leqq 1 \Rightarrow w_\rho(U^*TU) \leqq 1$$

for any unitary U, and the "unitary invariance" property (5) follows; indeed, it is clear that we have a more general statement: for $T \in \mathscr{B}(H_1)$

(13) $w_\rho(U^*TU) = w_\rho(T) \quad (U \text{ a unitary map from } H_2 \text{ onto } H_1).$

In what follows we shall also lay stress on the behavior of w_ρ with respect to orthogonal sums:

(14) $w_\rho(T_1 \oplus T_2) = \max\,(w_\rho(T_1), w_\rho(T_2))$

where $T_k \in \mathscr{B}(H_k)$ $(k = 1, 2)$; to see this we may note, for example that if $T_1, T_2 \in \mathscr{C}_\rho$ and U_1, U_2 are the corresponding ρ-dilations then $U_1 \oplus U_2$ provides a ρ-dilation for $T_1 \oplus T_2$ so that $T_1 \oplus T_2 \in \mathscr{C}_\rho$ also.

Some of the most interesting questions in this area ask about the action of norms such as w_ρ on products of operators. It has long seemed a reasonable conjecture, for example, that

(15) $w(ST) \leqq w(T)\,\|S\|$

whenever S and T commute (cf. (3) above). As far as we know, the best result in this direction is that presented in [**4**] by Ando and K. Okubo (who attribute important elements in their argument to M. J. Crabb):

(16) $w(ST) \leqq \frac{1}{2}(2 + 2\sqrt{3})^{1/2} w(T)\,\|S\| \quad (\leqq (1.169)w(T)\,\|S\|),$

whenever S and T commute.

2. A general class of operator norms.

2.1. In this section u represents a family of norms $\{u_H\}$, one for each separable Hilbert space under consideration. Each u_H is required to be a norm on $\mathscr{B}(H)$; for $T \in \mathscr{B}(H)$ we normally write simply $u(T)$ in place of the more correct $u_H(T)$. We shall always assume the normalization

(17) $u(I)\ (=u_H(I_H)) = 1$

where I_H is the identity operator on H. More essential to our concerns is the assumption that u is "unitarily invariant" in the following sense:

(18) $u(U^*TU) = u(T)$,

whenever $T \in \mathscr{B}(H_1)$ and U is a unitary operator from another Hilbert space H_2 onto H_1. In addition, we shall assume that the various u_H are linked together by the following requirements on orthogonal sums. If $A \in \mathscr{B}(H_1)$ and $0, B \in \mathscr{B}(H_2)$ (here 0 denotes the null operator), then

(19) $u(A \oplus 0) = u(A)$,

(20) $u(A \oplus A) = u(A)$,

and

(21) $u(A \oplus zB) = u(A \oplus B) \quad (z \in \mathbf{C}, |z| = 1)$.

2.2. The class of all norms satisfying the axioms of 2.1 is clearly convex, and from the facts collected in 1.2 it follows that the norms $u = w_\rho$ $(1 \leqq \rho \leqq 2)$ are included. In fact (see (14)), when $u = w_\rho$ a stronger relation holds for orthogonal sums:

(22) $u(A \oplus B) = \max(u(A), u(B))$.

One half of Proposition 1, below, shows that the norms of 2.1 satisfy (22) with an inequality, while Proposition 2 shows that norms satisfying (22) itself are rather special within the class: they are extreme points.

PROPOSITION 1. *If the Hilbert space H is decomposed as an orthogonal sum $H = H_1 \oplus H_2$ and $\begin{bmatrix} A & C \\ D & B \end{bmatrix}$ is the corresponding block representation of $T \in \mathscr{B}(H)$, then*

$$u(T) \geqq u(A \oplus B) \geqq \max(u(A), u(B)).$$

Proof. If U is the unitary operator $I_{H_1} \oplus - I_{H_2}$ we have

$$U^*TU = \begin{bmatrix} A & -C \\ -D & B \end{bmatrix}$$

https://doi.org/10.4153/CJM-1983-015-3 Published online by Cambridge University Press

so that, by (18),

$$2u(T) = u(T) + u(U^*TU) \geqq u(T + U^*TU) = u(2(A \oplus B)).$$

On the other hand, (21) ensures that $u(A \oplus (-B)) = u(A \oplus B)$ and so

$$2u(A \oplus B) \geqq u(2A \oplus 0) = 2u(A)$$

(recalling (19)). Finally, (18) and the obvious unitary map between $H_1 \oplus H_2$ and $H_2 \oplus H_1$ make it clear that $u(B \oplus A) = u(A \oplus B)$, so that $u(A \oplus B) \geqq u(B)$ also.

PROPOSITION 2. *If the norm u satisfies* (17), (18), *and* (22) *it cannot be written as a nontrivial convex combination*

$$(23) \quad u = (1 - t)u_0 + tu_1 \quad (0 < t < 1)$$

of distinct norms u_0, u_1 satisfying the axioms of 2.1.

Proof. Suppose that $u(A) \geqq u(B)$. Using Proposition 1 at several points, we note that (23) implies

$$\begin{aligned} u(A) &= u(A \oplus B) = (1 - t)u_0(A \oplus B) + tu_1(A \oplus B) \\ &\geqq (1 - t)u_0(A \oplus B) + tu_1(A) \geqq (1 - t)u_0(A) + tu_1(A) = u(A). \end{aligned}$$

Hence $u_0(A) = u_0(A \oplus B)$ $(\geqq u_0(B))$ under these conditions, i.e.,

$$u(A) \geqq u(B) \Rightarrow u_0(A) \geqq u_0(B).$$

Recalling (17), we see that $u_0(T) = 1$ wherever $u(T) = 1$, i.e., $u = u_0$. Clearly $u = u_1$ also so that $u_0 = u_1$.

In Section 5 we shall explore further the class of norms satisfying (22); here we continue with results that follow from the axioms of 2.1.

2.3. Under the hypotheses of 2.1, we next present a sequence of results that concern the behavior of u on operator products.

THEOREM 3. *For any $A, B \in \mathscr{B}(H)$,*

$$(24) \quad u(AB + B^*A) \leqq 2u(A)\|B\|.$$

Proof. If $V \in \mathscr{B}(H)$ is unitary it is easy to verify that

$$U = \tfrac{1}{2}\begin{bmatrix} I + V & I - V \\ I - V & I + V \end{bmatrix}$$

defines a unitary operator on $H \oplus H$, and that

$$U^*(A \oplus (-A))U = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

where $W = \frac{1}{2}(AV + V^*A)$. Recalling Proposition 1 and properties (18), (21), and (20) we have

$$u(W) \leqq u(U^*(A \oplus (-A))U) = u(A \oplus (-A)) = u(A),$$

so that $u(AV + V^*A) \leqq 2\,u(A)$ when V is unitary.

By homogeneity, (24) is equivalent to the statement that

(25) $u(AB + B^*A) \leqq 2u(A)$

whenever $\|B\| < 1$. By a theorem of T. W. Palmer (see Lemma 4, below) such a B may be written as a convex combination $\sum_1^n t_k V_k$ where each V_k is unitary. Hence (25) follows from the unitary case simply through the norm properties of u. Reduction to the unitary case can also be accomplished, without Palmer's theorem, by introducing a unitary dilation of B.

We shall refer to Palmer's result again on several occasions, so we state it below as a lemma. The original proof may be found in [**14**], and an interesting alternative approach is given in [**9**] (see §30). We remark that Palmer's result illuminates a number of phenomena in the general area of this paper: for example, Schatten's theorem (see [**16**]) that a norm u satisfying (6) also satisfies the inequality

(26) $u(ATB) \leqq \|A\|u(T)\|B\|$

follows immediately from Palmer's representation of the case $\|A\|$, $\|B\| < 1$.

LEMMA 4 (Palmer). *Let $\mathscr{A}$ be any initial B^*-algebra. Then any $a \in \mathscr{A}$ such that $\|a\| < 1$ is a (finite) convex combination of unitary elements of $\mathscr{A}$ (v in $\mathscr{A}$ is called unitary if $v^*v = vv^* = 1$).*

THEOREM 5. *If $A, B \in \mathscr{B}(H)$ double commute (i.e., $AB = BA$ and $AB^* = B^*A$) then $u(AB) \leqq u(A)\,\|B\|$.*

Proof. By homogeneity we need only show that $\|B\| < 1 \Rightarrow u(AB) \leqq u(A)$. Using Lemma 4, B may be written as a convex combination of unitaries U_k in the C^*-algebra generated by B (i.e., the algebra generated by I, B, and B^*). Since A, B double commute, A will commute with each U_k so that the desired inequality follows from the norm properties of u once we establish that

(27) $u(AU) = u(A)$

whenever U is a unitary operator commuting with A. We remark that here too reduction to the unitary case can be achieved without Palmer's result by introducing the appropriate unitary dilation of B.

