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SEMIPERMUTABILITY IN GENERALISED SOLUBLE GROUPS

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Abstract

Some classes of finitely generated hyperabelian groups defined in terms of semipermutability and S-semipermutability are studied in the paper. The classification of finitely generated hyperabelian groups all of whose finite quotients are PST-groups recently obtained by Robinson is behind our results. An alternative proof of such a classification is also included in the paper.

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1. Introduction

A subgroup H of a group G is said to be *permutable* in G if HK is a subgroup for every subgroup K of G. In recent years, some extensions of the concept of permutability, especially in finite groups, have been extensively studied. One of the most important is the S-permutability introduced by Kegel in his seminal paper [12]: a subgroup H of a periodic group G is *S-permutable* if HP is a subgroup of G for every Sylow subgroup P of G. S-permutable subgroups of finite groups are subnormal and nilpotent modulo their core [3, Theorem 1.2.14].

In the general finite universe, permutable subgroups are not subnormal in general. However, they are always ascendant by a result of Stonehewer [20]. Unfortunately, S-permutable subgroups of periodic groups are not ascendant in general: there exist locally nilpotent groups with nonascendant subgroups [11, Example 18.2.2].

Permutability and S-permutability, like normality, are not transitive in general. A group G is called a *T-group* (respectively, a *PT-group*) if normality (respectively, permutability) is a transitive relation, that is, if H is normal in K and K is normal in G, then H is normal in G (respectively, permutable). A periodic group G is called a *PST-group* if S-permutability is a transitive relation.

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By the above result of Kegel, a finite group G is a PT-group (respectively, a PST-group) if and only if every subnormal subgroup of G is permutable (respectively, S-permutable) in G. These classes of groups include the class of finite groups in which every subnormal subgroup is normal and they have been extensively studied in recent years. Many of the beautiful results on these classes of groups in the finite universe are presented in [3, Ch. 2]. Infinite soluble T-groups and PT-groups were studied in [16] and [5, 13, 14]. Locally finite PST-groups were studied in [4, 19].

In this context, Robinson [15] studied the impact of S-permutability in the structure of polycyclic groups and proved a nice characterisation of the finitely generated hyperabelian groups all of whose finite quotients are PST-groups.

Theorem 1.1 [15]. *Let G be a finitely generated hyperabelian group. Then every finite quotient of G is a PST-group if and only if G is one of the following:*

(i) *a finite soluble PST-group;*
(ii) *a nilpotent group;*
(iii) *a group of infinite dihedral type.*

A group of infinite dihedral type is, by definition, a group G in which the hypercentre H of G is a finite 2-group and the factor G/H is isomorphic with the dihedral group $\mathrm{Dih}(B)$ on a finitely generated, infinite abelian group B with no involutions. A useful characterisation for this kind of group is the following lemma contained in [15].

Lemma 1.2 [15]. *A group G is of infinite dihedral type if and only if it has a normal abelian subgroup A such that:*

(i) *A is a finitely generated, infinite abelian group containing no involutions;*
(ii) *G/A is a finite 2-group and $|G : C_G(A)| = 2$;*
(iii) *elements in $G/C_G(A)$ induce inversion in A.*

The purpose of this paper is to extend Robinson's theorem to some other classes of groups, defined by means of subgroup permutability properties which turned out to be of interest in the finite universe.

2. Preliminaries

As usual, if G is a periodic group, $\pi(G)$ denotes the set of the primes that divide the orders of the elements of G. A subgroup H of a periodic group G is called *semipermutable* (respectively, *S-semipermutable*) if HK is a subgroup of G for every subgroup K of G such that $\pi(H) \cap \pi(K) = \emptyset$ (respectively, HP is a subgroup of G for every Sylow p-subgroup P of G such that $p \notin \pi(H)$). A group in which semipermutability is a transitive relation is called a *BT-group*. In [21], the authors proved a beautiful theorem characterising the finite soluble BT-groups.

Theorem 2.1 [21]. *Let G be a finite group. The following statements are equivalent.*

(i) *G is a soluble BT-group.*
(ii) *Every subgroup of G is semipermutable.*
(iii) *Every subgroup of G is S-semipermutable.*
(iv) *G is a soluble PST-group with nilpotent residual L and, if p and q are distinct primes not dividing the order of L, then* $[P, Q] = 1$ *for every* $P \in \mathrm{Syl}_p(G)$ *and* $Q \in \mathrm{Syl}_q(G)$.

