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INFRASEQUENTIAL TOPOLOGICAL ALGEBRAS

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1. Introduction. The notion of sequential topological algebra was introduced by this author and Ng [3]. Among a number of results concerning these algebras, we showed that each multiplicative linear functional on a sequentially complete, sequential, locally convex algebra is bounded ([3], Theorem 1). From this it follows that every multiplicative linear functional on a sequential F-algebra (complete metrizable) is continuous ([3], Corollary 2). Here we introduce a weakened form of the "sequential" property, which we call "infrasequential" and show that the above-mentioned results hold. It is worth noting that the proof given here can be adapted to give an alternative proof of the results mentioned above. We also prove the continuity of an algebra homomorphism between any two such algebras. A number of permanence properties and examples of infrasequential algebras are given.

Throughout in this paper, we assume that a topological algebra used in the sequel is Hausdorff and is over the field of complex numbers.

2. Infrasequential algebras.

Definition. (i) Let A be a topological algebra. A is said to be *strongly sequential* if there exists a neighbourhood U of 0 such that for all $x \in U$, $x^k \to 0$ as $k \to \infty$. (cf: [3])

(ii) A is said to be *sequential* if for each sequence $\{x_n\}$, $x_n \to 0$ there exists $x_m \in \{x_n\}$ such that $x_m^k \to 0$ as $k \to \infty$. [3]

(iii) A is said to be *infrasequential* (resp. *weakly infrasequential*) if for each bounded set $B \subset A$ there exists $\lambda > 0$ such that for all $x \in B$, $(\lambda x)^n \to 0$ as $n \to \infty$ (resp. weakly).

Proposition 1. *With reference to the above definition, (i)$\Rightarrow$(ii)$\Rightarrow$(iii).*

Proof. (i)$\Rightarrow$(ii) ([3], Proposition 1).

(ii)$\Rightarrow$(iii): Suppose A satisfies (ii) but not (iii). Then there exists a bounded set $B \subset A$ and a sequence $\{x_n\} \subset B$ such that $(n^{-1}x_n)^k \not\to 0$ as $k \to \infty$ for all $n \geq 1$. But $n^{-1}x_n \to 0$ as $n \to \infty$ because B is bounded; this shows that A does not satisfy (ii), a contradiction.

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REMARK. The reverse implications in Proposition 1 need not hold, See Example 7 below.

THEOREM 1. *Let A be a sequentially complete, (weakly) infrasequential locally convex topological algebra. Then each multiplicative linear functional on A is bounded.*

Proof. Suppose there exists a multiplicative linear functional f on A which is not bounded. Let B be a bounded subset of A such that $f(B)$ is not a bounded subset of complex numbers. Since A is (weakly) infrasequential there exists a $\lambda>0$ such that $(\lambda x)^k \to 0$ (weakly) as $k\to\infty$ for all $x\in B$ and a sequence $\{x_n\}\subset B$ such that $|f(x_n)|\geq n$ for all $n\geq 1$. Choose a sufficiently large m such that $m>\lambda^{-1}$ and put $y=\lambda x_m$. Since $x_m\in B$, by assumption $y^k=(\lambda x_m)^k\to 0$ (weakly) as $k\to\infty$. Let C denote the closed absolutely convex hull of $\{y^k;\ k\geq 1\}$. Then

$$C=\left\{\sum_{k=1}^{\infty}\mu_k y^k : \sum_{k=1}^{\infty}|\mu_k|\leq 1\right\}.$$

C is clearly a (weakly) compact subset of A for both cases—weak infrasequential and infrasequential. Thus the linear span A_C of C is a Banach space with the norm $\|..\|$ defined by the Minkowski's functional of C.

We show that A_C is actually a commutative Banach algebra. For this, we verify that A_C is closed under products and the product is commutative.

