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PAPER

Wake flow past a submerged plate near a free surface

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Abstract
A wake model is pursued for potential flow past a submerged, finite-length plate that is perpendicular to a uniform, horizontal stream bounded above by a free surface. The effects of gravity are included along the free surface. The approach is to adopt an open-wake model such that the wake boundaries become parallel to the undisturbed stream at some (unknown) point downstream. Boundary integral equations are formed and then discretised along the wake boundaries and free surface in order to obtain a solution numerically. In terms of the dependency of the solution on various parameters, the problem will be formulated in two ways. First, for a given Froude number and ratio of the length of the vertical plate to the draft (the depth of the bottom of the vertical plate relative to the undisturbed free surface), the effect of the wake underpressure coefficient on the size of the wake will be considered. Then, the problem will be discussed where we instead (more naturally) fix the Froude number, draft and length of the vertical, submerged plate. The dependencies of the solution on these parameters will be analysed regarding the effects on several factors, including the size of the wake, the relative lengths of the upper and lower wake boundaries, and the resulting wake underpressure coefficient.

1. Introduction

We present a wake model for the flow past a submerged, finite-length plate. We assume that the flow is uniform and characterised by a constant velocity U far upstream. For simplicity, we assume that the plate is normal to the direction of the velocity U (with possible extension for plates inclined at an arbitrary angle discussed later in the concluding remarks). Figure 1 depicts this model. The flow is assumed to be steady, two-dimensional and irrotational, whilst the fluid is assumed to be inviscid and incompressible. Hence, we have potential flow. The effects of gravity will be included along the free surface that bounds the uniform stream – the exclusivity of which will be later justified. The open-wake model will be utilised for the approximation of the wake or 'closure' of the wake downstream. The investigations presented here are based on work produced for Chapter 5 of the PhD thesis of McLean [10].

Wake flows have been studied by many investigators, from different viewpoints. The most fundamental recent theoretical advances, carried out in the 1970s and 1980s, through a combination of numerical and asymptotic approaches concentrated on determining a self-consistent structure of the steady laminar wake at large values of a Reynolds number, based on body size. It was demonstrated that, in planar flow and in the absence of gravity, both the length and width of the wake scale linearly with the Reynolds number in the limit, both therefore significantly greater than the body size. The recirculating flow in the wake is characterised by having a constant vorticity, akin to so-called Prandtl–Batchelor flows. The theoretical work is described in the studies by Sadovskii [13], Smith [14] and Chernyshenko [1] and the numerical approaches in the work of Fornberg [3]. An excellent summary is presented by Fornberg and Elcrat [4] and there is more recent work, showing the subtleties of the analysis, by Vynnycky [16].

It is interesting that this structure emerges only as the Reynolds number passes around 400, which is numerically large, and beyond the value at which the wake flow is in fact likely to be strongly unsteady (which occurs at a value of around 100) resulting in significant mixing between the flow in the wake and the external flow passing the body. This mixing will cause the wake to close, in the sense that the mixing has essentially merged the external and wake fluid, removing the distinction between slow moving fluid in a trapped wake and faster moving flow outside. This closure is over a significantly shorter length scale, on the scale of the body itself. Although the wake may be steady on global scales, so that it is possible to define a sensible time-averaged wake, there will be significant unsteady or random mixing motions across the wake boundary. The study of this flow, using analytic techniques, on the assumption of steady flow requires a model of the wake closure, the effects of this unsteadiness and an acknowledgement that some average of the flow quantities are being modelled. Such models (along with detailed description of fully established wake flow) are discussed by Wu [19] and typically involve the use of potential flow theory. It is this approach that we follow here. Rather than taking the wake to be a constant vorticity of Prandtl–Batchelor type (as would be relevant if the wake remained steady and laminar), these approaches assume that the pressure in the wake take a constant value, as we shall do in the present study. Of particular note are the classical potential flow investigations of Kirchhoff [9] and Helmholtz [5] regarding the zero-gravity problem of a wake forming behind a finite-length plate that is submerged within an infinite fluid and is normal to the flow, with the pressure within the wake assumed to be ambient. In this case, a free streamline solution is obtainable through the use of conformal mappings.

To generalise the above zero-gravity problem such that the pressure within the wake can be freely chosen, a closure model can be employed. Many closure models have been proposed by different investigators, but the following qualitative similarity exists between them: the aim of a closure model is not to necessarily realistically capture the behaviour of the flow downstream but to facilitate the acquisition of a flow solution that is realistic in the near-field to the object behind which the wake forms. As mentioned earlier, discussion of several of the closure models (or wake approximations) proposed by various investigators is presented by Wu [19]. However, one of the simplest to implement is the open-wake model, which is utilised in the present study. This requires that the boundaries of the wake become parallel to the undisturbed stream at some point downstream. The idea for this model is suggested by Joukowsky [8] in terms of the introduction of a point along a free streamline before which we impose the usual dynamic boundary condition and after which the streamline continues in someway to downstream infinity. This undetermined way of continuing the free streamlines downstream is taken as the streamlines being parallel to the undisturbed stream by Roshko [12]. Wu [18] and Mimura [11] also employ the open-wake model in solving for more generalised obstacle shapes: Wu [18] formulates the problem for a general curved barrier whilst providing explicit solutions for a circular arc and a flat plate; and the work of Mimura [11] concerns an inclined flat plate. It should be noted that these previous studies neglect the effects of gravity, the plate is submerged in an infinite fluid and analytic solutions can be obtained through the use of conformal mappings.

An alternative approach is a closed-wake model (as opposed to the open-wake model discussed above) in which the wake is forced to be of finite length, with the streamlines on either side of the wake meeting to effect wake closure. Relevant work here is that of Hocking and Vanden-Broeck [7] for a constant pressure wake and that of Hocking [6] for a constant vorticity wake. Both of these studies incorporate the effects of gravity but only in a direction parallel to the flow so that symmetry is maintained. The problem considered in this paper does not allow such symmetric solutions.

