THE EXISTENCE OF PARAMETRIC SURFACE INTEGRALS.

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1. Introduction

In [2] we studied parametric n-surfaces (f, M^n), where M^n was a compact, oriented, topological n-manifold and f a continuous mapping of M^n into the real euclidean k-space R^k $(k \geqq n)$. A definition of bounded variation was given and, for each surface with bounded variation and each projection P from R^k to R^n, a signed measure:

$$\mu(Pf) = \mu_+(Pf) - \mu_-(Pf)$$

was constructed. This measure was used to define a linear type of surface integral:

$$(1)' \qquad \int_{(f,A)} g(x)\, dP(x),$$

over a "measurable" subset A of M^n, as the Lebesgue-Stieltjes integral:

$$\int_A gfd(Pf).$$

In [2] we were only able to prove (except for the special case $k = n$) that the integral (1)′ existed for a continuous g, by placing restrictions (in addition to bounded variation) on the surface (f, M^n). For example, when $k = n + 1$, it was assumed that the subset $f(M^n)$ of R^{n+1} had zero Lebesgue measure.

It is the purpose of this paper to remove this restriction for a special class of surfaces. It will be shown (Theorem 3.7) that, if $n = 2$, M^n is the euclidean 2-sphere

$$S^2 = \{x; x \in R^3 \text{ and } ||x|| = 1\}$$

and (f, S^2) is a surface in R^k $(k \geqq 2)$ such that, for each projection P from R^k to R^2, Pf has bounded variation on S^2, then any bounded, Borel-measurable, real-valued function g on $f(S^2)$ is integrable over (f, S^2) with respect to each projection P. The proof of this theorem depends strongly on some of the results of [1].

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The n-dimensional surface integral studied in [2] is a linear type of integral, hence when $n = 2$ it reduces to

(1) $$\iint_{(f,M^2)} g(x_1, x_2 \cdots, x_k)dx_i dx_j$$

so that (when $n = 2$) it is at best a special case of the general surface integral

(2) $$\iint_S F(x_1, x_2, x_3, dx_2, dx_3, dx_1, dx_3, dx_1 dx_2),$$

which has been defined by Cesari ([1], Appendix B) for every surface $S = (A, T)$, where A is an admissible subset of R^2, T is a continuous mapping of A into R^3 whose projections into the coordinate planes have bounded variation and F is a continuous function on $T(A) \times R^3$ with the property that $F(x, \lambda u) = \lambda F(x, u)$ for all $\lambda > 0$. The Cesari integral has been extended by J. Cecconi to surfaces (S^2, T), where T is a continuous mapping of the euclidean 2-sphere S^2 into R^3, whose projections have bounded variation. The question as to whether the integral (1) is equivalent to a special case of (2) has not yet been answered.

2. Notation

Unless otherwise stated, all concepts relating to parametric surfaces will be as defined in [2]. The real euclidean n-space is denoted by R^n. If $x \in R^n$, then x_j denotes the j-th co-ordinate of x; $(x)_j$ is thus a mapping from R^n to R^1. As in [2], P_j denotes the projection

$$P_j(x_1, \cdots, x_{n+1}) \equiv (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{n+1})$$

of R^{n+1} into R^n and, for $k \geqq n$, the symbol $\mathscr{P}_n^k$ is used to denote the collection of all projections P of R^k into R^n with the form

$$P(x_1, \cdots, x_k) \equiv (x_1, \cdots, x_{j_1-1}, x_{j_1+1}, \cdots, x_{j_2-1}, x_{j_2+1}, \cdots, x_{j_{k-n}}, x_{j_{k-n}+1}, \cdots, x_k).$$

The interior, frontier (or boundary), closure and complement of a subset A of a topological space are denoted by Int (A), Fr (A), $\bar{A}$ and $\sim A$, respectively. $\varnothing$ denotes the empty set. Lebesgue measure is denoted by m.