Turning to (27), let us first note that Theorem 3 implies that u is $\|\cdot\|$-continuous; in fact, replacing A by I in (24) and considering both

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$B = T$ and $B = iT$ we see that $u(T \pm T^*) \leqq 2\|T\|$, so that $u(2T) \leqq 4\,\|T\|$. Since we may $\|\cdot\|$-approximate U by sums of the form $\bigoplus_{k=1}^{n} z_k P_k$ where $|z_k| = 1$ and the P_k are spectral projections corresponding to U (so that $P_k A = AP_k$), (27) follows from the special case

$$u(\bigoplus_1^n z_k A_k) = u(\bigoplus_1^n A_k) \quad (\text{here } A_k = A|P_k H),$$

which may be obtained in turn by repeated applications of (21).

THEOREM 6. *For every $B \in \mathscr{B}(H)$, $w(B) \leqq u(B) \leqq \|B\|$.*

Proof. The second inequality follows directly from Theorem 5 (taking $A = I$). For the first, note that for each unit vector h in H, the block operator representation of B with respect to the decomposition $H = H_1 \oplus H_2$ where $H_1 = \text{span}\,\{h\}$ has the form

$$B = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

with $W = (Bh, h)I_{H_1}$. Hence, by Proposition 1, $|(Bh, h)| \leqq u(B)$. Recalling (1), we are done.

For each u we may define the quantity

(28) $\rho_u = \sup\,\{\|T\|/u(T) : 0 \neq T \in \mathscr{B}(H)\}$.

Our discussion in 1.2 (see (11) and (12)) shows that $\rho_u = \rho$ when $u = w_\rho$ $(1 \leqq \rho \leqq 2)$ and in particular that $\rho_w = 2$. It follows from Theorem 6 that $1 \leqq \rho_u \leqq 2$ for any u.

THEOREM 7. *Let $A, B, E \in \mathscr{B}(H)$, where E is Hermitean. Then*

(29) $u(AE \pm EA) \leqq 2u(A)w(E)$ *and*

(30) $u(AB \pm BA) \leqq 4u(A)w(B)$.

Proof. Theorem 3 yields (29) immediately upon putting successively $B = E$, $B = iE$ and recalling that, for Hermitean E, $w(E) = \|E\|$.

Any $B \in \mathscr{B}(H)$ may be expressed in terms of its "real" and "imaginary" parts: $B = E + iF$, where E and F are Hermitean. We thus obtain (30) from (29) by noting that, since $|\text{Re}(Bh, h)|, |\text{Im}(Bh, h)| \leqq |(Bh, h)|$, we have $w(E), w(F) \leqq w(B)$.

COROLLARY 8. *If A, B are commuting operators in $\mathscr{B}(H)$, then*

(31) $u(AB) \leqq 2u(A)w(B)$ *and*

(32) $u(AB) \leqq 2u(A)u(B)$.

Proof. Refer to (30) and Theorem 6.

We remark that (32), which is best possible for $u = w$, could be established in a quite elementary way for those u satisfying a power inequality (cf. [**11**, Theorem 2.11]).

3. Methods peculiar to the numerical radius in $\mathscr{B}(H)$. Corollary 8 shows that significant estimates for $u(AB)$ when A and B commute can sometimes be obtained by first considering $u(AB + BA)$ for unrelated A and B in $\mathscr{B}(H)$. However it doesn't seem clear how to carry out this program for estimates in terms of $u(A)\|B\|$. In fact, in the general context of Section 2 we seem to get no better estimate than the trivial

(33) $\quad u(AB + BA) \leqq 2\rho_u u(A)\|B\|,$

obtained from $\|AB + BA\| \leqq 2\|A\| \|B\|$, Theorem 6, and the definition of ρ_u. Since $\rho_w = 2$, this would yield simply

(34) $\quad w(AB + BA) \leqq 4w(A)\|B\|$

for arbitrary $A, B \in \mathscr{B}(H)$, and thereby

(35) $\quad w(AB) \leqq 2w(A)\|B\|$

when A and B commute. Recall that in fact we have, at worst,

(36) $\quad w(AB) \leqq 1.169w(A)\|B\|$

in the latter case (cf. (16)). In what follows we shall see that, by methods that so far seem restricted to the case $u = w$, the constant in (34) can be improved, but that (36) cannot be obtained in this way (i.e., the best constant exceeds 2×1.169).

LEMMA 9. *If $h_k \in H$ and $\|h_k\| \leqq 1$ $(k = 1, 2, \ldots, n)$, then for any $T \in \mathscr{B}(H)$*

$$|(Th_1, h_2) + (Th_2, h_3) + \ldots + (Th_{n-1}, h_n)| \leqq nw(T).$$

Proof. The left-hand side of the inequality is

$$\left|\int_0^{2\pi} (Th(\theta), h(\theta))e^{i\theta}d\theta/2\pi\right|,$$

where $h(\theta) = \sum_{k=1}^n e^{ik\theta}h_k$, and this is clearly dominated by

$$w(T)\int_0^{2\pi} \|h(\theta)\|^2 d\theta/2\pi = w(T)\sum_1^n \|h_k\|^2.$$

PROPOSITION 10. *For any $A, B \in \mathscr{B}(H)$*

$$w(AB + BA) \leqq 3w(A)\|B\|.$$

Proof. We need only observe that, if $\|B\| = 1$ and h_2 is any unit vector in H,

(37) $\quad |((AB + BA)h_2, h_2)| = |(Ah, h_2) + (Ah_2, h_3)|$

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where

$$\|h_1\| = \|Bh_2\| \leqq \|h_2\| = 1 \text{ and } \|h_3\| = \|B^*h_2\| \leqq \|h_2\| = 1.$$

Hence Lemma 9 ensures that (37) $\leqq 3w(A)$.

Remark. The constant 3 obtained in this proposition can be improved (see Theorem 11, below); we have presented it separately because it is based on a particularly simple method (Lemma 9) that could have analogues for other norms u. Moreover Lemma 9 may be of more general interest. The case $n = 2$, for example, yields immediately the familiar fact that $\|T\| \leqq 2w(T)$, and the inequality

$$w(TC + C^*T) \leqq 2w(T),$$

where C is any contraction (i.e., $\|C\| \leqq 1$), which is the case $u = w$ in Theorem 3, follows upon considering a unit vector h and the sequence

$$h_1 = Ch, h_2 = h, h_3 = Ch, h_4 = h, \ldots, h_{2m+1} = Ch,$$

then letting $m \to \infty$.

By somewhat more specialized methods we now obtain the best constant in inequalities of the type (34).

THEOREM 11. *For any* $A, B \in \mathscr{B}(H)$,

$$w(AB + BA) \leqq 2\sqrt{2}w(A)\|B\|.$$

In some cases, this is the best one can say.

Proof. First note that if $w(A) \leqq 1$ and $\|h\| = 1$ we have

$$(38) \quad \|Ah\|^2 + \|A^*h\|^2 \leqq 4.$$

This sort of inequality occurs in the work of Ando (see, e.g., [**1**, Theorem 3] and [**2**, Theorem 3.7]) who makes reference to related work of M. J. Crabb. The case (38) may be obtained by noting that

$$(39) \quad \operatorname{Re}(e^{i\theta}Ag(\theta), g(\theta)) \leqq \|g(\theta)\|^2$$

where

$$g(\theta) = \tfrac{1}{2}e^{-i\theta}A^*h + h + \tfrac{1}{2}e^{i\theta}Ah,$$

and integrating (39) over $[0, 2\pi]$ to obtain

$$\tfrac{1}{2}\|A^*h\|^2 + \tfrac{1}{2}\|Ah\|^2 \leqq \tfrac{1}{4}\|A^*h\|^2 + \|h\|^2 + \tfrac{1}{4}\|Ah\|^2.$$

From (38) it follows that $\|Ah\| + \|A^*h\| \leqq d\sqrt{2}$ so that when $w(A), \|B\|, \|h\| \leqq 1$ we have

$$|((AB + BA)h, h)| \leqq \|Bh\| \, \|A^*H\| + \|Ah\| \, \|B^*h\|$$
$$\leqq \|A^*h\| + \|Ah\| \leqq 2\sqrt{2}.$$

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It is easy to check that we have equalities for the simple case

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, B = \tfrac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \text{ (unitary)},$$

and

$$h = \tfrac{1}{2}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}.$$

It is perhaps surprising that the argument in the preceding theorem yields the best possible constant even though it ignores the geometric relationship between h, Bh, and B^*h. Theorem 12, below, takes this relationship into account and supplies improved inequalities for certain operators A. That theorem and Corollary 13, on the other hand, make it clear that only for very special A can one hope to establish $w(AB) \leqq w(A)\|B\|$ (when $AB = BA$) by first studying $w(AB + BA)$ with B unrestricted.