Applying Theorem 2.1, the class of all finite soluble BT-groups is subgroup and quotient closed.

In general, a semipermutable subgroup of a finite group G is not subnormal. For instance, in the symmetric group on three letters, a 2-Sylow subgroup is both semipermutable and S-semipermutable but it is not subnormal.

A group G is called an *SP-group* (respectively, an *SPS-group*) if every subnormal subgroup of G is semipermutable (respectively, S-semipermutable). A subgroup H of a periodic group G is said to be *seminormal* if it is normalised by every subgroup K of G such that $\pi(H) \cap \pi(K) = \emptyset$. A group in which every subnormal subgroup is seminormal is called an *SN-group*. In [6], Beidleman and Ragland proved the following result.

Theorem 2.2 [6]. *Let G be a finite soluble group. Then the following statements are equivalent.*

(i) *G is a PST-group.*
(ii) *G is an SP-group.*
(iii) *G is an SPS-group.*
(iv) *G is an SN-group.*

A group G is called an *SNT-group* if seminormality is a transitive relation. Ballester-Bolinches *et al.* [2, Theorem F] proved that any finite SNT-group is a PST-group but the converse does not hold in general [2, Example 1].

A subgroup H of a group G is said to be *SS-permutable* in G if H has a supplement K in G such that H permutes with every Sylow subgroup of K. Groups in which SS-permutability is a transitive relation are called *SST-groups*. In [8], it is proved that a finite SST-group is a BT-group but the converse is not true. The following theorem gives a criterion for a BT-group to be an SST-group.

Theorem 2.3 [8]. *Let G be a finite soluble BT-group with nilpotent residual L. Then the following statements are equivalent.*

(i) *G is an SST-group.*
(ii) *For every p-subgroup P of G with* $p \in \pi(G) \setminus \pi(L)$, *G has a subgroup* K_p *such that* $PK_p \in \mathrm{Syl}_p(G)$ *and* $[P, K_p] \leq O_p(G)$.

A group G is called an *MS-group* if the maximal subgroups of all the Sylow subgroups of G are S-semipermutable in G. Ballester-Bolinches *et al.* studied this class of groups (see [1]) and, moreover, they showed that the class of BT-groups and the class of MS-groups are not comparable.

The relationships between all the classes of groups listed above can be pictured by the following diagram, with the exception of MS-groups, which are incomparable with all the other ones.

$$\mathrm{SST} \Rightarrow \mathrm{BT} \Rightarrow \mathrm{SNT} \Rightarrow \mathrm{SN} \Leftrightarrow \mathrm{SP} \Leftrightarrow \mathrm{SPS} \Leftrightarrow \mathrm{PST}$$

3. Main results

Let $\mathfrak{X}$ be one of the classes SST, BT, SNT, SN, SP or SPS. In order to extend Robinson's theorem to finitely generated hyperabelian groups with all finite quotients belonging to $\mathfrak{X}$, we need to study the finite quotients of groups of infinite dihedral type. The next lemma shows that their structure is quite transparent.

Lemma 3.1 (see [15]). *Let G be a group of infinite dihedral type. If G/N is a finite quotient of G and L/N is the nilpotent residual of G/N, then L/N is a Hall $2'$-subgroup of G/N.*

Proof. By Lemma 1.2, G/N has a normal abelian subgroup T/N such that G/T is a 2-group, $|G/N : C_{G/N}(T/N)| = 2$ and the elements in $G/N \setminus C_{G/N}(T/N)$ induce inversion in T/N. Let $B/N = O_{2'}(T/N)$. Then G/B is a 2-group, so the nilpotent residual L/N of G/N is contained in B/N. On the other hand, since B/N has odd order and the elements of $G/N \setminus C_{G/N}(T/N)$ induce inversion in B/N, we have $B/N \leq L/N$. Thus, $B/N = L/N$ is the nilpotent residual of G/N. □

Next, we observe that all the finite quotients of a group of infinite dihedral type are BT-groups.

Lemma 3.2. *If G is a group of infinite dihedral type, then all its finite quotients are BT-groups.*

Proof. By [15, Lemma 3], every finite quotient of G is a soluble PST-group. Furthermore, if G/N is a finite quotient and L/N is the nilpotent residual of G/N, then G/L is a 2-group by Lemma 3.1. Therefore, G/N is a BT-group by Theorem 2.1. □

In [15, Proposition 2], it is shown that a group of infinite dihedral type is not a PST-group. However, the class of all groups of infinite dihedral type is a subclass of the class of all BT-groups, as shown in the following lemma.