If f is a continuous mapping of a compact Hausdorff space X into a Hausdorff space Y, following Cesari [1], we define a maximal continuum of constancy of f in X, to be a subset C of X such that $f(C)$ is a single point of Y and C is a component of $f^{-1}\{f(C)\}$.

The collection of all maximal continua of constancy for f in X will be denoted by $\Gamma(f, X)$. The members of this collection are mutually disjoint and their union is X. Each member of the collection is closed, hence compact.

3. The existence of the surface integral

Let f be a continuous mapping of S^2 into R^k, where $k \geqq 2$. Then (f, S^2) is a 2-surface. For each P of $\mathscr{P}_2^k$ and each point y of R^2, it is evident that each member of $\Gamma(f, S^2)$ is either contained in a component of $(Pf)^{-1}(y)$ or does not intersect $(Pf)^{-1}(y)$. For each P of $\mathscr{P}_2^k$, let $Y^{(P)}$ denote the subset of R^2 consisting of all those points y for which the components of $(Pf)^{-1}(y)$ are members of $\Gamma(f, S^2)$.

3.1. THEOREM. *If for all $P \in \mathscr{P}_2^k$, Pf has bounded variation on S^2, then for all $P \in \mathscr{P}_2^k$, $R^2 \sim Y^{(P)}$ has zero measure.*

PROOF. (i) When $k = 2$. In this case the theorem is trivial.

(ii) When $k = 3$. $\mathscr{P}_2^3$ consists of the three projections P_1, P_2, P_3. Consider the unit square

$$A = \{(u, v); 0 \leqq u \leqq 1, 0 \leqq v \leqq 1\}.$$

Let χ be a continuous mapping of A onto S^2 such that $\chi|\text{Int}\,(A)$ is a homeomorphism and $\chi\{\text{Fr}\,(A)\}$ is a single point of S^2. Let T_1, T_2, T_3 be continuous mappings from A to R^2 defined by

$$T_i = P_i f\chi.$$

By II 1.10 of [2], the function $e(T_i, S^2, y)$ is measurable on R^2 with respect to y and its integral is finite. But it follows from II 1.8 of [2] and 12.1 of [1], that $e(T_i, S^2, y)$ is $\geqq$ the function $N(y; T_i, A)$, hence $N(y; T_i, A)$ has a finite integral so that by 12.3 of [1], each T_i has bounded variation in the sense of [1]. Hence, by 16.9 (iii) of [1], there exists for each $i = 1, 2, 3$ a subset X_i of R^2 such that $R^2 \sim X_i$ has zero measure and, for all $y \in X_i$, the components of $T_i^{-1}(y)$ are members of $\Gamma(f\chi, A)$. Now

$$\Gamma(f, S^2) = \{\chi(C); C \in \Gamma(f\chi, A)\}$$

and for each $y \in R^2$, the components of $(P_i f)^{-1}(y)$ are the sets $\chi(D)$, where D is a component cf $T_i^{-1}(y)$. Hence for all $y \in X_i$ the components of $(P_i f)^{-1}(y)$ are elements of $\Gamma(f, S^2)$.
Thus

$$X_i \subseteqq Y^{(P_i)}$$

so that $R^2 \sim Y^{(P_i)}$ has zero measure.

(iii) When $k > 3$. Let P be an arbitrary fixed member of $\mathscr{P}_2^k$. We have to show that

$$R^2 \sim Y^{(P)} \tag{1}$$

has zero measure.

Let $\mathscr{Q}$ denote the collection of all those members Q of $\mathscr{P}_3^k$ for which there exists a projection R of $\mathscr{P}_2^3$ with

$$P = RQ. \tag{2}$$

For each $Q \in \mathscr{Q}$, let

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(3) $$Z^{(Q)}$$

denote the subset of R^2 consisting of all those points y for which the components of $(Pf)^{-1}(y)$ are members of $\Gamma(Qf, S^2)$.

Since $\mathscr{Q}$ is a finite, non-empty collection, the validity of (1) now follows immediately from Lemmas 3.2 and 3.3, which appear below.