THEOREM 12. *For any* $A \in \mathscr{B}(H)$,

$$\sup_{\|B\|\leqq 1} w(AB + BA) = \sup_{\|h\|=1} 2\sqrt{|(Ah, h)|^2 + R^2(h)}$$

where

$$R(h) = \tfrac{1}{2}\{\|Ah - (Ah, h)h\| + \|A^*h - (A^*h, h)h\|\}.$$

Proof. It is an easy consequence of Lemma 4 that

$$\sup_{\|B\|\leqq 1} w(AB + BA) = \sup\{w(AU + UA): U \text{ unitary}\}.$$

Furthermore, if $\|u\| = 1$ and U is unitary

$$|((AU + UA)h, h)| = |(Af, h) + (Ah, g)|$$

where

$$(40) \quad \|f\| = \|h\| = \|g\| = 1 \text{ and } (f, h) = (h, g).$$

On the other hand, the condition (40) ensures that

$$\|zf + wh\| = \|zh + wg\| \quad (z, w \in \mathbf{C})$$

and hence that there exists a unitary operator $U \in \mathscr{B}(H)$ such that $f = Uh$ and $h = Ug$. Clearly then

$$\sup_{\|B\|\leqq 1} w(AB + BA) = \sup\{|(Af, h) + (Ah, g)|: (40) \text{ holds}\}.$$

The condition (40) may be expressed as follows: $\|h\| = 1, f = \alpha h \oplus \beta f_1$, $g = \bar{\alpha}h \oplus \gamma g_1$ where f_1 and g_1 are arbitrary unit vectors orthogonal to h, and $\alpha, \beta, \gamma \in \mathbf{C}$ are subject only to the restrictions

$$|\alpha|^2 + |\beta|^2 = |\alpha|^2 + |\gamma|^2 = 1.$$

Since, with this representation,

$$|(Af, h) + (Ah, g)| = |2\alpha(Ah, h) + \beta(f_1, A^*h) + \bar{\gamma}(Ah, g_1)|$$

we see that

$$\sup_{\|B\| \leqq 1} w(AB + BA) = \sup\{2t|(Ah, h)| + s(Q(A^*h) + Q(Ah)) : \|h\| = 1 = t^2 + s^2\}$$

where

$$Q(x) = \sup\{|(x, h_1)| : \|h_1\| = 1, h_1 \perp h\}.$$

Clearly $Q(x) = \|x - (x, h)h\|$, so that

$$\sup_{\|B\| \leqq 1} w(AB + BA) = \sup 2\{t|(Ah, h)| + sR(h) : t^2 + s^2 = 1 = \|h\|\}$$

and the theorem follows by a routine argument.

COROLLARY 13. *For any* $A \in \mathscr{B}(H)$ *such that* $w(A) > r(A)$,

$$\sup_{\|B\| \leqq 1} w(AB + BA) > 2w(A).$$

Proof. Let h_n be unit vectors such that $(Ah_n, h_n) \rightarrow \lambda$ and $|\lambda| = w(A)$. Since $\lambda \notin \sigma(A)$ (and $\lambda \notin \sigma(A^*)$), $R(h_n) \nrightarrow 0$ and our conclusion follows from the formula of Theorem 12.

In the following proposition we note a "noncommutative" version of Ando's argument (see [**1**]) that

$$w(AB) \leqq \sqrt{2}w(A)\|B\| \text{ when } AB = BA.$$

PROPOSITION 14. *For any* $A, B \in \mathscr{B}(H)$ *and* $h \in H$ *such that* $\|h\| = 1$,

$$\sqrt{|(ABh, h)(BAh, h)|} \leqq \sqrt{2}w(A)\|B\|.$$

Proof. Consider the operators $A \otimes B$ and $B \otimes A$ in $\mathscr{B}(H \otimes H)$. By (30), with $u = w$ and A replaced by $A \otimes B$ and B by $B \otimes A$,

$$w((AB \otimes BA) + (BA \otimes AB)) \leqq 4w(A \otimes B)w(B \otimes A).$$

But $w(A \otimes B) \leqq w(A)\|B\|$; this may be checked readily using dilation theory or seen as a special case of Proposition 17, below. Since $\|h \otimes h\| = 1$,

$$|(((AB \otimes BA) + (BA \otimes AB))(h \otimes h), (h \otimes h))| \leqq 4w^2(A)\|B\|^2,$$

that is

$$2|(ABh, h)(BAh, h)| \leqq 4w^2(A)\|B\|^2.$$

Remark. Alternatively, this result may be conveniently obtained from the inequality (38).

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4. Continuity conditions. In this section we discuss a group of results that appear to depend on a sort of "continuity" properly for the norm u:

(41) $u(A^{(\infty)}) = u(A)$,

where $A^{(\infty)}$ denotes the countable orthogonal sum $A \oplus A \oplus \ldots$. Hence, in what follows we assume (41) along with the usual axioms of 2.1. Note however that (20) now becomes redundant; in fact, if $A^{(n)}$ denotes the orthogonal sum of n copies of A, there is an obvious unitary equivalence between $(A^{(n)})^{(\infty)}$ and $A^{(\infty)}$ so that (41) and (18) imply a more general version of (41): $u(A^{(n)}) = u(A)$ for n finite or $n = \infty$.

Note that in this set-up u is actually determined by its action on any particular separable infinite-dimensional Hilbert space H, since for $A \in \mathscr{B}(H_0)$, where H_0 is finite-dimensional, the operator $A^{(\infty)}$ is unitarily equivalent to an operator on H.

It is clear that our expanded axiom system again determines a class of operator norms that is convex and includes the w_ρ $(1 \leqq \rho \leqq 2)$.

Later we shall need the observation that the continuity condition (41) can be expressed in terms of tensor products.

LEMMA 15. *If u satisfies the conditions of this section,*

$$u(A \otimes I) = u(I \otimes A) = u(A)$$

for any $A \in \mathscr{B}(H_1)$ and $I = I_{H_2} \in \mathscr{B}(H_2)$.

Proof. If n is the dimension of H_2 it is easy to see that $A \otimes I$ (or $I \otimes A$) is unitarily equivalent to $A^{(n)}$ so that we need only refer to (18) and the general version of (41) discussed above.

We have defined, by (28), the quantity ρ_u associated with a norm u, and have noted that $\rho_u = \rho$ when $u = w_\rho$. In fact, (12) shows that when $u = w_\rho$ we may compute ρ_u as $(u(T))^{-1}$ for any T such that $\|T\| = 1$ and $T^2 = 0$. The following theorem gives a related result for more general u.

THEOREM 16. *For any u satisfying the conditions of this section*

(42) $\rho_u = (u(Z))^{-1}$, *where* $Z = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \in \mathscr{B}(H \oplus H)$.

Proof. Since $\|Z\| = 1$ it is clear from the definition (28) that $\rho_u \geqq (u(Z))^{-1}$. It remains to show that

(43) $u(T) \geqq \|T\| u(Z) \quad (T \in \mathscr{B}(H))$.

Let us first argue that

(44) $u(T) \geqq u\left(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right)$.