Let $u,v\in A_C$. Then $u=\sum_{k=1}^{\infty}\mu_k y^k$ and $v=\sum_{k=1}^{\infty}\nu_k y^k$, where $\sum_{k=1}^{\infty}|\mu_k|\leq 1$ and $\sum_{k=1}^{\infty}|\nu_k|\leq 1$. It is easy to verify that

$$uv=\sum_{k=1}^{\infty}\left[\sum_{i=1}^{k}\mu_k\nu_{k-i}\right]y^{k+1}$$

and

$$\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}|\mu_k\nu_{k-i}|\right)\leq 1.$$

This shows that $CC\subset C$ and clearly $\|uv\|\leq\|u\|\,\|v\|$. It is also easy to check that $uv=vu$. Thus A_C is a commutative Banach algebra and so the restriction of f on A_C, being a multiplicative linear functional, is continuous with respect to the norm topology. Therefore $|f(z)|\leq\|z\|$ for all $z\in A_C$. But then

$$\begin{aligned}1<\lambda m\leq\lambda\,|f(x_m)|=|f(\lambda x_m)|=|f(y)|\\ \leq\sup_{z\in C}|f(z)|\leq 1,\end{aligned}$$

is absurd. Hence f must be bounded.

COROLLARY 1. *Every sequentially complete, bornological, (weakly) infrasequential locally convex algebra is functionally continuous* [5].

Proof. This follows from the fact that each bounded linear functional on a bornological space is continuous.

COROLLARY 2. *Every (weakly) infrasequential, Frèchet algebra is functionally continuous.*

REMARK. The author gratefully acknowledges the suggestion of I. Tweddle for the above proof and part of Example 7 below.

The following fact is obvious if f is a continuous linear multiplicative functional. In general, however, we have:

PROPOSITION 2. *Let A be a sequentially complete locally convex topological algebra and f a multiplicative linear functional. If $x \in A$ is such that $x^n \to 0$ as $n \to \infty$, then $|f(x)| < 1$ and hence $f(x^n) \to 0$ as $n \to \infty$.*

Proof. Suppose $|f(x)| \geq 1$. Put $y = -x/f(x)$. Then $y^n \to 0$ as $n \to \infty$ and $f(y) = -1$. Let C denote the closure of the absolutely convex hull of $\{y^n; n \geq 1\}$. Then as in the proof of Theorem 1 above, C is compact and A_C, the linear span of C, is a commutative Banach algebra and so the restriction of f on A_C is continuous. Since $y^n \in A_C$, $y^n \to 0$, we have $f(y^n) \to 0$ as $n \to \infty$. But $f(y^n) = [f(y)]^n = (-1)^n \not\to 0$, a contradiction. Thus $|f(x)| < 1$.

Now we prove sequential continuity of any algebra homomorphism between certain sequentially complete locally convex algebras. Precisely, we have:

THEOREM 2. *Let A, B be two sequentially complete locally convex algebras such that A is sequential and B satisfies the following condition:*

($\acute{C}$): *For all sequences $\{y_n\} \subset B$, $y_n \neq 0$, $y_n \not\to 0$ there exists a sequence $\{f_m\}$ of multiplicative linear functionals on B such that* $\inf |f_m(y_n)| = \varepsilon > 0$.

Then each homomorphism $T: A \to B$ is sequentially continuous.

Proof. Suppose T is not sequentially continuous. Then there exists a sequence $\{x_n\} \subset A$, $x_n \to 0$ but $Tx_n \not\to 0$. Put $y_n = \frac{1}{2}Tx_n$. We may assume that $y_n \neq 0$ for all $n \geq 1$. Then by hypothesis there exists a sequence $\{f_m\}$ of multiplicative linear functionals on B such that $\inf |f_m(y_n)| = \varepsilon > 0$. But then $|f_m \circ T(\varepsilon^{-1}x_n)| = |f_m(\varepsilon^{-1}Tx_n)| \geq 2$ for all m, $n \geq 1$. Put $z_n = \varepsilon^{-1}x_n$. Then $z_n \to 0$ as $n \to \infty$. Since A is sequential, there exists $z \in \{z_n\}$, $z = \varepsilon^{-1}x_k$ (for some k) such that $z^i \to 0$ as $i \to \infty$. Now for f_m, $f_m \circ T$ is a multiplicative linear functional on A, because T is an algebra homomorphism. But then by Proposition 2, $|f_m \circ T(z)| = |f_m(\varepsilon^{-1}Tx_k)| < 1$ and by construction $|f_m(\varepsilon^{-1}Tx_k)| \geq 2$. This is impossible. Hence $Tx_n \to 0$ and T is sequentially continuous.