Lemma 3.3. *If G is a group of infinite dihedral type, then every subgroup of G is semipermutable. In particular, G is a BT-group.*

Proof. Applying Lemma 1.2, G has a normal finitely generated infinite abelian subgroup A containing no involutions such that G/A is a finite 2-group, $|G : C_G(A)| = 2$ and the elements in $G \setminus C_G(A)$ induce inversion in A.

In particular, every subgroup of A is normal in G and, if D is the torsion subgroup of A, then $\pi(G) = \{2\} \cup \pi(D)$. Clearly, if A is torsion-free, every subgroup of G is semipermutable. Hence, we may assume that D is not trivial.

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Let x, y be elements of G of order p^α and q^β, respectively, with p and q different prime numbers. Since p and q are different, one of them must belong to $\pi(D)$. Assume that $p \in \pi(D)$, so that $x \in A$. In this case, $\langle x \rangle$ is a normal subgroup of G and therefore $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$. □

Since in a group of infinite dihedral type all finite quotients are BT-groups, we can prove that its finite quotients are actually SST-groups using Theorem 2.3.

Lemma 3.4. *Let G be a finite group with a normal abelian subgroup A such that:*

(i) *G/A is a 2-group and $|G : C_G(A)| = 2$;*
(ii) *elements in $G \setminus C_G(A)$ induce inversion in A.*

Then G is an SST-group.

Proof. By Theorem 2.1, G is a BT-group. Therefore, $G = L \rtimes M$, where $M \in \mathrm{Syl}_2(G)$ and L is the nilpotent residual of G. In particular, $\pi(G) \setminus \pi(L) = \{2\}$. Clearly, $C_G(A) \leq C_G(L)$. If $G = C_G(L)$, then G is nilpotent and so G is an SST-group. Hence, we may assume that G is not nilpotent and $C_G(A) = C_G(L)$.

First, suppose that $O_2(G) = 1$. Since $M \cap C_G(L)$ is a normal 2-subgroup of M, it follows that $M \cap C_G(L) = \{1\}$ and M has order 2. If P is a 2-subgroup of G, then $P \in \mathrm{Syl}_2(G)$ and choosing $K_2 = 1$ we see that $[P, K_2^L] \leq O_2(G)$.

Now suppose that $O_2(G)$ is not trivial. Since the factor group $G/O_2(G)$ satisfies the same hypotheses as G, it follows from the previous argument that $|M : O_2(G)| = 2$. Let P be a 2-subgroup of G. Without loss of generality, we may assume that $P \leq M$. If $M = PO_2(G)$, then we may choose $K_2 = O_2(G)$ and clearly $[P, K_2^L] \leq O_2(G)$. Otherwise, if $P \leq O_2(G)$, let $K_2 = M$, so that $[P, K_2^L] \leq [O_2(G), M^L] \leq O_2(G)$. □

Corollary 3.5. *Let G be a group of infinite dihedral type. Then every finite homomorphic image of G is an SST-group.*

Theorem 3.6. *Let G be a finitely generated hyperabelian group. Then every finite quotient of G is an SST-group if and only if G is one of the following:*

(i) *a finite soluble SST-group;*
(ii) *a nilpotent group;*
(iii) *a group of infinite dihedral type.*

Proof. Since any finite soluble SST-group is a PST-group, if every finite quotient of G is an SST-group, then the result follows by Robinson's theorem.

Conversely, if G is finite or nilpotent, then trivially G is an SST-group. If G is a group of infinite dihedral type, the assertion follows from Corollary 3.5. □

Bearing in mind the relation between the classes SST, BT, SNT, SN, SP and SPS in the finite universe, the following theorem is a direct consequence of Robinson's result and Theorem 3.6.

Theorem 3.7. *Let $\mathfrak{X}$ be one of the classes BT, SNT, SN, SP or SPS and let G be a finitely generated hyperabelian group. Then every finite quotient of G is an $\mathfrak{X}$-group if and only if G is one of the following:*

(i) *a finite soluble $\mathfrak{X}$-group;*
(ii) *a nilpotent group;*
(iii) *a group of infinite dihedral type.*

We bring the section to a close by studying the finitely generated hyperabelian MS-groups.