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Now (20) and (21) ensure that $u(T) = u(T \oplus T) = u(T \oplus (-T))$, and since the operators

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} I & 0 \\ 0 & i \end{bmatrix}$$

are unitary, we have, by (18),

$$u\left(\begin{bmatrix} 0 & T \\ T & 0 \end{bmatrix}\right) = u(U^*(T \oplus (-T))U) = u(T \oplus (-T)) \quad \text{and}$$

$$u\left(\begin{bmatrix} 0 & T \\ -T & 0 \end{bmatrix}\right) = u\left(i\begin{bmatrix} 0 & T \\ -T & 0 \end{bmatrix}\right) = u\left(V^*\begin{bmatrix} 0 & T \\ T & 0 \end{bmatrix}V\right) = u\left(\begin{bmatrix} 0 & T \\ T & 0 \end{bmatrix}\right).$$

Thus

$$u(T) = u\left(\begin{bmatrix} 0 & T \\ T & 0 \end{bmatrix}\right) = u\left(\begin{bmatrix} 0 & T \\ -T & 0 \end{bmatrix}\right),$$

so that

$$2u(T) \geqq u\left(\begin{bmatrix} 0 & 2T \\ 0 & 0 \end{bmatrix}\right),$$

and we have (44).

Next we shall show that

$$(45) \quad u\left(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right) = \|T\|u(Z).$$

Note first that whenever U, V are unitary in $\mathscr{B}(H)$, $W = U \oplus V$ is unitary in $\mathscr{B}(H \oplus H)$ so that (18) implies that

$$(46) \quad u\left(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right) = u\left(W^*\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}W\right) = u\left(\begin{bmatrix} 0 & U^*TV \\ 0 & 0 \end{bmatrix}\right).$$

Using Lemma 4 and the norm properties of u in an obvious way we obtain from (46) the more general statement

$$(47) \quad u\left(\begin{bmatrix} 0 & XTY \\ 0 & 0 \end{bmatrix}\right) \leqq \|X\|u\left(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right)\|Y\| \quad (T, X, Y \in \mathscr{B}(H)).$$

Consider the polar decomposition of T: $T = X|T|$ where X is a partial isometry and $|T| = X^*T$. By (47), we have

$$u\left(\begin{bmatrix} 0 & |T| \\ 0 & 0 \end{bmatrix}\right) \leqq \|X^*\|u\left(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right) \leqq \|X^*\|\,\|X\|u\left(\begin{bmatrix} 0 & |T| \\ 0 & 0 \end{bmatrix}\right)$$

and it follows that

$$(48) \quad u\left(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right) = u\left(\begin{bmatrix} 0 & |T| \\ 0 & 0 \end{bmatrix}\right) \quad (T \in \mathscr{B}(H)).$$

Moreover, (47) shows that

$$u\left(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right) \leqq \|T\| u(Z),$$

so that in verifying (45) it remains only to show that

$$u\left(\begin{bmatrix} 0 & |T| \\ 0 & 0 \end{bmatrix}\right) \geqq \|T\| u(Z).$$

Now $|T|$ may be approximated in norm by orthogonal sums of the form $\bigoplus_1^n r_k P_k$ where each P_k is a nonzero orthogonal projection and $r_1 = \|T\| \geqq r_k$ ($k = 1, 2, \dots, n$). By the norm continuity of u (see Theorem 6, for example), we need only show that

$$u\left(\begin{bmatrix} 0 & \bigoplus_1^n r_k P_k \\ 0 & 0 \end{bmatrix}\right) \geqq r_1 u(Z).$$

Since $\|P_1\| = 1$, (47) ensures that

$$u\left(\begin{bmatrix} 0 & \bigoplus_1^n r_k P_k \\ 0 & 0 \end{bmatrix}\right) \geqq r_1 u\left(\begin{bmatrix} 0 & P_1 \\ 0 & 0 \end{bmatrix}\right),$$

and we have only to show that

$$u\left(\begin{bmatrix} 0 & P \\ 0 & 0 \end{bmatrix}\right) = u(Z)$$

for any nonzero orthogonal projection $P \in \mathscr{B}(H)$.

Because there is clearly a unitary similarity between

$$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^{(\infty)} \quad \text{and} \quad \begin{bmatrix} 0 & A^{(\infty)} \\ 0 & 0 \end{bmatrix},$$

(18) and (41) ensure that

$$u\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) = u\left(\begin{bmatrix} 0 & A^{(\infty)} \\ 0 & 0 \end{bmatrix}\right)$$

for any $A \in \mathscr{B}(H)$. In view of this we may replace P by $P^{(\infty)}$ if necessary and assume that both PH and $(I - P)H$ are infinite dimensional. In this case there is a unitary operator U from $M = (I - P)H$ onto $L = PH$. With respect to the decomposition $H = L \oplus M$ we have

$$V = \begin{bmatrix} 0 & U \\ U^* & 0 \end{bmatrix} \text{ unitary and } P = \begin{bmatrix} I_L & 0 \\ 0 & 0 \end{bmatrix}.$$

Using (46),

$$u\left(\begin{bmatrix} 0 & P \\ 0 & 0 \end{bmatrix}\right) = u\left(\begin{bmatrix} 0 & PV \\ 0 & 0 \end{bmatrix}\right)$$

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and, since $PV = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}$,

$$u\left(\begin{bmatrix} 0 & P \\ 0 & 0 \end{bmatrix}\right) = u\left(\begin{bmatrix} 0 & 0 & 0 & U \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

where the last block operator is written with respect to the decomposition $H \oplus H = L \oplus M \oplus L \oplus M$. Using (18) in connection with the unitary operator that exchanges the two copies of M we obtain

$$u\left(\begin{bmatrix} 0 & P \\ 0 & 0 \end{bmatrix}\right) = u\left(\begin{bmatrix} 0 & U & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right),$$

and (19) allows us to replace this by $u\left(\begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}\right)$.

Finally, let U_L and U_M be unitary operators from H onto L, M respectively, and consider the unitary $X = U_L \oplus U_M$ mapping $H \oplus H$ onto $L \oplus M$. Choosing U_L, U_M so that $U_L = UU_M$ and applying (18) once more we have

$$u\left(\begin{bmatrix} 0 & P \\ 0 & 0 \end{bmatrix}\right) = u\left(\begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}\right) = u\left(X^*\begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}X\right) = u(Z).$$

It only remains to remind the patient reader that (44) and (45) yield (43).

PROPOSITION 17. *For any* $A, B \in \mathscr{B}(H)$,

$$u(A)w(B) \leqq u(A \otimes B) \leqq u(A)\|B\|.$$

Proof. Given a unit vector h in H let L be the span of h and let $M = L^\perp$. With respect to the decomposition

(49) $H \otimes H = (H \otimes L) \oplus (H \otimes M)$

we have

$$A \otimes B = \begin{bmatrix} A \otimes (Bh, h)I & X \\ Y & Z \end{bmatrix},$$

so that Proposition 1 and Lemma 15 yield

(50) $u(A \otimes B) \geqq u(A \otimes (Bh, h)I) = u(A)|(Bh, h)| \quad (\|h\| = 1).$

Hence (in view of (1)) $u(A)w(B) \leqq u(A \otimes B)$.

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For the second inequality, note that the operators $A \otimes I$ and $I \otimes B$ double commute so that, by Theorem 5 and Lemma 15,

$$u(A \otimes B) \leqq u(A \otimes I)\|I \otimes B\| = u(A)\|B\|.$$

COROLLARY 18. *If B is normal,* $u(A \otimes B) = u(A)\|B\|$.

In [**1**, Theorem 4] Ando introduced an interesting technique using tensor products to prove the inequality

$$w(AB) \leqq \sqrt{2}w(A)\|B\|$$

for all commuting operators A and B. For the norm w this inequality was later improved, as we have discussed (see (16), in particular). In the theorem that follows we see that Ando's idea can be modified so that it yields the $\sqrt{2}$ result for any norm u satisfying the blanket conditions of this section and also the condition

$$(51) \quad (u(T))^2 \leqq u(T \otimes T) \quad (T \in \mathscr{B}(H)).$$

It seems unclear which u in our class satisfy (51) although it is easy to see that $u(\cdot) = \|\cdot\|$ and $u = w$ have this property and that it is inherited by convex combinations such as $u(T) = \frac{1}{2}(\|T\| + W(T))$. Before proving the $\sqrt{2}$ result we establish that the w_ρ also have this property; in the case of w_ρ, Ando and Okubo [**4**, Theorem 2] have achieved constants better than $\sqrt{2}$, using methods that seem unrelated to tensor products.