Also of relevance to the flow of interest in the present study is the work of Faltinsen and Semenov [2] where a closed cavity model is used for the flow past a hydrofoil close to the free surface. The relevance of this study lies in the fact that wake and cavity potential flows are often studied simultaneously since their physical manifestations can be similarly approximated: a (super-cavitating) flow is as described earlier for the wake, but instead a vapour-filled bubble forms behind the obstacle, which is still a region of slow recirculation (and so relatively constant pressure). Returning to the work of Faltinsen and Semenov [2], it should be noted that they propose a more mathematically complex closure: it is assumed that a curvilinear contour closes the upper boundary of the cavity; the upper and lower boundaries meet at some point downstream to physically close the cavity; and the shape of the closure of the lower boundary

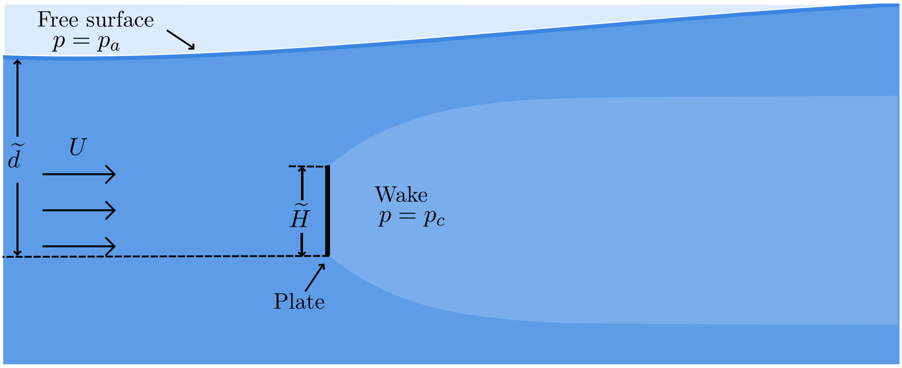

Figure 1. *Illustration of the open-wake model for flow past a submerged, finite-length plate that is normal to the oncoming flow bounded above by a free surface.*

is found from the solution. Boundary integral equations are obtained and solved numerically. It is found that, as the depth of submergence decreases or gravity increases, there is a decrease in the length of the cavity. Faltinsen and Semenov [2] state that this closure model is proposed to more realistically simulate the potential flow at the end of the cavity. There is good agreement with experimental results, but this is also often the case with the employment of more simple closure models, at least in terms of quantities relevant to the local neighbourhood of the plate.

Analogous to the wake flow problem studied in the present work are ploughing flows as investigated by Tuck and Vanden-Broeck [15]. The difference is that the ploughing flows concern flow around a semi-infinitely long, solid block that is submerged in a uniform, horizontal stream bounded above by a free surface: the top and bottom sides of the object in the ploughing flows case contrast with the upper and lower wake boundaries of the present study. The approach used in the present study is similar to that utilised by Tuck and Vanden-Broeck [15]. The results regarding the ploughing flows will be used to validate our solutions presented here in the limit as the beginning points for the wake approximation downstream approach the edges of the plate that is normal to the flow, that is, recovering the problem of flow past a submerged, semi-infinitely long block.

2. Formulation of the problem

The flow is as depicted in Figures 1 and 2a. We have a plate BB' (of length $\widetilde{H}$) that is normal to a uniform, horizontal stream of speed U; and this stream is bounded above by a free surface EF. The plate is submerged with the lower edge of the plate at B' at a distance $\widetilde{d}$ below the level of the free surface of the undisturbed stream (i.e. the level of the free surface far upstream). We will refer to $\widetilde{d}$ as the draft of the flow. At some point not known *a priori*, there is a stagnation point S along the plate BB' and we let AS denote the separation streamline. Note that, since we utilise the open-wake model, the upper and lower streamlines that bound the wake become horizontal (i.e. parallel to the undisturbed stream) at some points downstream. Let C and C' denote these points (not known *a priori*) at which the flow angle becomes zero along the bounding streamlines of the wake. Throughout this work, BC and $B'C'$ will be referred to as the upper and lower wake boundaries, respectively. The horizontal parts (i.e. where the flow angle is zero) of the bounding streamlines of the wake are denoted by CD and $C'D'$.

As discussed in the introduction, the pressure within the wake is assumed to be constant and we take the pressure along the wake boundaries to be this same constant, say p_c. Whilst gravity is included on the free surface, we will neglect the effects of gravity on the wake boundaries. This can be justified by noting that a wake contains fluid of the same density as that of the rest of the flow, but it is at (relative) rest and so we have hydrostatic pressure within the wake. The Bernoulli condition tells us that

$$p + \rho g y = p_c \tag{1}$$

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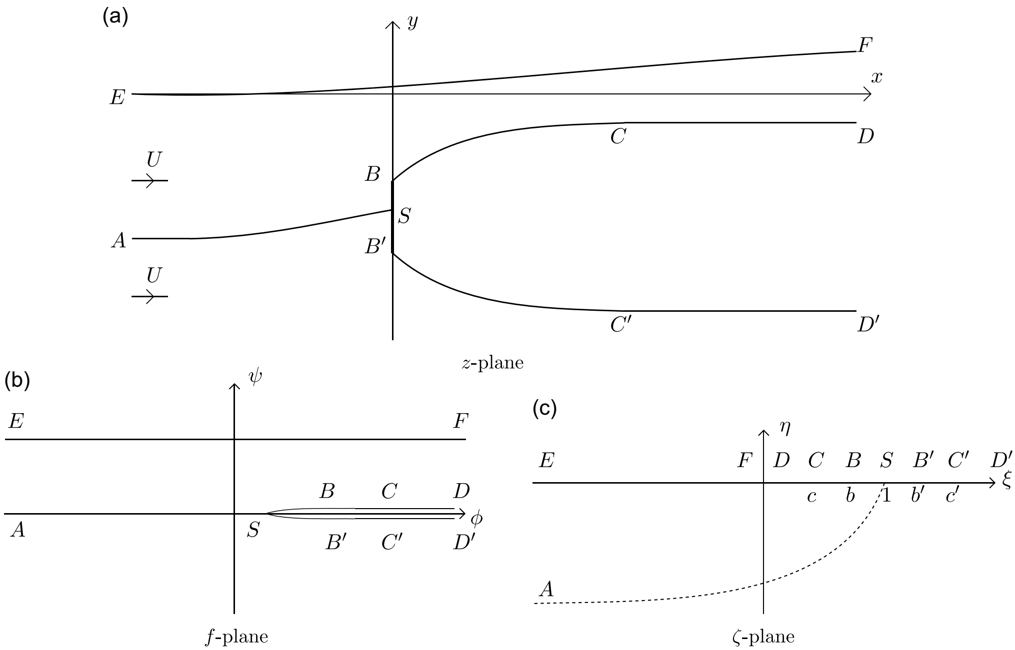

Figure 2. *Complex planes for the open-wake model with a free surface bounding the flow above.*

within the wake, where p denotes the pressure, ρ is the density and g is the gravitational acceleration. Then, through use of (1) to eliminate the pressure p from the Bernoulli condition, along the wake boundaries we have

$$\frac{1}{2}\rho q^2 + p_c = \widetilde{C}, \tag{2}$$

where $\widetilde{C}$ is a constant and q denotes the speed. Hence, gravity does not appear in this boundary condition and so we can neglect the effects of gravity along the boundaries of the wake.