Lemma 3.8. *If G is a group of infinite dihedral type, then all its finite quotients are MS-groups.*

Proof. Let G/N be a finite quotient of G and let L/N be the nilpotent residual of G/N. By [15, Lemma 3], G/N is a soluble PST-group and, by Lemma 3.1, L/N is a normal abelian Hall $2'$-subgroup of G/N. Then conditions (iv) and (v) of [1, Theorem 3.1] are trivially satisfied and so G/N is an MS-group by [1, Theorem 3.2]. □

In the proof of the main theorem for MS-groups, we used some results proved in [10] about polycyclic groups whose finite quotients are T_0-groups.

Here, a finite group G is called a *T_0-group* if the factor group $G/\Phi(G)$ over the Frattini subgroup is a T-group.

Theorem 3.9. *Let G be a polycyclic group. Then every finite quotient of G is an MS-group if and only if G is one of the following:*

(i) *a finite soluble MS-group;*
(ii) *a nilpotent group;*
(iii) *a group of infinite dihedral type.*

Proof. Assume that every finite quotient of G is an MS-group. If $G/\Phi(G)$ is abelian, then any maximal subgroup of G is normal, so that any finite quotient of G is a nilpotent group. By [18, 5.4.18], G is nilpotent.

If $G/\Phi(G)$ is finite, then only finitely many primes are possible for the indices of maximal subgroups and so G has no infinite abelian factors. Then G is finite. In particular, G is a finite MS-group.

Hence, we may assume that G is an infinite polycyclic group, such that the Frattini quotient group $G/\Phi(G)$ is an infinite nonabelian group. By [7, Theorem C], any finite MS-group is a T_0-group. Hence, by [10, Theorem C], G is the semidirect product of an abelian group A by a cyclic group $\langle t\rangle$ of order 2. Let N be a normal subgroup of G of finite index. Since A is a maximal subgroup of G, we have that $G = AN$ or $N \le A$. In the first case, the quotient G/N is abelian. So, assume that $N \le A$. In this case, $G/N = A/N \rtimes \langle t\rangle N/N$ is a finite T_0-group whose nilpotent residual is abelian. By [7, Lemma 4], G/N is a PST-group. Thus, all finite quotients of G are PST-groups and, therefore, by [15, Theorem], G is either nilpotent or a group of infinite dihedral type. □

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THEOREM 3.10. *Let G be a finitely generated hyperabelian group whose finite quotients are MS-groups. Then G is polycyclic.*

PROOF. We follow the proof of [15, Theorem] and use the same notation. We may assume, arguing by contradiction, that G is just nonpolycyclic. Note that the group $\overline{G}$ obtained there is, in our case, a finite MS-group whose nilpotent residual is abelian. Since $\overline{G}$ is a T_0-group, it follows that $\overline{G}$ is a PST-group by [7, Lemma 4]. Then the contradiction follows as in [15, Theorem]. □

COROLLARY 3.11. *Let G be a finitely generated hyperabelian group. Then every finite quotient of G is an SST-group if and only if G is one of the following:*

(i) *a finite soluble MS-group;*
(ii) *a nilpotent group;*
(iii) *a group of infinite dihedral type.*

4. A proof of the theorem of Robinson

We propose here an alternative proof for Robinson's theorem for polycyclic groups [15, Proposition 3]. Our proof depends on [10, Theorem D] and the following lemma.

LEMMA 4.1. *Let G be an infinite nonnilpotent supersoluble group whose finite quotients are PST-groups. Then the hypercentre $Z_\infty(G)$ of G does not have finite index in G.*

PROOF. Let F be the Fitting subgroup of G. Applying [18, 5.4.10], G/F is a finite abelian group. Assume that $G/Z_\infty(G)$ is finite. Then, by [17, Theorem 4.21], there exists a positive integer k such that $\gamma_k(G) = \gamma_\infty(G)$ is finite. Since G is not nilpotent, there exist a prime p and a positive integer i such that G/F^{p^i} is not nilpotent (otherwise, $\gamma_\infty(G)$ would be contained in every p'-component of F by [15, Lemma 1] and G would be nilpotent). Then G/F^{p^j} is not nilpotent for all $j \geq i$. Let N_j be a normal subgroup of finite index in G which is maximal with respect to $F \cap N_j = F^{p^j}$. Then G/N_j is not nilpotent. By [3, Theorem 2.1.8], the nilpotent residual of G/N_j is an abelian Hall subgroup of G/N_j contained in FN_j/N_j with noncentral chief factors. Hence, $Z_\infty(G) \leq N_j$ and so $Z_\infty(G) \leq F \cap N_j = F^{p^j}$. Thus, $Z_\infty(G) \leq \bigcap_{j>i} F^{p^j}$, which is a finite subgroup of F by [15, Lemma 1]. This contradiction proves the lemma. □