PROPOSITION 19. *For any $\rho \in [1, 2]$ and $T \in \mathscr{B}(H)$*

$$(w_\rho(T))^2 \leqq w_\rho(T \otimes T).$$

Proof. For w_2 $(=w)$ one can work directly through the relation (1). For $\rho \in [1, 2)$, we shall use the criterion (10).

By homogeneity it is enough to show that $w_\rho(T) \leqq 1$ whenever $w_\rho(T \otimes T) \leqq 1$. The latter inequality means, in view of (10), that

$$|(2 - \rho)z((T \otimes T)(u \otimes u), (v \otimes v)) + (\rho - 1)((u \otimes u), (v \otimes v))| \leqq 1$$

whenever $|z| = 1 = \|u\| = \|v\|$. Choosing z so as to maximize the expression on the left, we see that

$$(2 - \rho)|(Tu, v)|^2 + (\rho - 1)|(u, v)|^2 \leqq 1.$$

It is an elementary consequence of the Cauchy-Schwarz inequality that

$$(2 - \rho)|(Tu, v)| + (\rho - 1)|(u, v)| \leqq 1 \quad (\|u\| = \|v\| = 1).$$

Clearly, then, $\|(2 - \rho)zTu + (\rho - 1)u\| \leqq 1$ whenever $|z| = 1$ and $\|u\| = 1$, so that (10) tells us that $w_\rho(T) \leqq 1$.

THEOREM 20. *If u satisfies* (51) *in addition to the blanket assumptions of this section, then*

$$u(AB) \leqq \sqrt{2}u(A)\|B\|$$

whenever A and B commute.

Proof. Since $A \otimes B$ and $B \otimes A$ commute, Corollary 8 ensures that

$$u(AB \otimes BA) \leqq 2u(A \otimes B)u(B \otimes A).$$

Hence, by (51),

$$(u(AB))^2 \leqq 2u(A \otimes B)u(B \otimes A).$$

Since $B \otimes A$ is clearly unitarily equivalent to $A \otimes B$ we have, by (18),

$$(u(AB))^2 \leqq 2(u(A \otimes B))^2,$$

and we invoke Proposition 17 to complete the argument.

In [**10**] R. Bouldin developed methods that allowed him to prove that $w(AB) \leqq w(A)\|B\|$ provided A, B commute and B is an isometry or satisfies certain weaker, technical conditions. Subsequently Ando showed how to extend such results to other w_ρ (see, for example, Corollary 2.3 in [**2**]). In our next theorem we present a version of these ideas that applies to our axiomatic set-up. We appear to need an additional condition of "continuity" for the norm u:

(52) $\{T \in \mathscr{B}(H): u(T) \leqq 1\}$ is closed under strong limits.

This condition is evidently satisfied when $u = w_\rho$ $(1 \leqq \rho \leqq 2)$ upon consideration of (9) and (10). It is also easy to see that (52) is preserved under formation of convex combinations of norms.

THEOREM 21. *If u is a norm satisfying* (52) *in addition to the blanket conditions of this section, then*

$$u(ST) \leqq u(S)\|T\|$$

whenever S commutes with T and with T^*T. In particular, $u(ST) \leqq u(S)$ whenever T is an isometry commuting with S.

Proof. (a) Consider first the case where $T(\in \mathscr{B}(H))$ is an isometry, i.e., $T^*T = I$. By a standard construction (see e.g. [**17**, Sections I.1 and I.2]) T may be extended to a unitary operator U on a Hilbert space K containing H as a subspace and such that $\bigcup_{n=1}^{\infty} U^{-n}H$ is dense in K. One easily checks that a linear map $\hat{S}$ is consistently defined on $\bigcup_{n=1}^{\infty} U^{-n}H$ by setting

$$\hat{S}U^{-n}h = U^{-n}Sh \quad (h \in H).$$

Evidently $\|\hat{S}\| \leqq \|S\|$ so that $\hat{S}$ may be extended by continuity to all of

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K. Let $\hat{S}$ denote this extension also. It is easy to see that $\hat{S}$ extends S to K and that $\hat{S}$ commutes with U. Furthermore, if S_0 denotes the operator $S \oplus 0$ with respect to the decomposition $K = H \oplus (K \ominus H)$, we see that $\hat{S}$ is the strong limit of the sequence $\{U^{-n}S_0U^n\}$. Appealing to (52), (18), and (19), we have

$$u(\hat{S}) \leqq \sup_n u(U^{-n}S_0U^n) = u(S_0) = u(S).$$

Now Theorem 5 (or (27)) ensures that $u(\hat{S}U) = u(\hat{S})$, and since ST is the restriction of $\hat{S}U$ to H, we have $u(ST) \leqq u(\hat{S}U)$ by Proposition 1. Combining these facts, we conclude that $u(ST) \leqq u(S)$.

(b) Next, let us assume only that S commutes with T^*T (as well as with T itself). By homogeneity we may consider the case where $\|T\| \leqq 1$, and we introduce the isometric dilation V of T to $H^{(\infty)}$ defined by

$$V(h_1 \oplus h_2 \oplus h_3 \oplus \dots) = Th_1 \oplus (I - T^*T)^{1/2}h_1 \oplus h_2 \oplus h_3 \oplus \dots.$$

By our assumptions it is clear that $S^{(\infty)}$ commutes with V so that $u(S^{(\infty)}V) \leqq u(S^{(\infty)})$ by part (a) of our proof. The block form of $S^{(\infty)}V$ with respect to the decomposition $H^{(\infty)} = H \oplus (H \oplus H \oplus \dots)$ is

$$\begin{bmatrix} ST & * \\ * & * \end{bmatrix}$$

where the stars indicate irrelevant entries. By Proposition 1 and (41) we have

$$u(ST) \leqq u(S^{(\infty)}V) \leqq u(S^{(\infty)}) = u(S).$$

5. Interpolation and attenuation. In this section we wish to concentrate on the subclass $\mathcal{N}_*$ of norms determined by the axioms (17), (18), and (22). As we have seen in Proposition 2, this subclass forms part of the set of extreme points for the larger, convex class $\mathcal{N}$ of norms satisfying (19), (20) and (21) in place of (22). Two norms u_0 and u_1 in $\mathcal{N}_*$ may of course be joined by the line of their convex combinations in $\mathcal{N}$. Here we point out that by the well-known process of interpolation u_0 and u_1 may be joined by an arc lying entirely in $\mathcal{N}_*$. We shall also introduce a process we call "attenuation" of a given norm $u \in \mathcal{N}_*$; this yields a family $u^\alpha (\alpha \in [0, 1])$ of norms in $\mathcal{N}_*$ such that $u^1 = u$ and $u^\alpha(T)$ decreases as α decreases.

For the purposes of this section we fix the following notation: if u_0 and u_1 are two norms in $\mathcal{N}_*$ and $0 \leqq \alpha \leqq 1$ we denote by u_α the usual interpolated norm defined by

(53) $\quad u_\alpha(T) = \inf\{\|f\| : f \in \mathcal{F} \text{ and } f(\alpha) = T\} \quad (T \in \mathcal{B}(H)),$

where $\mathcal{F}$ is the family of all bounded holomorphic $\mathcal{B}(H)$-valued functions on the strip $\mathscr{S} = \{z \in \mathbf{C}: 0 \leqq \operatorname{Re} z \leqq 1\}$ and

(54) $\|f\| = \max\{\sup_{t\in\mathbf{R}} u_0(f(it)), \sup_{t\in\mathbf{R}} u_1(f(1+it))\}$.

It is well-known that this construction will produce a family of norms on $\mathscr{B}(H)$ and that the notation $u_\alpha(\alpha \in [0, 1])$ is consistent.

Moreover, given any $u \in N_*$ and $\alpha \in (0, 1]$, we define a function $u^\alpha: \mathscr{B}(H) \to \mathbf{R}^+$ by

(55) $u^\alpha(T) = \inf\{r > 0: u(\alpha z(T/r) + (1-\alpha)I) \leqq 1 \text{ when } |z| \leqq 1\}$.

Clearly $u^\alpha(T) \leqq u(T)$ for each α and we shall see that $u^\alpha(T)$ is non-increasing as α decreases so that we may define $u^0(T)$ as $\lim_{\alpha\downarrow 0} u^\alpha(T)$. We venture to call this process "attenuation" of u; the attenuation of $\|\cdot\|$ yields the operator radii $w_\rho(1 \leqq \rho \leqq 2)$ and in fact $\|T\|^\alpha = w_{2-\alpha}(T)$ (see Section 1.2 and (10) in particular).

