J. Austral. Math. Soc. (Series A) **53** (1992), 219–228

TOTAL CHROMATIC NUMBER OF GRAPHS OF HIGH DEGREE, II

H. P. YAP and K. H. CHEW

(Received 30 January 1990; revised 31 October 1990)

Communicated by L. Caccetta

Abstract

We prove Theorem 1: suppose G is a simple graph of order n having $\Delta(G) = n - k$ where $k \geq 5$ and $n \geq \max(13, 3k - 3)$. If G contains an independent set of $k - 3$ vertices, then the TCC (Total Colouring Conjecture) is true. Applying Theorem 1, we also prove that the TCC is true for any simple graph G of order n having $\Delta(G) = n - 5$. The latter result together with some earlier results confirm that the TCC is true for all simple graphs whose maximum degree is at most four and for all simple graphs of order n having maximum degree at least $n - 5$.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 05 C 15.
Keywords and phrases: total colouring, total chromatic number.

1. Introduction

Throughout this paper, all graphs are finite, simple and undirected. Let G be a graph. We denote its vertex set, edge set, complementary graph, chromatic index, the minimum degree of its vertices, and the maximum degree of its vertices by $V(G)$, $E(G)$, $\overline{G}$, $\chi_1(G)$, $\delta(G)$, and $\Delta(G)$ respectively. A well-known theorem of Vizing says that $\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1$. A graph G is said to be *Class* 1 (respectively *Class* 2) if $\chi_1(G) = \Delta(G)$ (respectively $\Delta(G)+1$). If $x \in V(G)$, we denote by $N_G(x)$ (or simply $N(x)$) the neighbourhood of x and $d_G(x)$ (or simply $d(x)$) the degree of x. A vertex x of G is called a *major vertex* if $d(x) = \Delta(G)$. A vertex y of G is called a *minor vertex* if y is not a major vertex. The subgraph of G induced by all its major vertices is called the *core* of G and is denoted by G_Δ. If $F \subseteq E(G)$, then $G - F$ is the graph obtained from G by deleting F from G. If $S \subseteq V(G)$, then $G[S]$ and $G - S$ denote the subgraphs of G induced by S and $V(G)\backslash S$ respectively. However, we write $G[v_1, v_2, \dots, v_k]$ instead of $G[\{v_1, v_2, \dots, v_k\}]$. We write $H \leq G$ if H is an induced subgraph of G. A subset S of vertices of G is said to be *independent* (in G) if no two vertices of S are adjacent in G. A graph G is *maximal* if for any two nonadjacent vertices of G at least one of them is a major vertex. The null graph of order m and the complete graph of order n are denoted by O_m and K_n respectively.

A *total colouring* π of a graph G is a mapping $\pi: V(G) \cup E(G) \to \{1, 2, \dots\}$ such that

(i) no two adjacent vertices or edges have the same image;
(ii) the image of each vertex x is distinct from the images of its incident edges.

The *total chromatic number* $\chi_T(G)$ of a graph G is the smallest integer k such that G has a total colouring π having image set $\{1, 2, \dots, k\}$.

From the definition of total chromatic number, it is clear that $\chi_T(G) \geq \Delta(G) + 1$. Behzad [1] and Vizing [9, 11] independently made the following conjecture.

TOTAL COLOURING CONJECTURE. *For any graph G, $\chi_T(G) \leq \Delta(G) + 2$.*

This conjecture was proved for complete graphs by Behzad, Chartrand and Cooper [2]; for graphs G having $\Delta(G) \leq 3$ by Rosenfeld [7] and Vijayaditya [8]; for graphs G having $\Delta(G) = 4$ by Kostochka [6]; for complete 3-partite graphs, complete balanced r-partite graphs by Rosenfeld [7]; for complete r-partite graphs by Yap [13]; for regular graphs of high degree by Chetwynd and Hilton [5]; and for graphs of order n having $\Delta(G) \geq n - 4$ by Yap, Wang and Zhang [14]. The main results of this paper are stated in the abstract above. The proof technique we use here is basically the same as that used in [14]. However, we introduce a few new techniques in settling some special cases when n is small.

2. Preliminary results

We shall apply the following results.

THEOREM 2.1 (Rosenfeld [7], Vijayaditya [8] and Kostochka [6]). *For any graph G having $\Delta(G) \leq 4$, $\chi_T(G) \leq \Delta(G) + 2$.*

LEMMA 2.2. *For any subgraph* H *of a graph* G, $\chi_T(H) \le \chi_T(G)$.

Lemma 2.2 requires no proof. This lemma is often applied implicitly.

LEMMA 2.3. *If* G_Δ *is a forest, then* G *is Class* 1.

Lemma 2.3 follows from some of Vizing's results (see for instance, [12, Theorem 3.3 and Corollary 3.6]). [14, Lemma 2.2] can be restated as follows.

LEMMA 2.4. *Let* G *be a graph of order* n *and let* $\Delta = \Delta(G)$. *If* G *contains an independent set* S *where* $|S| \ge n - \Delta - 1$, *then* $\chi_T(G) \le \Delta + 2$.

THEOREM 2.5 (Yap, Wang and Zhang [14]). *For any graph* G *of order* n *having* $\Delta(G) \ge n - 4$, $\chi_T(G) \le \Delta(G) + 2$.

THEOREM 2.6 (Chetwynd and Hilton [3]). *Let* G *be a connected graph of order* n *with three major vertices. Then* G *is Class* 2 *if and only if each of the three major vertices is of degree* $n - 1$ *and the remaining vertices have degree* $n - 2$.

THEOREM 2.7. *Let* G *be a bipartite graph with bipartition* (X, Y). *Then* G *contains a matching that saturates every vertex in* X *if and only if* $|N(S)| \ge |S|$ *for all* $S \subseteq X$.