3.2 LEMMA.

(4) $$\bigcap_{Q \in \mathscr{Q}} Z^{(Q)} \subseteqq Y^{(P)}.$$

PROOF. Let y be an arbitrary point of the left hand side of (4) and C an arbitrary component of $(Pf)^{-1}(y)$. It will be sufficient to prove that

(5) $$C \in \Gamma(f, S^2).$$

Suppose that $C \notin \Gamma(f, S^2)$. Then $f(C)$ is not a single point of R^k, hence there exist two points a, b of S^2 such that

(6) $$f(a) \neq f(b).$$

Let $i_1 < i_2$ be positive integers such that

$$P(x_1, x_2, \cdots, x_k) \equiv (x_{i_1}, x_{i_2}).$$

Since $Pf(C) = y$, it follows that

$$\{f(a)\}_i = \{f(b)\}_i$$

for $i = i_1, i_2$, hence by (6) there exists an $i_3 \neq i_1, i_2$ such that

(7) $$\{f(a)\}_{i_3} \neq \{f(b)\}_{i_3}.$$

Let Q^* be the projection of $\mathscr{P}_3^k$ defined by

(8) $$Q^*(x_1, \cdots, x_k) \equiv (x_{j_1}, x_{j_2}, x_{j_3}),$$

where j_1, j_2, j_3 are the numbers i_1, i_2, i_3 arranged in ascending order of magnitude. Then

(9) $$Q^* \in \mathscr{Q}$$

and by (7) and (8),

$$Q^*f(a) \neq Q^*f(b),$$

so that $Q^*f(C)$ is not a single point of R^3. Therefore

(10) $$C \notin \Gamma(Q^*f, S^2).$$

But since y lies in the left-hand side of (4) it follows from (3) and (9), that

$$C \in \Gamma(Q^*f, S^2).$$

This contradicts (10), hence the lemma is proved.

3.3 LEMMA. *For all* $Q \in \mathscr{Q}$,

$$R^2 \sim Z^{(Q)}$$

has zero measure.

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PROOF. Let Q' be an arbitrary fixed member of $\mathcal{Q}$. $(Q'f, S^2)$ is evidently a 2-surface in R^3 and each of

$$P_i Q'f \qquad i = 1, 2, 3$$

has bounded variation on S^2. Therefore, if W_i $(i = 1, 2, 3)$ denotes the subspace of R^2 consisting of all points y for which the components of $(P_i Q'f)^{-1}(y)$ are members of $\Gamma(Q'f, S^2)$, it follows from (ii) that

(11) $$R^2 \sim W_i$$

has zero measure.

But by (2),

$$P = P_i Q'$$

for one value of i—say i_0. Take an arbitrary point y of W_{i_0}. Then

$$(P_{i_0} Q'f)^{-1}(y) = (Pf)^{-1}(y),$$

so that the components of $(Pf)^{-1}(y)$ are members of $\Gamma(Q'f, S^2)$; hence, by (3), $y \in Z^{(Q')}$.

Thus

$$W_{i_0} \subseteqq Z^{(Q')},$$

so that, by (11), $R^2 \sim Z^{(Q')}$ has zero measure.

3.4 THEOREM. *If K is a compact subset of a metric space R, C is a component of K and D is a closed subset of K that does not intersect C, then there exists a closed subset H of R such that*

$$C \subseteqq \text{Int}\,(H)$$
$$H \cap D = \varnothing$$

and

$$K \cap \text{Fr}\,(H) = \varnothing.$$

(The interior and frontier are taken in R).