PROPOSITION 22. *Each of the interpolated norms $u_\alpha(\alpha \in [0, 1])$ is in $\mathscr{N}_*$.*

Proof. Concerning (17), it is clear that $u_\alpha(I) \leqq 1$ (let $f(z) \equiv 1$). On the other hand, suppose that $f \in \mathscr{F}$ and $f(\alpha) = I$. Consider, for any unit vector $h \in H$, the function $g(z) = (f(z)h, h)$; $g: \mathscr{S} \to \mathbf{C}$ and is bounded and holomorphic on $\mathscr{S}$ so that, by the "maximum principle for $\mathscr{S}$" (Phragmen-Lindelöf theorem),

$$1 = |g(\alpha)| \leqq \sup\{|g(z)|: \operatorname{Re} z = 0 \text{ or } 1\}.$$

Hence

$$1 \leqq \sup\{w(f(z)): \operatorname{Re} z = 0 \text{ or } 1\}$$

so that, in view of Theorem 6 (applied to u_0 and u_1) we have $\|f\| \geqq 1$. It follows that $u_\alpha(I) \geqq 1$.

That (18) holds for each u_α is clear from the fact that, invoking (18) for u_0 and u_1, we have $\|U^*f(\cdot)U\| = \|f\|$ for any $f \in \mathscr{F}$ and unitary U.

If $A \in \mathscr{B}(H_1)$, $B \in \mathscr{B}(H_2)$, and $r > \max(u_\alpha(A), u_\alpha(B))$, we have $f, g \in \mathscr{F}$ (with values in $\mathscr{B}(H_1)$, $\mathscr{B}(H_2)$ respectively) such that $\|f\|$, $\|g\| < r$, $f(\alpha) = A$, and $g(\alpha) = B$. Consideration of the function ϕ defined by $\phi(z) = f(z) \oplus g(z)$ makes it clear that $u_\alpha(A \oplus B) \leqq r$; hence

$$u_\alpha(A \oplus B) \leqq \max(u_\alpha(A), u_\alpha(B)).$$

To verify the reverse inequality, consider any $f \in \mathscr{F}$ (with values in $\mathscr{B}(H_1 \oplus H_2)$) and make the block operator decomposition

$$f(z) = \begin{bmatrix} f_{11}(z) & f_{12}(z) \\ f_{21}(z) & f_{22}(z) \end{bmatrix}.$$

Clearly $f_{11}, f_{22} \in \mathscr{F}$ (with respect to the appropriate spaces) and, by

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Proposition 1, $\|f_{11}\|, \|f_{22}\| \leqq \|f\|$. Finally, if $f(\alpha) = A \oplus B$, we must have $f_{11}(\alpha) = A$ so that $u_\alpha(A) \leqq \|f_{11}\| \leqq \|f\|$, and similarly $u_\alpha(B) \leqq \|f\|$. Hence

$$\max(u_\alpha(A), u_\alpha(B)) \leqq u_\alpha(A \oplus B),$$

completing the verification of (22) for u_α.

PROPOSITION 23. *If u is a norm in $\mathcal{N}_*$ then so is each of the "attenuations" u^α. Furthermore, if $u_1, u_2 \in \mathcal{N}_*$ are such that $u_1 \geqq u_2$ (i.e., $u_1(T) \geqq u_2(T)$ for each $T \in \mathcal{B}(H)$) then $u_1^\alpha \geqq u_2^\alpha$ for each $\alpha \in (0, 1]$. For every $u \in \mathcal{N}_*$, $u^0 = w$.*

Proof. To see that (55) defines a norm on $\mathcal{B}(H)$ we simply observe that the set

$$C(\alpha) = \{T \in \mathcal{B}(H) : u(\alpha zT + (1 - \alpha)I) \leqq 1 \text{ whenever } |z| \leqq 1\}$$

is certainly convex and "circled" (i.e., $T \in C(\alpha), |w| \leqq 1 \Rightarrow wT \in C(\alpha)$). Note that

$$T \in C(\alpha) \Leftrightarrow u^\alpha(T) \leqq 1.$$

Evidently $I \in C(\alpha)$, $(1 + \epsilon)I \notin C(\alpha)$ for any $\epsilon > 0$, and $U^*C(\alpha)U = C(\alpha)$ for any unitary U (use (18) for u). Properties (17) and (18) follow for u^α. The assumption (22) for u makes it clear that $A \oplus B \in C(\alpha)$ (for the space $H_1 \oplus H_2$, where $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$) if, and only if, $A, B \in C(\alpha)$ for their respective spaces. Hence $u^\alpha(A \oplus B) \leqq 1$ if, and only if, $u^\alpha(A), u^\alpha(B) \leqq 1$. By homogeneity, (22) follows for u^α.

If $u_1 \geqq u_2$, it is clear that the corresponding convex bodies satisfy $C_1(\alpha) \subset C_2(\alpha)$, so that $u_1^\alpha(T) \leqq 1 \Rightarrow u_2^\alpha(T) \leqq 1$, and homogeneity ensures that $u_1^\alpha \geqq u_2^\alpha$.

Finally, $\|\cdot\| \geqq u$ and $u^\alpha \geqq w$ by Theorem 6 so that

$$w_{2-\alpha} = \|\cdot\|^\alpha \geqq u^\alpha \geqq w$$

and, since

$$\lim_{\alpha \downarrow 0} w_{2-\alpha} = w_2 = w,$$

we must also have

$$u^0 = \lim_{\alpha \downarrow 0} u^\alpha = w.$$

PROPOSITION 24. *For any $u \in \mathcal{N}_*$ and $\alpha, \beta \in [0, 1]$ we have $(u^\alpha)^\beta = u^{\alpha\beta}$. In particular, $u^\alpha(T)$ decreases with α.*

Proof. Since $u^0 = w$ for any $u \in \mathcal{N}_*$ (see Proposition 23), we restrict our attention to $\alpha, \beta \in (0, 1]$. Suppose that $(u^\alpha)^\beta(T) \leqq 1$, so that

$$u^\alpha(\beta zT + (1 - \beta)I) \leqq 1 \quad \text{whenever } |z| \leqq 1.$$

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It follows, in particular, that

$$u(\alpha(\beta zT + (1 - \beta)I) + (1 - \alpha)I) \leqq 1$$

and, in view of the fact that $\alpha(1 - \beta) + (1 - \alpha) = 1 - \alpha\beta$, we have verified that $u^{\alpha\beta}(T) \leqq 1$. Hence,

$$(u^\alpha)^\beta(T) \leqq 1 \Rightarrow u^{\alpha\beta}(T) \leqq 1.$$

It will be clear from the foregoing discussion that to reverse the implication it is necessary to show that

$$(56) \quad u(\alpha z_1(\beta z_2 T + (1 - \beta)I) + (1 - \alpha)I) \leqq 1 \quad (|z_1|, |z_2| \leqq 1)$$

follows from

$$(57) \quad u(\alpha\beta zT + (1 - \alpha\beta)I) \leqq 1 \quad (|z| \leqq 1).$$

But (56), for particular z_1 and z_2 may be expressed as

$$(58) \quad u(|z_1 z_2|\alpha\beta T + wI) \leqq 1$$

where

$$w = \alpha(1 - \beta)|z_1|e^{i\theta} + (1 - \alpha)e^{i\phi},$$

and θ, ϕ are appropriate arguments. Since

$$|w| \leqq \alpha(1 - \beta) + (1 - \alpha) = 1 - \alpha\beta,$$

there are arguments θ_1, θ_2 such that

$$w = \tfrac{1}{2}(e^{i\theta_1}(1 - \alpha\beta) + e^{i\theta_2}(1 - \alpha\beta)).$$

Moreover, letting $z = |z_1 z_2|e^{-i\theta_k}$ in (57), we obtain

$$u(|z_1 z_2|\alpha\beta T + e^{i\theta_k}(1 - \alpha\beta)I) \leqq 1 \quad (k = 1, 2)$$

so that (58) follows by the triangle inequality for u. An appeal to homogeneity completes the proof that $(u^\alpha)^\beta = u^{\alpha\beta}$.