We will now formulate the overall problem. The flow is as described above and shown in Figure 2a where $z = x + \mathrm{i}y$ with the x-axis set along the level of the undisturbed free surface and the y-axis is set along the vertical plate BB'. Recall that we employ the open-wake model, so the flow angle is zero along CD and $C'D'$. The effect of gravity is included on the free surface EF that bounds the flow above and we assume constant atmospheric pressure $p = p_a$ along this free surface. On the other hand, gravity is not included along the wake boundaries BC and $B'C'$; and the pressure is taken to be constant (say, $p = p_c$) along these boundaries. We normalise the velocity such that the uniform, horizontal stream is of unit speed and normalise the distances such that π is the normalised magnitude of the net volume flux of the flow above the plate. We denote the characteristic length for the latter normalisation by L and note that the normalised depth of the dividing streamline far upstream is such that it lies along $y = -\pi$. Then, the Bernoulli condition on the free surface EF is

$$\frac{1}{2}q^2 + \frac{y}{F^2} = \frac{1}{2}, \tag{3}$$

where

$$F = \frac{U}{\sqrt{gL}} \tag{4}$$

is the Froude number. Along the wake boundaries BC and $B'C'$, we have

$$q^2 = K + 1, \tag{5}$$

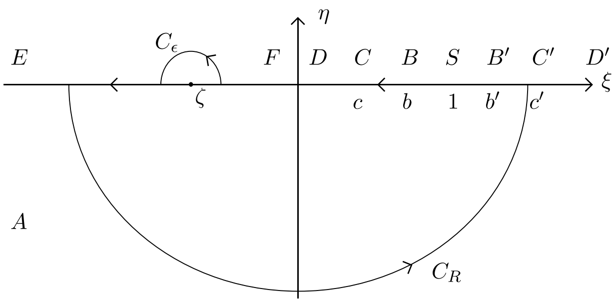

Figure 3. *ζ-plane with contour $\widetilde{C} = C_R \cup [R, \zeta + \epsilon] \cup C_\epsilon \cup [\zeta - \epsilon, -R]$.*

where

$$K = \frac{p_\infty - p_c}{\frac{1}{2}\rho U^2} \tag{6}$$

is the wake underpressure coefficient and p_∞ denotes the pressure of the stream in the far-field. The wake underpressure coefficient is a non-dimensional parameter that characterises the flow regime, that is, whether we have a fully or partially established wake. In particular, $K = 0$ corresponds to the formation of wake at ambient pressure. Note that, in the present study, the wake is assumed to be fully established with the plate edges B and B' of the plate being the separation points.

We let f denote the complex potential for the flow and $f = \phi + \mathrm{i}\psi$, where ϕ is the velocity potential and ψ is the streamfunction. We set $\psi = 0$ along the dividing streamline and $\psi = \pi$ along the free surface (since we have a net volume flux of π for the flow between the dividing streamline and the free surface). The velocity potential at the stagnation point is set to be $\phi = 1$. Then, there is a branch cut along the positive ϕ-axis for $\phi > 1$. For illustration, the f-plane is as depicted in Figure 2b. We also define an intermediate complex plane by $\zeta = \xi + \mathrm{i}\eta$ via the relation:

$$f = \zeta - \log \zeta. \tag{7}$$

Therefore, the flow region maps to the lower-half of the ζ-plane. In particular, the free surface EF maps to the negative ξ-axis; and the wake boundaries BC and $B'C'$, the downstream horizontal approximations CD and $C'D'$, and the vertical plate BB' all map to the positive ξ-axis. Also note that, by the given mapping (7), $\zeta = 1$ corresponds to the stagnation point S (that is situated at some point along the vertical plate in the physical plane). The positions in the ζ-plane corresponding to the edges of the plate and the points at which the zero flow angle begins along the streamlines bounding the wake are unknown and left to be determined. By b, b', c and c' we denote these unknown positions corresponding to B, B', C and C' in the ζ-plane, respectively.

Finally, we let $\Omega = \tau - \mathrm{i}\theta$ where Ω is defined such that $\Omega = \log(\mathrm{d}f/\mathrm{d}z)$. Then, we apply the Cauchy integral formula to $\Omega(\zeta)$ for $\zeta \in \mathbb{R}$ with a semi-circular contour $\widetilde{C} = C_R \cup [R, \zeta + \epsilon] \cup C_\epsilon \cup [\zeta - \epsilon, -R]$ in the lower-half of the ζ-plane, where C_R is the semi-circular arc of radius R centred at the origin, C_ϵ is the semi-circular arc of radius ϵ centred at ζ and the intervals $[R, \zeta + \epsilon]$ and $[\zeta - \epsilon, -R]$ are along the real ζ-axis. The contour is traversed anticlockwise, and we consider the limit as $R \to +\infty$. This contour is depicted in Figure 3. Then, for $\zeta \in \mathbb{R}$, on the contour, we obtain that

$$\tau(\zeta) - \mathrm{i}\theta(\zeta) = -\frac{1}{\pi \mathrm{i}}\, P.V. \int_{-\infty}^{\infty} \frac{\tau(\sigma) - \mathrm{i}\theta(\sigma)}{\sigma - \zeta}\, \mathrm{d}\sigma, \tag{8}$$

where $P.V.$ denotes the Cauchy principal value. Here, we have utilised the fact that we have horizontal flow of unit speed in the far-field; hence, τ and θ tend towards to zero in the far-field. Taking the real part of (8), we find

$$\tau(\zeta) = \frac{1}{\pi}\, P.V. \int_{-\infty}^{\infty} \frac{\theta(\sigma)}{\sigma - \zeta}\, \mathrm{d}\sigma. \tag{9}$$

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Then, by using the known flow angles along the vertical plate BB' and the downstream, horizontal parts CD and $C'D'$ of the streamlines bounding the wake, the expression can be rewritten as:

$$\tau(\zeta) = \frac{1}{\pi}\left[\int_{-\infty}^{0}\frac{\theta(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma + \int_{c}^{b}\frac{\theta(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma + \int_{b'}^{c'}\frac{\theta(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma\right] + \frac{1}{2}\log\left|\frac{(1-\zeta)^2}{(b-\zeta)(b'-\zeta)}\right|. \tag{10}$$