PROOF. There is no component of K that intersects both C and D. Also C is closed. Therefore, by [3] (9.3) p. 15, there exist closed subsets F, G of K such that

(1) $$C \subseteqq F, \qquad D \subseteqq G,$$

(2) $$F \cap G = \varnothing$$

and

(3) $$K = F \cup G.$$

Since a metric space is normal, there exist open sets U, V of R such that

(4) $$F \subseteqq U, \qquad G \subseteqq V$$

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and

(5) $$U \cap V = \varnothing.$$

Put

(6) $$H = \bar{U}.$$

By (1), (4) and (6),

$$C \subseteqq \text{Int}\,(H).$$

It follows from (5) and (6), that

(7) $$H \cap V = \varnothing,$$

hence by (1) and (4),

$$H \cap D = \varnothing.$$

By (4) and (6)

$$F \subseteqq \text{Int}\,(H)$$

and by (4) and (7)

$$G \subseteqq R \sim H,$$

hence by (3)

$$K \subseteqq \text{Int}\,(H) \cup (R \sim H)$$

so that

$$K \cap \text{Fr}\,(H) = \varnothing.$$

3.5. THEOREM, *If. for all $P \in \mathscr{P}_2^k$, Pf has bounded variation on S^2, and if U is an open set of R^k, then for each $P \in \mathscr{P}_2^k$, $f^{-1}(U)$ is measurable $\mu_+(Pf)$ and $\mu_-(Pf)$.*

PROOF. Let P be an arbitrary fixed member of $\mathscr{P}_2^k$. We have to show that

(1) $$f^{-1}(U)$$

is measurable $\mu_+(Pf)$ and $\mu_-(Pf)$. Throughout the proof we will denote these two measures simply by μ_+ and μ_-.

For each positive integer r, denote by $\mathscr{I}_r$ the collection of those open squares of R^2 which have the form

$$\{(x, y); s2^{-r} < x < (s+1)2^{-r}, t2^{-r} < y < (t+1)2^{-r}\}$$
$$s, t = 0, \pm 1, \pm 2, \cdots.$$

Let

(2) $$Z = \{R^2 \sim Y^{(P)}\} \cup \bigcup_{r=1}^{\infty} \bigcup_{I \in \mathscr{I}_r} \text{Fr}\,(I).$$

By 3.1

(3) $$m(Z) = 0$$

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Denote by $\mathscr{D}_r$ the collection of those (open) sets of S^2 each of which is a component of a set $(Pf)^{-1}(I)$, $I \in \mathscr{I}_r$, and each of which is contained in $f^{-1}(U)$. Put

(4) $$\mathscr{D} = \bigcup_{r=1}^{\infty} \mathscr{D}_r$$

Since S^2 is separable, each $\mathscr{D}_r$ is countable, so that $\mathscr{D}$ is also countable. Furthermore, for each $D \in \mathscr{D}$, there exists an r and an $I \in \mathscr{I}_r$ such that $Pf\{\text{Fr}\,(D)\} \subseteqq \text{Fr}\,(I)$. Hence D is a member of the ring $\mathscr{R}(Pf)$ defined in [2] II 1.2. Thus, each $D \in \mathscr{D}$ is measurable μ_+ and μ_-, so that the subset

(5) $$A = \bigcup_{D \in \mathscr{D}} D$$

of S^2 is measurable μ_+ and μ_-. Evidently

(6) $$A \subseteqq f^{-1}(U).$$

We will now prove that

(7) $$f^{-1}(U) \subseteqq A \cup (Pf)^{-1}(Z).$$

To prove this, let a be an arbitrary point of $f^{-1}(U)$ that is not in $(Pf)^{-1}(Z)$. Then

(8) $$f(a) \in U$$

and

(9) $$Pf(a) \notin Z.$$

By (2) and (9), $Pf(a) \in Y^{(P)}$ so that the components of

(10) $$(Pf)^{-1}Pf(a)$$

are members of $\Gamma(f, S^2)$. Let C be the component of $(Pf)^{-1}Pf(a)$ that contains a.