PROPOSITION 25. *If the power inequality*

$$u(T^n) \leqq (u(T))^n \quad (n = 1, 2, \dots ; T \in \mathscr{B}(H))$$

is satisfied for $u = u_0$ *and* $u = u_1$, *it is also satisfied for* $u = u_\alpha$ $(\alpha \in [0, 1])$. *More generally, if* $p(z)$ *is any polynomial such that, for* $u = u_0$ *and* $u = u_1$,

$$u(T) \leqq 1 \Rightarrow u(p(T)) \leqq 1 \quad (T \in \mathscr{B}(H))$$

then the same implication holds for $u = u_\alpha(\alpha \in [0, 1])$.

Proof. If $u_\alpha(T) < 1$ there is some $f \in \mathscr{F}$ such that $f(\alpha) = T$ and $u_0(f(it)) \leqq 1$, $u(f(1 + it)) \leqq 1$ for all $t \in \mathbf{R}$. Clearly $p \circ f \in \mathscr{F}$ and our assumption ensures that $\|p \circ f\| \leqq 1$. Since $p \circ f(\alpha) = p(T)$, we must have $u_\alpha(p(T)) \leqq 1$.

It seems reasonable to suggest that when $u_0 = w_{\rho_0}$ and $u_1 = w_{\rho_1}$ $(1 \leqq \rho_0 \leqq \rho_1 \leqq 2)$ the interpolated norm u_α is also of the type w_ρ; Proposition 28 (below) shows that if this is so the value of ρ must be $\rho_0^{(1-\alpha)}\rho_1^{\alpha}$. We do not see how to verify this suggestion, but the following result is supporting evidence, in view of the fact that

$$\rho_0^{(1-\alpha)}\rho_1^{\alpha} \leqq (1-\alpha)\rho_0 + \alpha\rho_1$$

and $w_\rho(\cdot)$ is decreasing in ρ.

PROPOSITION 26. *If $u_0 = w_{\rho_0}$ and $u_1 = w_{\rho_1}$, then*

$$w_{\rho(\alpha)}(T) \leqq u_\alpha(T) \quad (T \in \mathscr{B}(H))$$

where $\rho(\alpha) = (1-\alpha)\rho_0 + \alpha\rho_1$.

Proof. It is convenient for this argument to use the following criterion for $w_\rho(S) \leqq 1$: $r(S) \leqq 1$ and

$$\text{(59)} \quad \operatorname{Re}\,((I - \zeta S)^{-1}h, h) = 1 - \rho/2 \quad (\|h\| = 1, |\zeta| < 1).$$

This criterion is available whenever $\rho \leqq 2$; see, e.g., [**17**, §I, 11, (11.4)].

Suppose that $u_\alpha(T) < 1$. Then there exists $f \in \mathscr{F}$ such that $f(\alpha) = T$ and, in view of (59):

$$\text{(60)} \quad \begin{aligned} &\operatorname{Re}\,((I - \zeta f(it))^{-1}h, h) \geqq 1 - \rho_0/2 \quad \text{and} \\ &\operatorname{Re}\,((I - \zeta f(1+it))^{-1}h, h) \geqq 1 - \rho_1/2 \end{aligned}$$

whenever $t \in \mathbf{R}$, $\|h\| = 1$, and $|\zeta| < 1$.

Since $w \leqq w_{\rho_0}, w_{\rho_1}$, we have $w(f(z)) \leqq 1$ for $z \in \partial\mathscr{S}$, so that by a standard application of the maximum principle in $\mathscr{S}$, $w(f(z)) \leqq 1$ for all $z \in \mathscr{S}$. It is then elementary (see, e.g., [**9**; §15, Lemma 1]) that

$$\text{(61)} \quad \|(I - \zeta f(z))^{-1}\| \leqq (1 - |\zeta|)^{-1} \quad (|\zeta| < 1).$$

Fix h, ζ such that $\|h\| = 1 > |\zeta|$, and define F by setting

$$F(z) = \exp\,(1 - \tfrac{1}{2}((1-z)\rho_0 + z\rho_1) - ((I - \zeta f(z))^{-1}h, h))$$

for $z \in \mathscr{S}$. By (61) and the fact that

$$1 \leqq \operatorname{Re}\,((1-z)\rho_0 + z\rho_1) \leqq 2,$$

it is clear that $F \in \mathscr{F}$. Moreover, (60) ensures that $|F(z)| \leqq 1$ for $z \in \partial\mathscr{S}$. Using the maximum principle again we see that $|F(\alpha)| \leqq 1$, so that

$$\operatorname{Re}\,((I - \zeta T)^{-1}h, h) \leqq 1 - \frac{\rho(\alpha)}{2}.$$

Since this is true for each h, ζ such that $\|h\| = 1 > |\zeta|$, we conclude from (59) that $w_{\rho(\alpha)}(T) \leqq 1$.

Since both $w_{\rho(\alpha)}$ and u_α are homogeneous, the argument above shows that $w_{\rho(\alpha)}(T) \leqq u_\alpha(T)$ for every $T \in \mathscr{B}(H)$.

COROLLARY 27. *For a fixed* $T \in \mathscr{B}(H)$, $\log w_\rho(T)$ *is convex as a function of* ρ, *for* $1 \leqq \rho \leqq 2$.

Remark. This corollary gives a different approach to a result of Ando and K. Nishio (see [**3**] or [**2**, Theorem 3.5]), who showed that $\log w_\rho(T)$ is convex for all $\rho \in (0, \infty)$.

Proof. Let $u_0 = w_{\rho_0}$, $u_1 = w_{\rho_1}$ for $1 \leqq \rho_0 \leqq \rho_1 \leqq 2$. By a general feature of the interpolation process (see, e.g., [**13**, Chapter IV, 1.2]),

$$u_\alpha(T) \leqq (u_0(T))^{(1-\alpha)}(u_1(T))^\alpha$$

so that by Proposition 26

$$w_{(1-\alpha)\rho_0+\alpha\rho_1}(T) \leqq (w_{\rho_0}(T))^{(1-\alpha)}(w_{\rho_1}(T))^\alpha.$$

PROPOSITION 28. *Let* u_0 *and* u_1 *satisfy the additional condition* (41), *and let* ρ_α *denote the quantity* $\rho_{(u_\alpha)}$ *as defined by* (28). *Then*

$$\rho_\alpha = \rho_0^{(1-\alpha)}\rho_1^{\alpha}.$$

Proof. By the general feature of interpolation mentioned during the previous proof,

$$u_\alpha(Z) \leqq (u_0(Z))^{(1-\alpha)}(u_1(Z))^\alpha,$$

where

$$Z = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \in \mathscr{B}(H \oplus H).$$

Recalling Theorem 16, we have

$$\rho_\alpha^{-1} \leqq (\rho_0^{-1})^{(1-\alpha)}(\rho_1^{-1})^\alpha,$$

i.e.,

$$\rho_0^{(1-\alpha)}\rho_1^\alpha \leqq \rho_\alpha.$$

To obtain the reverse inequality, suppose that $\epsilon > 0$ and that $f:\mathscr{S} \to \mathscr{B}(H \oplus H)$ is a function in $\mathscr{F}$ with $f(\alpha) = Z$ and $\|f\| \leqq \rho_\alpha^{-1} + \epsilon$. Write

$$f(z) = \begin{bmatrix} f_{11}(z) & f_{12}(z) \\ f_{21}(z) & f_{22}(z) \end{bmatrix}.$$

Since $\|f_{12}(z)\| \leqq \|f(z)\|$, it is clear that $f_{12} \in \mathscr{F}$. It is a direct consequence of the Hadamard 3-line theorem that

$$1 = \|f_{12}(\alpha)\| \leqq (\sup_{t\in\mathbf{R}} \|f_{12}(it)\|)^{(1-\alpha)}(\sup_{t\in\mathbf{R}}\|f_{12}(1+it)\|)^\alpha.$$

By the definition of ρ_0,

$$\rho_0 u_0(f(it)) \geqq \|f(it)\| \geqq \|f_{12}(it)\|;$$

similarly for ρ_1. Hence

$$\begin{aligned} 1 &\leqq \rho_0^{(1-\alpha)}\rho_1^\alpha(\sup_{t\in\mathbf{R}} u_0(f(it)))^{(1-\alpha)}(\sup_{t\in\mathbf{R}} u_1(f(1+it)))^\alpha \\ &\leqq \rho_0^{(1-\alpha)}\rho_1^\alpha\|f\| \leqq \rho_0^{(1-\alpha)}\rho_1^\alpha(\rho_\alpha^{-1}+\epsilon). \end{aligned}$$