Note that one of the integrals in (10) is a Cauchy principal value, depending on the value of ζ. We recall that the speed of the flow can be expressed as $q = \exp(\tau)$ and that the dynamic boundary condition along the wake boundaries is given by (5). It follows that, along BC and $B'C'$ where $\zeta \in \mathbb{R}$ such that $c < \zeta < b$ or $b' < \zeta < c'$, we have

$$\frac{1}{2}\log(K+1) = \frac{1}{\pi}\left[\int_{-\infty}^{0}\frac{\theta(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma + \int_{c}^{b}\frac{\theta(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma + \int_{b'}^{c'}\frac{\theta(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma\right] + \frac{1}{2}\log\left|\frac{(1-\zeta)^2}{(b-\zeta)(b'-\zeta)}\right|. \tag{11}$$

We also look to obtain an integral equation that is valid along the free surface EF. First, we differentiate the Bernoulli condition (3) along EF with respect to ζ, so we have

$$\frac{\mathrm{d}\tau}{\mathrm{d}\zeta}\mathrm{e}^{2\tau} + \frac{1}{F^2}\frac{\mathrm{d}y}{\mathrm{d}\zeta} = 0. \tag{12}$$

Note that we can express the derivative that appears in the second term above as:

$$\frac{\mathrm{d}y}{\mathrm{d}\zeta} = \mathrm{e}^{-\tau(\zeta)}\left(1-\frac{1}{\zeta}\right)\sin(\theta(\zeta)). \tag{13}$$

Then, using this along with (12) and the fact that $\tau \to 0$ as $\xi \to -\infty$, we find

$$\tau(\zeta) = \frac{1}{3}\log\left(1 - \frac{3}{F^2}\int_{-\infty}^{\zeta}\left(1-\frac{1}{\xi}\right)\sin(\theta(\xi))\,\mathrm{d}\xi\right). \tag{14}$$

Finally, eliminating τ between (10) and (14), then along EF where $\zeta \in \mathbb{R}$ s.t. $-\infty < \zeta < 0$, we look to satisfy

$$\begin{aligned}\frac{1}{3}\log\left(1 - \frac{3}{F^2}\int_{-\infty}^{\zeta}\left(1-\frac{1}{\xi}\right)\sin\theta(\xi)\,\mathrm{d}\xi\right) &= \frac{1}{\pi}\left[\int_{-\infty}^{0}\frac{\theta(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma + \int_{c}^{b}\frac{\theta(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma\right.\\ &\quad\left. + \int_{b'}^{c'}\frac{\theta(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma\right] + \frac{1}{2}\log\left|\frac{(1-\zeta)^2}{(b-\zeta)(b'-\zeta)}\right|.\end{aligned} \tag{15}$$

Note that this is a Nekrasov-type equation [17]. Again, we remark that one of the integrals on the right-hand side of (15) is a Cauchy principal value, depending on the value of ζ.

Before summarising the problem and discretising to obtain a numerical solution, we first discuss the flow far upstream where, along the free surface, we have $\xi \to -\infty$. Considering (10) as $\xi \to -\infty$, we have that

$$\begin{aligned}\tau(\xi) &\sim \frac{1}{\xi}\left(-\frac{1}{\pi}\int_{-\infty}^{\infty}\theta(\sigma)\,\mathrm{d}\sigma\right) + \cdots\\ &\sim \frac{1}{\xi}\left[-\frac{1}{\pi}\left(\int_{-\infty}^{0}\theta(\xi)\mathrm{d}\xi + \int_{c}^{b}\theta(\xi)\mathrm{d}\xi + \int_{b'}^{c'}\theta(\xi)\mathrm{d}\xi\right) - 1 + \frac{1}{2}(b+b')\right] + \cdots\\ &\sim \frac{\beta}{\xi} + \cdots,\end{aligned} \tag{16}$$

where we let β denote the constant coefficient of the ξ^{-1} term. From this behaviour of τ far upstream, the Bernoulli condition (3) along the free surface and the fact that $x \sim \xi$ as $\xi \to -\infty$ (since we have unit horizontal velocity of the undisturbed flow), we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} \sim \frac{\beta F^2}{x^2} \text{ as } x \to -\infty \tag{17}$$

and hence

$$\theta(\xi) \sim \frac{\beta F^2}{\xi^2} \text{ as } \xi \to -\infty, \tag{18}$$

where β is the constant as defined above.

3. Numerical approach

Now, we detail the numerical scheme. We discretise the problem by introducing $N+1$ mesh points along each of the wake boundaries and along the free surface, that is, we take

$$\begin{aligned} \xi_j^U - \log \xi_j^U &= b - \log b + \frac{j-1}{N}\left(c - b + \log \frac{b}{c}\right), \text{ upper wake boundary } BC \\ \xi_j^L - \log \xi_j^L &= b' - \log b' + \frac{j-1}{N}\left(c' - b' + \log \frac{b'}{c'}\right), \text{ lower wake boundary } B'C' \\ \xi_j^{FS} - \log |\xi_j^{FS}| &= X_- + \frac{j-1}{N}(X_+ - X_-), \text{ free surface } EF \end{aligned} \tag{19}$$

where $j = 1, 2, \ldots, N+1$ and where the finite range (X_-, X_+) is used to truncate the range $(-\infty, \infty)$ of the velocity potential. Then, the unknowns of the problem are the values of the flow angle θ at each of these mesh points, that is, we aim to find $\theta_j^{FS} = \theta(\xi_j^{FS})$, $\theta_j^U = \theta(\xi_j^U)$ and $\theta_j^L = \theta(\xi_j^L)$ for $j = 1, 2, \ldots, N+1$. Recall that the positions corresponding to B, B', C and C' in the ζ-plane which are b, b', c and c', respectively, are unknowns. We also look to find the constant β of the decay of τ far upstream, as defined through (16). Therefore, overall we have $3N+8$ unknowns.

For the equations in these unknowns, we introduce N collocation points along the free surface and wake boundaries defined by:

$$\begin{aligned} \xi_j^{U_{\text{inter}}} - \log \xi_j^{U_{\text{inter}}} &= b - \log b + \frac{j-\frac{1}{2}}{N}\left(c - b + \log \frac{b}{c}\right), \text{ upper wake boundary } BC \\ \xi_j^{L_{\text{inter}}} - \log \xi_j^{L_{\text{inter}}} &= b' - \log b' + \frac{j-\frac{1}{2}}{N}\left(c' - b' + \log \frac{b'}{c'}\right), \text{ lower wake boundary } B'C', \\ \xi_j^{FS_{\text{inter}}} - \log |\xi_j^{FS_{\text{inter}}}| &= X_- + \frac{j-\frac{1}{2}}{N}(X_+ - X_-), \text{ free surface } EF, \end{aligned} \tag{20}$$