Then $f(C) = f(a)$; hence C does not intersect the closed set $f^{-1}(R^2 \sim U)$. Consequently C is a component of

$$K = [(Pf)^{-1}\{Pf(a)\}] \cup f^{-1}(R^2 \sim U).$$

By putting $D = f^{-1}(R^2 \sim U)$ and applying Theorem 3.4, one can see that there exists a closed subset H of S^2 such that

(11) $$C \subseteqq \text{Int}\,(H),$$
$$H \cap f^{-1}(R^2 \sim U) = \varnothing$$

and

$$\text{Fr}\,(H) \cap [(Pf)^{-1}\{Pf(a)\} \cup f^{-1}(R^2 \sim U)] = \varnothing.$$

Then

(12) $$H \subseteqq f^{-1}(U)$$

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and

(13) $$Pf(a) \notin Pf\{\mathrm{Fr}\,(H)\}.$$

By (13), there exists a positive integer r' such, that $2^{-r+\frac{1}{2}}$ is less than the distance between $Pf(a)$ and $Pf\{\mathrm{Fr}(H)\}$. By (9), there exists an $I \in \mathscr{I}_{r'}$ with $Pf(a) \in I$ and

(14) $$I \cap Pf\{\mathrm{Fr}\,(H)\} = \varnothing.$$

Let E be the component of $(Pf)^{-1}(I)$ containing a. Then E does not intersect $\mathrm{Fr}\,(H)$, hence $E \subseteqq H$, so that by (12),

(15) $$E \subseteqq f^{-1}(U).$$

By (15), $E \in \mathscr{D}$, hence by (5), $a \in A$.
Thus (7) is true.

It follows from (3) and [2] II 1.16 that

$$\mu_+\{(Pf)^{-1}(Z)\} = \mu_-\{(Pf)^{-1}(Z)\} = 0,$$

so that by (5), (6) and (7), $f^{-1}(U)$ is measurable μ_+ and μ_-. This completes the proof.

3.6. Theorem. *If, for all* $P \in \mathscr{P}_2^k$, *Pf has bounded variation on* S^2, *and if* U *is a Borel set of* R^k, *then for each* $P \in \mathscr{P}_2^k$, $f^{-1}(U)$ *is measurable* $\mu_+(Pf)$ *and* $\mu_-(Pf)$.

Proof. Let P' be a fixed projection of $\mathscr{P}_2^k$. Denote by $\mathscr{R}$, the σ-ring consisting of all those subsets of S^2 that are measurable $\mu_+(P'f)$ and $\mu_-(P'f)$. Let $\mathscr{S}$ be the collection of all those subsets A of R^k, such that $f^{-1}(A) \in \mathscr{R}$. $\mathscr{S}$ is a σ-ring; because, if $A_1, A_2, \cdots \in \mathscr{S}$, then

$$f^{-1}\{\bigcup_i A_i\} = \bigcup_i f^{-1}(A_i) \in \mathscr{R},$$

and if $A, B \in \mathscr{S}$ with $B \subseteqq A$, then

$$f^{-1}(A \sim B) = f^{-1}(A) \sim f^{-1}(B) \in \mathscr{R}.$$

It follows from 3.5, that every open set of R^k is a member of $\mathscr{S}$; hence the Borel set U is in $\mathscr{S}$; i.e. $f^{-1}(U) \in \mathscr{R}$. Thus $f^{-1}(U)$ is measurable $\mu_+(P'f)$ and $\mu_-(P'f)$.

3.7. Theorem. *If, for all* $P \in \mathscr{P}_2^k$, *Pf has bounded variation on* S^2 *and if* g *is a bounded, Borel-measurable, real-valued function on* $f(S^2)$, *then for each* $P \in \mathscr{P}_2^k$, *the surface integral*

(1) $$\int_{(f,S^2)} g(x)dP(x)$$

exists.

Proof. In [2] II 2.2 the surface integral (1) is defined to be the Lebesgue-

Stieltjes integral

$$\int_{S^2} gfd\mu(Pf) = \int_{S^2} gfd\mu_+(Pf) - \int_{S^2} gfd\mu(Pf)$$

and it follows immediately from 3.6, that this latter integral exists.

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