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Since this holds for every $\epsilon > 0$,

$$\rho_\alpha \leqq \rho_0^{(1-\alpha)}\rho_1^{\alpha}.$$

6. The Banach algebra setting. There is a natural way to define the numerical radius $w_{\mathscr{A}}(a)$ for an element a of any unital Banach algebra $\mathscr{A}$. The set $\Sigma(\mathscr{A})$ of "normalized states" of $\mathscr{A}$ is defined by:

$$\Sigma(\mathscr{A}) = \{\phi \in \mathscr{A}^*: \|\phi\| = \phi(1) = 1\},$$

and we set

$$(62)\quad w_{\mathscr{A}}(a) = \sup\{|\phi(a)|: \phi \in \Sigma(\mathscr{A})\}.$$

Given any unit vector h in a Hilbert space H, it is clear that ϕ_h, defined by $\phi_h(T) = (Th, h)$, is a normalized state on $\mathscr{B}(H)$; such a state ϕ_h is sometimes called a "spatial" state. Evidently $w(T) \leqq w_{\mathscr{B}(H)}(T)$. Although not every state in $\Sigma(\mathscr{B}(H))$ is spatial, it turns out that the spatial states are rich enough so that $w(T) = w_{\mathscr{B}(H)}(T)$; hence the $w_{\mathscr{A}}$ concept is a generalization to Banach algebras of the numerical radius as we have discussed it in this paper. On the other hand, $\Sigma(\mathscr{A})$ is rich enough in any Banach algebra to ensure that

$$(63)\quad \|a\| \leqq e w_{\mathscr{A}}(a) \quad (a \in \mathscr{A}).$$

An excellent account of this material (and much more) may be found in [**8**, **9**].

Our interest in inequalities such as (3) and (16) and in the role played there by commutativity makes it natural to examine the following constants, defined for any (unital) Banach algebra $\mathscr{A}$ (note that we shall drop the subscript in the notation $w_{\mathscr{A}}$ when it is obvious which algebra is involved):

$$c_1(\mathscr{A}) = \sup\{w(ab)/w(a)\|b\|: 0 \neq a, b \in \mathscr{A} \text{ and } ab = ba\};$$
$$C_1(\mathscr{A}) = \sup\{w(ab)/w(a)\|b\|: 0 \neq a, b \in \mathscr{A}\};$$
$$c_2(\mathscr{A}) = \sup\{w(ab)/w(a)w(b): 0 \neq a, b \in \mathscr{A} \text{ and } ab = ba\};$$
$$C_2(\mathscr{A}) = \sup\{w(ab)/w(a)w(b): 0 \neq a, b \in \mathscr{A}\}.$$

In the spirit of (28) we shall also define $\rho(\mathscr{A})$ by

$$\rho(\mathscr{A}) = \sup\{\|a\|/w(a): 0 \neq a \in \mathscr{A}\}.$$

Remarks. A closely related quantity has been introduced by Bonsall and Duncan (see [**8**, p. 43]); their numerical index $n(\mathscr{A}) = (\rho(\mathscr{A}))^{-1}$. The inequality (63) makes it clear that, for all $\mathscr{A}$, $1 \leqq \rho(\mathscr{A}) \leqq e$, and it is known (see [**9**, §32, Theorem 4]) that any value in this range is possible. In our earlier notation $\rho(\mathscr{B}(H))$ is ρ_w and we have seen that this value is 2.

There are some obvious relations among the constants c_1, C_1, c_2, C_2, and

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ρ; they are no deeper than such observations as

$$w(ab) \leqq \|ab\| \leqq \|a\| \, \|b\| \leqq \rho(\mathscr{A})w(a)\|b\|,$$

and they are summarized in the following proposition.

PROPOSITION 29. *For any unital Banach algebra* $\mathscr{A}$,

$$\begin{array}{ccccccc} 1 & \leqq & c_1(\mathscr{A}) & \leqq & C_1(\mathscr{A}) & \leqq & \rho(\mathscr{A}) \\ & & \wedge\!\parallel & & \wedge\!\parallel & & \\ 1 & \leqq & c_2(\mathscr{A}) & \leqq & C_2(\mathscr{A}) & \leqq & (\rho(\mathscr{A}))^2. \end{array}$$

When $\mathscr{A} = \mathscr{B}(H)$, we have more exact information. Easy examples show that

$$C_1(\mathscr{B}(H)) = 2(= \rho(\mathscr{B}(H))) \text{ and}$$
$$C_2(\mathscr{B}(H)) = 4(= (\rho(\mathscr{B}(H)))^2).$$

It is known that $c_2(\mathscr{B}(H)) = 2$ (see [**11**, Theorem 2.1]) and there are good reasons to suspect that $c_1(\mathscr{B}(H)) = 1$ (recall (16) and Theorem 21, for example).

Our final result shows that commutativity has not at all the same effect on these constants in the general Banach algebra setting as it has for $\mathscr{B}(H)$. We recall that the "projective tensor product" $\mathscr{A} \otimes_p \mathscr{A}$ is obtained from the algebraic tensor product $\mathscr{A} \otimes \mathscr{A}$ by completion with respect to the norm

$$(64) \quad \|x\| = \inf \left\{ \sum_1^n \|a_k\| \, \|b_k\| : x = \sum_1^n a_k \otimes b_k \right\}.$$

This norm is a "cross-norm", i.e., $\|a \otimes b\| = \|a\| \, \|b\|$. The structure is made into a unital Banach algebra by means of a product satisfying the relation $(a \otimes b)(c \otimes d) = ac \otimes bd$.

PROPOSITION 30. *For any unital Banach algebra* $\mathscr{A}$,

$$c_1(\mathscr{A} \otimes_p \mathscr{A}) \geqq C_1(\mathscr{A}) \quad \textit{and} \quad c_2(\mathscr{A} \otimes_p \mathscr{A}) \geqq C_2(\mathscr{A}).$$

Proof. First note that, for products in $\mathscr{A} \otimes_p \mathscr{A}$, we have

$$(65) \quad w(ab) \leqq w(a \otimes b) \quad (a, b \in \mathscr{A}).$$

To see this observe that, since the map $(a, b) \mapsto ab$ is bilinear, there is a linear map $F: \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ such that $F(a \otimes b) = ab$; moreover $\|F(x)\| \leqq \|x\|$ since $x = \sum_1^n a_k \otimes b_k$ implies

$$\|F(x)\| = \left\| \sum_1^n a_k b_k \right\| \leqq \sum_1^n \|a_k\| \, \|b_k\|,$$

and $\|x\|$ is defined by (64). Thus F extends by continuity to $\mathscr{A} \otimes_p \mathscr{A}$ and F in the extended sense also satisfies $\|F\| \leqq 1$. Since, in addition, $F(1 \otimes 1) = 1$, we have:

$$s \in \Sigma(\mathscr{A}) \Rightarrow s_1 = s \circ F \in \Sigma(\mathscr{A} \otimes_p \mathscr{A})$$

so that the relation $s(ab) = s_1(a \otimes b)$ makes (65) clear.

Next observe that $w(a) = w(a \otimes 1)$ for any $a \in \mathscr{A}$, since $w(a) \leqq w(a \otimes 1)$ follows from (65) and a state $s_0 \in \Sigma(\mathscr{A})$ is defined by $s_0(x) = s(x \otimes 1)$ for any state $s \in \Sigma(\mathscr{A} \otimes_p \mathscr{A})$, so that $w(a) \geqq w(a \otimes 1)$ also. Similarly we see that $w(a) = w(1 \otimes a)$.

Now if $r < C_1(\mathscr{A})$ we have some $a, b \in \mathscr{A}$ such that $a, b \neq 0$ and $w(ab) \geqq r\, w(a)\|b\|$. Consider $x = a \otimes 1$ and $y = 1 \otimes b$; these elements commute in $\mathscr{A} \otimes_p \mathscr{A}$ and

$$w(xy) = w(a \otimes b) \geqq w(ab) \geqq r\, w(a)\|b\| = r\, w(x)\|y\|.$$

Thus $c_1(\mathscr{A} \otimes_p \mathscr{A}) \geqq r$ and we conclude that $c_1(\mathscr{A} \otimes_p \mathscr{A}) \geqq C_1(\mathscr{A})$. The second inequality of the proposition may be proved in a similar manner.

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https://doi.org/10.4153/CJM-1983-015-3 Published online by Cambridge University Press