for $j = 1, 2, \ldots, N$. Satisfying the condition (15) at the collocation points along the free surface and satisfying the condition (11) at the collocation points along the upper and lower wake boundaries leads to $3N$ equations. In order to obtain these $3N$ equations, care is required to evaluate the integrals that appear in (11) and (15) since, for a given value of ζ that corresponds to some point along the free surface EF or the wake boundaries BC and $B'C'$, one of the integrals in each equation will be a Cauchy principal value. Note that the collocation points (20) are the midpoints between the mesh points (19), so the point to emphasise is that the collocation and mesh points do not coincide. Also, we assume that $\theta(\xi)$ is stepwise linear on $[\xi_j, \xi_{j+1}]$ for $j = 1, 2, \ldots, N$ (i.e. on each interval of consecutive mesh points (19)). Then, by integrating over each of these intervals (which make up the intervals for integration that appear in (11) and (15)), we can obtain an expression to (analytically) approximate the Cauchy principal value. The approximation here is due to the aforementioned assumption of stepwise linear behaviour of the flow angle. This expression can then be evaluated numerically at each collocation point.

So far, we have $3N$ equations in the $3N+8$ unknowns. We also have several known values of the flow angle θ. Along the upper wake boundary, we know that

$$\theta(b) = \theta_1^U = \frac{\pi}{2} \quad \text{and} \quad \theta(c) = \theta_{N+1}^U = 0. \tag{21}$$

Along the lower wake boundary, we have

$$\theta(b') = \theta_1^L = -\frac{\pi}{2} \quad \text{and} \quad \theta(c') = \theta_{N+1}^L = 0. \tag{22}$$

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Along the free surface, we take

$$\theta_1^{FS} = \frac{\beta F^2}{(\xi_1^{FS})^2} \quad \text{and} \quad \theta_{N+1}^{FS} = 0. \tag{23}$$

The first condition of the above (23) is obtained by evaluating (18) at the furthest mesh point upstream; and by the second condition of (23), we impose zero flow angle at the furthest mesh point downstream along the free surface, which is appropriate for supercritical flow. We also have that the constant β is constrained by:

$$\beta = -\frac{1}{\pi}\left(\int_{-\infty}^{0} \theta(\xi)\mathrm{d}\xi + \int_{c}^{b} \theta(\xi)\mathrm{d}\xi + \int_{b'}^{c'} \theta(\xi)\mathrm{d}\xi\right) - 1 + \frac{1}{2}(b+b'). \tag{24}$$

This is as derived earlier in (16). Finally, we set the ratio of the (normalised) length H of the vertical plate to the (normalised) draft d by imposing a given value for H/d. In order to impose the value of this parameter, note that expressions to evaluate the values of H and d are required. We have that $y_{C'} = -\pi(1+\beta)$, where $y_{C'}$ denotes the y-value at C'. This expression can be derived by noting that the depth of the dividing streamline far upstream (i.e. point A in Figure 2a) is $-\pi$. Also, the $\pi\beta$ contribution to the expression for $y_{C'}$ can be obtained by finding the difference between the y-values far upstream at A and far downstream at D'. This is achieved by integrating around a large arc in the lower-half of the ζ-plane, taking the limit as the radius of this arc tends to infinity and noting that $\Omega \sim \beta/\zeta$ in the far-field. The draft d of the plate can then be determined by integrating from the point C' on the lower wake boundary to the bottom B' of the plate. Also, the length H of the plate can be calculated by evaluating

$$H = y_B - y_{B'}. \tag{25}$$

Overall, we have $3N+8$ equations in the $3N+8$ unknowns.

To solve, we specify the values of the following parameters: Froude number F, wake underpressure coefficient K and ratio H/d. It is worth noting that, in the current presentation of the method of solution, the problem is posed with the wake underpressure coefficient K as a chosen parameter and the draft is obtained from the solution. This allows for convenient comparison of the results with those found by Tuck and Vanden-Broeck [15] in their work on the similar ploughing flows where the bluff body is a semi-infinite, rectangular block (instead of the plate and wake that we investigate in the present study). However, the problem can be reworked so that the draft is prescribed and K is left to be found – this is a more natural way to consider the physical problem of the submerged plate placed at some given distance beneath the level of the undisturbed free surface. This will be considered later in the discussion of results.

Finally, before presenting the results, we introduce Froude numbers based on the draft d of the plate and based on the flow quantities that characterise the channel between the free surface and the upper wake boundary. This will facilitate a quantitative examination of the results. We let q_∞ and h_∞ denote the non-dimensional speed and width (respectively) of the channel that forms above the wake. Then, we let

$$F_d = \frac{F}{\sqrt{d}}, \tag{26}$$

which is a Froude number defined based on the draft; and we let

$$F_h = \frac{Fq_\infty}{\sqrt{h_\infty}}, \tag{27}$$

which is a Froude number defined based on the downstream channel above the wake. These Froude numbers are as utilised by Tuck and Vanden-Broeck [15].

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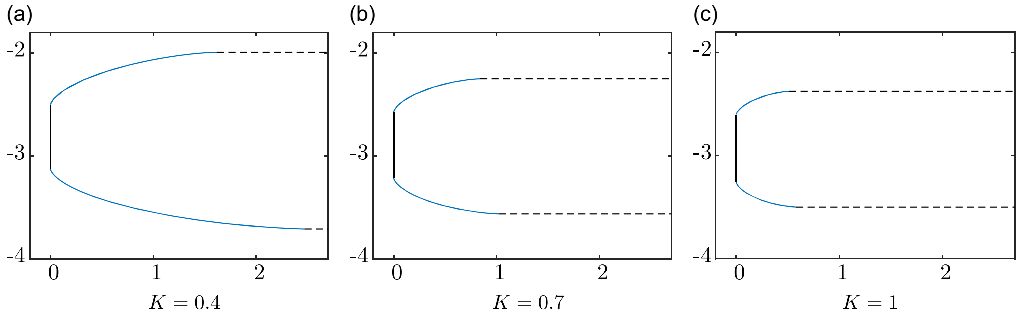

Figure 4. Profiles to show the wake boundaries for $N = 200$, $F = 4$ and $H/d = 0.2$. Note that the undisturbed free surface is set along $y = 0$. The dashed lines are the horizontal sections CD and C′D′ of the streamlines bounding the wake.