3. Condition ($\acute{C}$). Condition ($\acute{C}$) in Theorem 2 was stated first for real Fréchet algebras in ([2]). There is a misprint, viz $\inf |f_n(y_n)| = \varepsilon > 0$ should be $\inf |f_m(y_n)| = \varepsilon > 0$ as is in Theorem 2 above. Condition ($\acute{C}$) is a variation of a similar property, condition (C) for real Banach algebras, introduced in ([1]).

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Precisely, condition (C) states: for any sequence $\{x_n\}$, $\|x_n\| \geq 1$, there exist $\varepsilon > 0$ and a sequence of real multiplicative linear functionals $\{f_m\}$ such that $\inf|f_m(x_n)| \geq \varepsilon$. Using real Banach algebras B satisfying condition (C), we were able to show that each algebra homomorphism of a real Fréchet algebra A into B is continuous ([2], Theorem 1). In particular, each real multiplicative linear functional on a *real* Fréchet algebra is continuous, thus answering a long-standing Michael's question [5] for *real* Fréchet algebras. The problem which is stated for complex Fréchet algebras still remains.

The following consequence of condition $(\acute{C})$ for Fréchet algebras is noteworthy.

PROPOSITION 3. *Let A be a Fréchet algebra and let $M(A)$ denote the set of all non-zero multiplicative linear functionals on A. If A satisfies condition $(\acute{C})$ then for each $x \in A$, the spectral radius, $\rho(x) = \sup_{f \in M(A)} |f(x)| = \infty$.*

Proof. Suppose A satisfies condition $(\acute{C})$. Let x be an element in A such that $|f(x)| \leq \alpha < \infty$ for all $f \in M(A)$. Define $\{y_n\}$, where $y_{2n+1} = x$ and $y_{2n} = (1/n)x$. Then $y_n \neq 0$ and $y_n \not\to 0$. If $\{f_m\}$ is a sequence of multiplicative linear functionals on A satisfying condition $(\acute{C})$, then $\inf|f_m(y_n)| \leq \inf|f_n(y_n)| \leq \inf(\alpha/n) = 0$. This violates condition $(\acute{C})$. Hence $\rho(x) = \infty$.

COROLLARY 3. *Let A be a locally convex algebra. Then condition $(\acute{C})$ cannot hold under any one of the following situations:*

(i) *A is a Banach algebra*
(ii) *A is a Q-algebra, (cf:* [5])

Proof. (i) Since for each $f \in M(A)$, $|f(x)| \leq \|x\|$ and so $\rho(x) < \infty$(cf. [6]).

(ii) For each $x \in A$, the spectrum $\sigma(x)$ of each $x \in A$ is compact ([6]) and so $\rho(x) < \infty$.

4. Permanence properties and examples.

PROPOSITION 4. *Let $\mathcal{A}$ be the class of all infrasequential topological algebras A. Then $\mathcal{A}$ is closed under the following operations:*

(i) *subalgebras*
(ii) *finite products*
(iii) *quotients if in addition A is metrizable*
(iv) *unitization i.e. $A^+ = A \times \mathbb{C}$,* [cf: 6].

Proof. (i) If B is a subslgebra of A, then each bounded subset of B in the induced topology is also bounded in A and the result follows from the definition.

(ii) If $A = \Pi_{i=1}^n A_i$, where each A_i is infrasequential, then a set M of A is bounded iff $M \subset B = \{(x_1, \ldots, x_n) \in A,\ x_i \in B_i\}$ where each B_i is a bounded subset of A_i. There are $\lambda_i > 0$, $i = 1, \ldots, n$ such that $(\lambda_i x_i)^n \to 0$ as $n \to \infty$ for each $i = 1, \ldots, n$. If $\lambda = \min_{1 \le i \le n} \lambda_i > 0$, then it is clear that for each $x \in B$, $(\lambda x)^i = (\lambda x_1, \ldots, \lambda x_n)^i = ((\lambda x_1)^i, \ldots, (\lambda x_n)^i) \to 0$ as $i \to \infty$. This proves (ii).