THEOREM 4.2. *Let G be an infinite polycyclic group and let F be its Fitting subgroup. If every finite quotient of G is a PST-group, then either G is nilpotent or the following conditions are satisfied.*

(i) *$Z_\infty(G)$ is a 2-group.*
(ii) *$F/Z_\infty(G)$ is an abelian group containing no involutions.*
(iii) *$|G : F| = 2$ and every element of $G \setminus F$ induces inversion in $F/Z_\infty(G)$.*

PROOF. Assume that G is not nilpotent. Since any finite PST-group is supersoluble, G is supersoluble by a result of Baer. By [18, 5.4.10], G/F is a finite abelian group.

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Let N be any normal subgroup of G of finite index and let $\overline{G} = G/N$. Then $\overline{G}$ is a PST-group. By [3, Theorem 2.1.8], the nilpotent residual $\overline{A}$ of $\overline{G}$ is an abelian Hall subgroup of G. Note that $\overline{A} \le \overline{G}' \le \overline{F}$. By [9, Ch. IV, Theorem 5.18], $\overline{A}$ is complemented by a Carter subgroup $\overline{D}$ of $\overline{G}$. Let $\overline{C} = \overline{F} \cap \overline{D}$; then $\overline{F} = \overline{A} \times \overline{C}$ and $\overline{C} \le C_{\overline{D}}(\overline{A})$. Therefore, $\overline{C}$ is contained in the hypercentre $Z_\infty(\overline{G})$ of $\overline{G}$ by [9, Ch. IV, Theorem 6.14] and $\overline{G}/\overline{C}$ is a T-group. Hence, $\overline{G}/Z_\infty(\overline{G})$ is a T-group and $\overline{G}$ is a T_1-group.

Applying Lemma 4.1, $G/Z_\infty(G)$ is infinite. By [10, Theorem D], $G/Z_\infty(G)$ is an extension of an abelian group $A/Z_\infty(G)$ containing no involutions by a cyclic subgroup of order 2 such that the elements of $G \setminus A$ invert all the elements of $A/Z_\infty(G)$. Since $A/Z_\infty(G)$ is abelian, A is nilpotent and $A = F$. In particular, the 2-component F_2 of F is contained in $Z_\infty(G)$ and $|G : F| = 2$.

Applying [18, 5.2.10], G/F' is not nilpotent and F/F' is the Fitting subgroup of G/F'. By [18, 5.2.6], F/F' is infinite. Let p be an odd prime and assume that G/F^{p^i} is nilpotent for some positive integer i. Let $j > i$. Since the nilpotent residual of G/F^{p^j} is a Hall subgroup of G/F^{p^j} contained in F^{p^i}/F^{p^j}, it follows that G/F^{p^j} is nilpotent. Let N_j be a normal subgroup of finite index of G which is maximal with respect to $F \cap N_j = F'F^{p^j}$. Then G/N_j is nilpotent. Since FN_j/N_J is a p-group, it follows that G/N_j is a p-group. Moreover, $G = FN_j$. Since F/F' is infinite, F/F' contains a nontrivial torsion-free subgroup M/F' such that M is a normal subgroup of G and G/M is finite. If $j \ge i$, $[M, G] = [M, N_j] \le M \cap N_j = M^{p^j} \pmod{F'}$. Then $[M/F', G/F']$ is finite by [15, Lemma 1]. Hence, $[M/F', G/F'] = 1$ and $M/F' \le Z(G/F')$. Then $G/F'/Z(G/F')$ is finite, contrary to Lemma 4.1.

Therefore, G/F^{p^i} is a nonnilpotent PST-group and F/F^{p^i} is the nilpotent residual of G/F^{p^i} for each odd prime p and positive integer i. Since G/F^{p^i} acts on F/F^{p^i} as a power automorphism by conjugation [3, Theorem 2.1.8] and the only power automorphism of order 2 is the inversion, the elements of $G \setminus F$ invert all the elements of F/F^{p^i}. Since $\bigcap_{r>2} (\bigcap_{i>0} F^{r^i}) = \bigcap_{r>2} F_{r'} = F_2$ by [15, Lemma 2], every element of $G \setminus F$ induces inversion on F/F_2 so that $Z(G/F_2) = 1$ and $F_2 = Z_\infty(G)$. □

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