4. Results

We begin analysis of the results by first looking solely at the wake boundaries. We find that, as the wake underpressure coefficient K increases, the points at which the zero flow angle begins along the streamlines bounding the wake approach the edges B and B' of the vertical plate and hence, the size of the wake decreases – this is clear in Figure 4. This is qualitatively as can be observed in the zero-gravity case where the open-wake model is employed for the wake flow past a plate that is still normal to the flow but is submerged in an infinite fluid, that is, in the case investigated by Roshko [12]. This can be understood by considering the fact that a larger value of K corresponds to a smaller constant pressure p_c within the wake. Then, there is a larger centripetal force acting upon the streamlines, resulting in greater curvature and we attain a zero flow angle closer to the vertical plate. Hence, we have a smaller wake. This observation validates our results with respect to previous studies where gravity is neglected and allows for the following further comparison with the results concerning gravitational ploughing flows by Tuck and Vanden-Broeck [15].

Figure 5 shows plots of F_h against F_d for two different given values for the wake underpressure coefficient and for various given values of the ratio H/d. Note that these Froude numbers are determined here in post-processing once the solution has been obtained. These plots are similar to those presented by Tuck and Vanden-Broeck [15]. In Figure 5, as $H/d \to 0$, F_h varies linearly with F_d. In the ploughing flows case of Tuck and Vanden-Broeck [15], as $H/d \to 0$, it is found that $F_h = F_d$ which is to be expected since it is the zero-disturbance limit. To recover these results in the model of the present study, the limit as $K \to \infty$ is also required. This is apparent since, in Figure 5, the plots of F_h against F_d (approximately) translate horizontally to the left as the wake underpressure coefficient increases; and since increasing K decreases the length of the wake boundaries then, as $K \to \infty$, we are left with a vertical plate and horizontal streamlines that begin at the plate corners – hence approaching a semi-infinite, rectangular block as studied by Tuck and Vanden-Broeck [15]. We also approach a linear relationship between F_h and F_d in Figure 5 for each value of H/d for large given values of F. Note that this corresponds to all of the Froude numbers tending towards infinity, that is, for small gravity. Unlike the ploughing flows case of Tuck and Vanden-Broeck [15], we do not simply obtain $F_h \approx F_d$ for small gravity since the pressure difference between the stream and within the wake in our case means that the velocity upstream and the velocity downstream in the channel above the wake cannot be equal. Quantitatively, this is due to the wake underpressure coefficient being non-zero. From Figure 5, it is also clear that, for supercritical flow downstream (i.e. $F_h > 1$), there is a minimum value for F_d (the Froude number based on the draft of the plate).

We now present the results obtained following a small change to the formulation of the problem since it is more natural to set the draft d instead of the wake underpressure coefficient K. In brief summary, we set the Froude number F, (normalised) length H of the plate and (normalised) draft d; and we leave K

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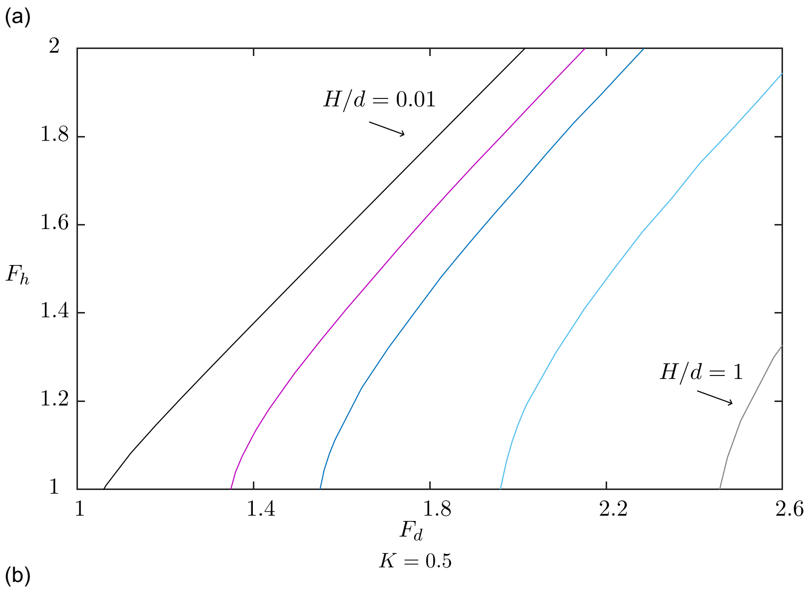

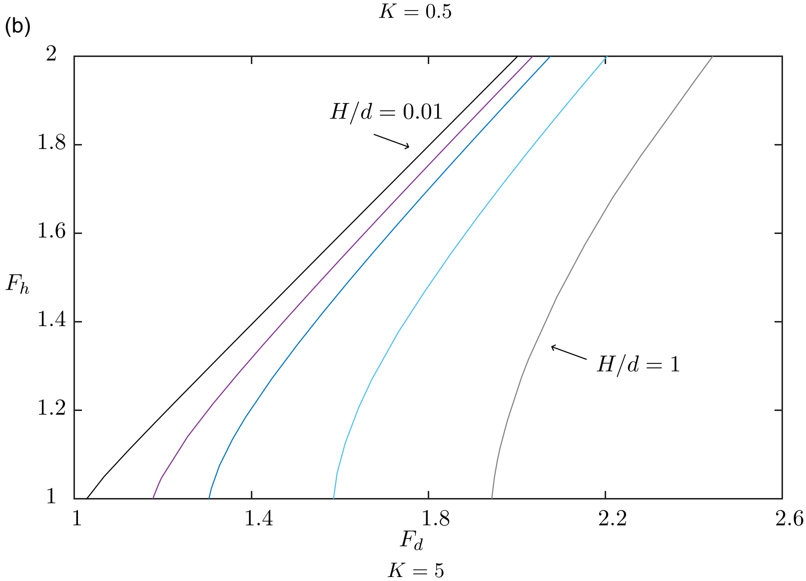

Figure 5. *A plot of F_h against F_d for $N = 200$ and various given values for H/d. From left to right, the lines correspond to setting $H/d = 0.01, 0.1, 0.2, 0.5$ and 1.*

as an unknown to be found. First, we consider the effect of the Froude number. The profiles of Figure 6 depict the wake boundaries and free surface for a given draft d and length H of the plate but with different given Froude numbers. In the case of Figure 6a, we have $F = 8$ where we obtain $K = 0.92$; and in Figure 6b, we have $F = 4$ where we obtain $K = 0.24$. Hence, a smaller Froude number results in a smaller value for K. Earlier, we discussed the correlation between the wake underpressure coefficient K and size of the wake. Therefore, we have that a smaller Froude number results in a larger wake. We also find that as the Froude number F decreases, the ratio between the length of the upper and lower wake boundaries changes. For large Froude numbers, the upper wake boundary is longer than the lower wake boundary, as is the case in Figure 6a. For smaller Froude numbers (i.e. increased gravity), the upper is shorter than the lower wake boundary, as is the case in Figure 6b. Another point of note is that as the Froude number F decreases from $F = 8$ to $F = 4$, the downstream height of the free surface increases. At