(iii) Under the assumption, A/M is metrizable and it is easy to see that it is infrasequential.

(iv) Routine verification.

Following are the examples of infrasequential algebras:

Example 1. Let $\mathbb{C}^{(N)}$ denote the algebra of finite sequences with pointwise addition and multiplication, endowed with the strict inductive limit topology defined by the strictly increasing sequence of Euclidean spaces, $\{\mathbb{C}^n\}$. It is a complete, barrelled, bornological, non-metrizable topological algebra. If B is bounded in $\mathbb{C}^{(N)}$, then $B \subset \mathbb{C}^n$ for some n. It is clear that $\mathbb{C}^n$ is infrasequential and so there exists $\lambda > 0$ such that $(\lambda x)^k \to 0$ as $k \to \infty$. Hence $\mathbb{C}^{(N)}$ is infrasequential. Actually, it is strongly sequential.

Example 2. Let $C_0(\mathbb{R})$ be the algebra of all continuous complex-valued functions with compact supports, endowed with the strict inductive limit topology defined by the Banach algebras $C[-n, n]$ of continuous functions on compact intervals $[-n, n]$. Then $C_0(\mathbb{R})$ is a non-metrizable complete locally m-convex topological algebra which is infrasequential by being strongly sequential.

Example 3. The test space $\mathcal{D}(\mathbb{R})$ of Schwarz distributions, endowed with the strict inductive limit of Fréchet algebras is infrasequential by being strongly sequential.

Example 4. Every normed algebra is infrasequential.

Example 5. The algebra $C[0, 1]$ of continuous complex valued functions endowed with the uniform topology on countable compact sets is infrasequential by being sequential (cf. [4]).

Example 6. The algebra ℓ_1 of absolutely convergent series endowed with the convolution multiplication and the weak topology $\sigma(\ell_1, \ell_\infty)$ is infrasequential by being sequential.

Example 7. Let $A = C^*(\mathbb{R})$ denote the algebra of all continuous bounded complex-valued functions with the strict topology β[7] defined by a family $\{p_\phi : \phi \in C_0(\mathbb{R})\}$ of seminorms, where $C_0(\mathbb{R}) = \{\phi \in C^*(\mathbb{R}) : \phi$ vanishes at $\infty\}$ and

$$p_\phi(f) = \sup_{x \in \mathbb{R}} |f(x)\phi(x)|.$$

Since $p_\phi(fg) \le p_{\sqrt{\phi}}(f)p_{\sqrt{\phi}}(g)$ for $f, g \in A$, A is a locally convex algebra. If $f_n(x) = 0$ for $x < n-1$, $= x - n + 1$ for $n - 1 \le x \le n$ and $= 1$ for $x > n$, then

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for no f_n in the sequence $\{f_n\} \subset A$, $f_n^k \to \infty$, i.e. A is not sequential for the same arguments as for $C(\mathbb{R})$ ([3], p. 500) in view of the fact that a sequence converges in the β-topology iff it is uniformly bounded and converges uniformly on each compact subset of $\mathbb{R}$ ([7], Theorem 1(v)). But by Theorem 1(iii) [7] each subset of A is β-bounded iff it is uniformly bounded. Now let B be a β-bounded subset of A then there is $\mu > 0$ such that $|f(x)| \leq \mu$ for all $f \in B$ and $x \in \mathbb{R}$. If $\lambda = (1+\mu)^{-1}$, then $(\lambda f)^k \to 0$ as $k \to \infty$ for all $f \in B$ and so A is infrasequential.

On the other hand, the algebra $C[0, \Omega)$ of all continuous functions on the space of all ordinals $[0, \Omega)$ less than the first uncountable ordinal Ω with the order-topology, endowed with the compact-open topology is a complete locally m-convex sequential [cf: [4]) algebra which is not strongly sequential because if it were strongly sequential then it has to be a Q-algebra which it is not (cf: [5], pp. 49, 77 and [3], theorem 2).

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