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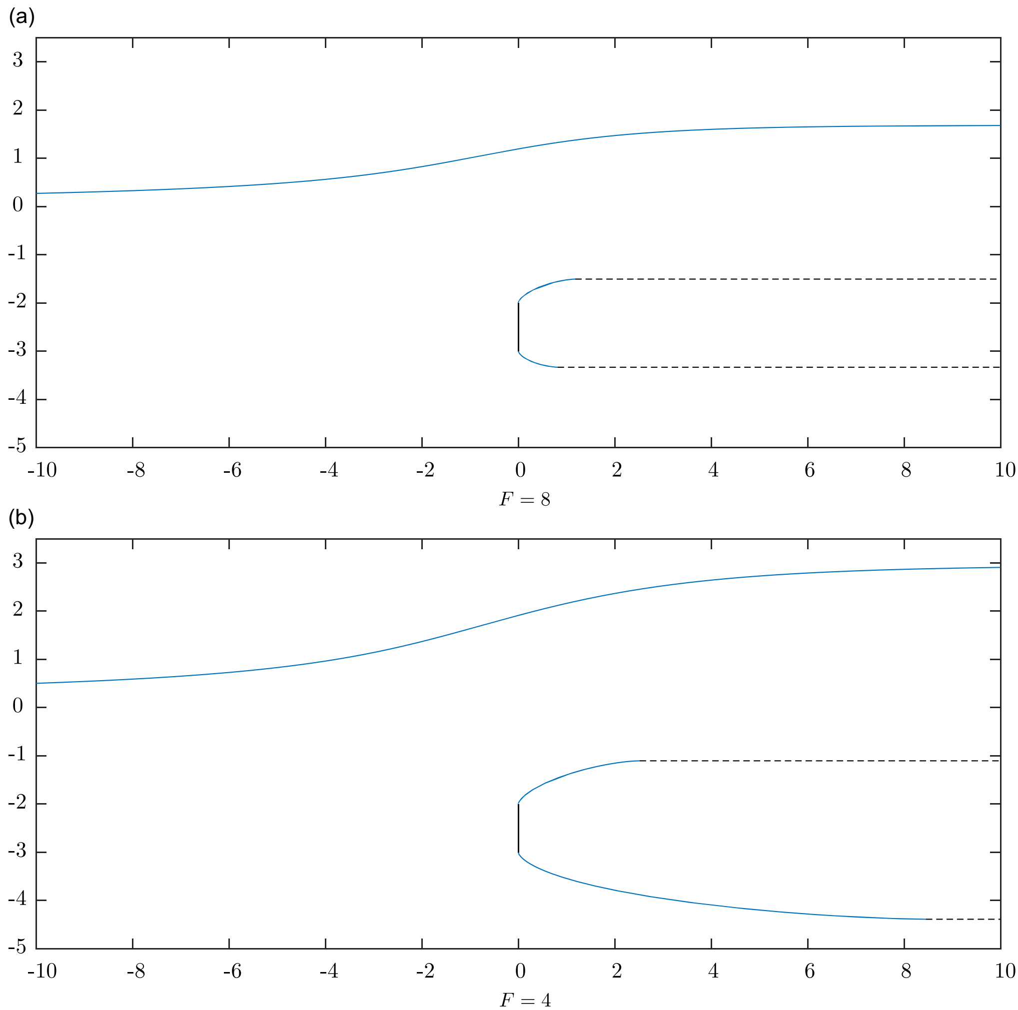

Figure 6. *Profiles to show the wake boundaries and free surface for $N = 200$, $H = 1$ and $d = 3$. Note that the undisturbed free surface is set along $y = 0$. The dashed lines are the horizontal sections CD and C′D′ of the streamlines bounding the wake.*

this point, it is worth recalling that the effects of gravity have been neglected from the wake boundaries. As mentioned above, the smaller Froude number results in a smaller wake underpressure coefficient which correlates to a larger wake and hence a greater disturbance to the free surface.

The effect of varying H is also discussed. We see that, for a given Froude number F and draft d, as the length H of the vertical plate decreases, the width of the wake obviously decreases but the length of the wake increases. This is evident in Figure 7, particularly in Figure 7a and 7b. Similarly, the decreased effect on the free surface of a shorter plate (i.e. smaller H) is also clear. This is due to the vertical plate and wake becoming more slender as H decreases, and so the effect of the body and wake on the flow diminishes. It is worth noting that in the majority of the profiles presented here, it appears that the undisturbed level of the free surface (i.e. $y = 0$) is not reached until very far upstream due to the slow decay of Ω far upstream. However, in the case shown in Figure 7c where the plate is small, we can clearly observe the recovery of the undisturbed flow far upstream.

We also discuss the effect of the draft on the wake flow. For a given Froude number and length of the plate, we find that as d decreases, the resulting wake underpressure coefficient decreases and the size of

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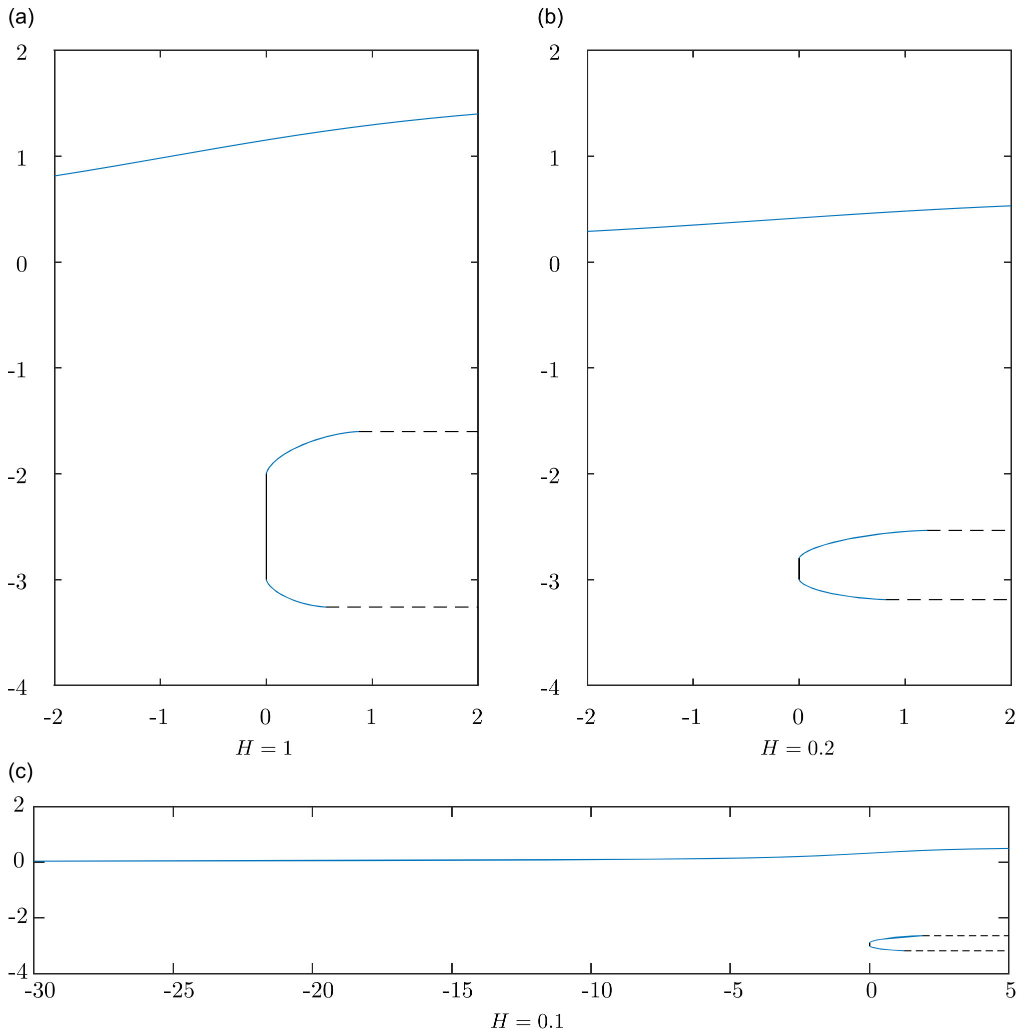

Figure 7. *Profiles to show the wake boundaries and free surface for $N = 200$, $F = 10$ and $d = 3$. Note that the undisturbed free surface is set along $y = 0$. The dashed lines are the horizontal sections CD and C′D′ of the streamlines bounding the wake.*

the wake increases. This is observable from Figure 8 where for $d = 2.9$ and $d = 2.8$ we obtain $K = 0.78$ and $K = 0.57$, respectively. There is also, as expected in the case of a smaller draft, an increased effect on the free surface. There is greater displacement of the free surface from its undisturbed level. The increase in length of the upper wake boundary, in particular, can be understood by noting that we have a given magnitude of the net volume flux of flow above the plate and for a given Froude number (i.e. a given gravitational dominance), we have a set curvature of the jet. Therefore, if the plate is closer to the undisturbed level of the free surface, then a zero flow angle on the upper wake boundary will be attained further from the plate edge. Hence, we have a longer upper wake boundary with a decrease in draft.

Finally, regarding the quality of the numerical solutions, there are several points to consider. Due to the nature of the formulation and method of solution leading to discretised boundary integral equations involving the unknown flow angle along the wake boundaries and free surface, the accuracy of the integrations that are involved should be noted. As discussed earlier, the Cauchy principal values are evaluated by use of an analytical approximation, made possible by the fact that the mesh and collocation

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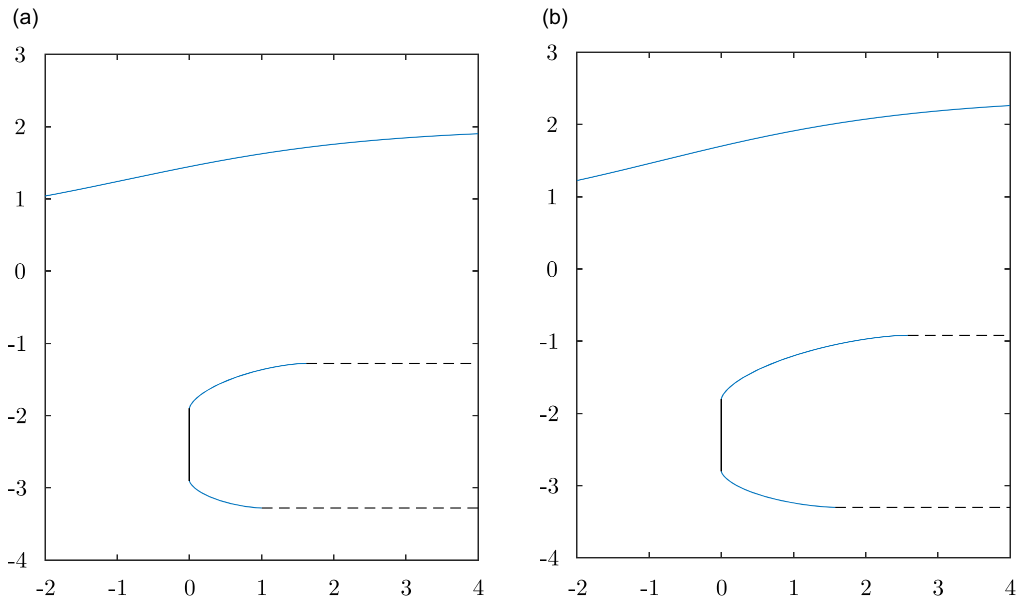

Figure 8. *Profiles to show the wake boundaries and free surface for $N = 200$, $F = 10$ and $H = 1$. Note that the undisturbed free surface is set along $y = 0$. The dashed lines are the horizontal sections CD and C′D′ of the streamlines bounding the wake.*

points do not coincide. Elsewhere in the method where numerical integration is required, the `integral` function of MATLAB is utilised, performed with accuracy of order 10^{-6}. Regarding the overall solution method presented here, we repeated the calculations with various values of N and found that the results were indistinguishable within graphical accuracy for sufficiently large N. All of the results presented here have been obtained by setting $N = 200$, which is sufficiently large to obtain these graphically equivalent solutions.

5. Concluding remarks

The model and results presented here can be used to better understand the flow past a submerged, finite-length plate that is normal to the stream with a wake forming behind the object. Additionally, it is worth emphasising that the open-wake model utilised here is relatively simple to implement, highlighting the potential use and applicability of this approach for investigating wake flows forming behind more arbitrarily-shaped obstacles. In particular, minimal modifications to the calculations and the code for implementing the numerical scheme would be required to extend this work to plates at arbitrary angles of attack to the stream. The numerical results have been validated by comparison with the case where we have zero gravity and infinite fluid; and the results have also been validated by comparison with the ploughing flows of Tuck and Vanden-Broeck [15]. In a more natural formulation of the problem where we set the Froude number, draft and length of the submerged plate, the dependencies of the solution on these parameters have been discussed. Further work will generalise the problem to allow for subcritical, wave solutions.

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Competing interests. The authors declare none.

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