(A proof of Theorem 2.7 can be found in many textbooks on graph theory.)

THEOREM 2.8 (Chetwynd and Hilton [4]). *Let* G *be a connected graph having four major vertices. Then* G *is Class* 2 *if and only if for some integer* n, *one of the following holds (for definition of valency-list, see* [12]):

(i) *the valency-list of* G *is* $(2n-2)^{2n-3}(2n-1)^4$;

(ii) *the valency-list of* G *is* $(2n-2)(2n-1)^{2n-4}(2n)^4$;

(iii) G *contains a bridge* e *such that* $G - e$ *is the union of two graphs* G_1 *and* G_2 *where* $\Delta(G_1) \le 2m - 1$ *for some integer* $m < n$ *and the valency-list of* G_2 *is either* $(2m-2)(2m-1)^{2m-4}(2m)^4$ *or* $(2m-1)^{2m-2}(2m)^3$.

3. Proof of main results

We shall now apply the above preliminary results to prove the main theorem (Theorem 3.2). We prove a general result first.

LEMMA 3.1. *Let G be a graph of order n having $\Delta(G) = n - k$, $k \geq 5$ and $n \geq 3k - 4$. If G is maximal and $G \geq O_{k-2}$, then $\chi_T(G) \leq \Delta(G) + 2$.*

PROOF. By Lemma 2.4, we can assume that $G \not\geq O_{k-1}$. Let $T = \{x_1, x_2, \dots, x_{k-2}\}$ be a set of independent vertices in G. Let V_1 be the set of major vertices of G. Suppose $u, v \in V(G)\backslash T$ are not adjacent in G. Let M_1 be a matching in $H = G - T$ satisfying

(1) all the major vertices in $\{u, v\}$ are M_1-saturated;

(2) $|V(M_1) \cap V_1|$ is maximum among all the matchings in H satisfying (1).

We now prove that $V_1\backslash T$ contains at most one M_1-unsaturated vertex. Suppose otherwise. Let $z_1, z_2 \in V_1\backslash T$ be M_1-unsaturated. Since $\Delta(G) = n - k$, z_1 is adjacent to at least $n - 2k + 2$ vertices $a_1, a_2, \dots, a_{n-2k+2}$ in H. Clearly each a_i is M_1-saturated. Let $b_i \in V(H)$ be such that $a_ib_i \in M_1$, $i \in I = \{1, 2, \dots, n - 2k + 2\}$. Clearly $b_i \neq z_2$ for any $i \in I$ but b_i can be a_j for some $j \neq i$. Suppose $z_2b_i \in E(G)$ for some $i \in I$. Then replacing a_ib_i in M_1 by $\{z_1a_i, z_2b_i\}$ we obtain a larger matching M_1' satisfying (1) but contradicting (2). Hence $z_2b_i \notin E(G)$, for all $i \in I$. (We call this argument the "*Enlarge Matching Argument.*") Since $z_1, b_1, b_2, \dots, b_{n-2k+2} \notin N(z_2)$, we have $n - k = d(z_2) \leq n - (n - 2k + 2 + 2) = 2k - 4$. Hence $n = 3k - 4$, and any other vertex of G is adjacent to z_2. Let $A = \{a_1, a_2, \dots, a_{n-2k+2}\}$, $B = \{b_1, b_2, \dots, b_{n-2k+2}\}$. Suppose $A \neq B$. We first consider the case $A \cap B \neq \varnothing$. By symmetry, there are vertices $c_i, c_j \in N(z_2)$ such that $c_ic_j \in M_1$. Let C be such a set of vertices. By the "Enlarge Matching Argument," each $b_i \in B\backslash A$ is not adjacent to any $b_r \in B$ and any $c_s \in C$. It is also not adjacent to z_1 and z_2. Hence $d(b_i) \leq (n - 1) - (n - 2k + 1) - 4 = 2k - 6$. However replacing a_ib_i in M_1 by z_1a_i we have another matching M_1' satisfying conditions (1) and (2) and thus $b_i \in V_1$. Now $2k - 6 \geq d(b_i) = n - k$ contradicts the assumption that $n \geq 3k - 4$. Hence $A \cap B = \varnothing$. Now since $(n - 2k + 2) + 1 \geq k - 1$ and $G[b_1, b_2, \dots, b_{n-2k+2}, z_2] \geq O_{k-1}$, we have another contradiction. Hence $V_1\backslash T$ contains at most one M_1-unsaturated vertex, say z.

Let $G_1 = G - M_1$, $H_1 = G_1 - \{u, v\}$ and $V_2 = \{h \in V(H_1) | d_{G_1}(h) \geq \Delta - 1\}$. Since G is maximal and $G[T] = O_{k-2}$, T contains at most one minor vertex. Now let $Y = V(G)\backslash(T \cup \{u, v\})$. For any $S \subseteq T$ where S consists of only major vertices, let $s = |S|$ and $r = |N(S) \cap Y|$. Then $s(n - k - 2) \leq r(k - 2)$ from which it follows that $r \geq s$. Hence by Theorem 2.7, H_1 contains a matching M_2 satisfying

(3) z and all the major vertices in T are M_2-saturated;

(4) $|V(M_2) \cap V_2|$ is maximum among all matchings in H_1 satisfying (3).

We now show that V_2 contains at most two M_2-unsaturated vertices. Suppose otherwise. Let $w_1, w_2, w_3 \in V_2$ be M_2-unsaturated. Then w_1 is adjacent to at least $n-k-3$ vertices $a_1, a_2, \dots, a_{n-k-3}$ in H_1. Clearly each a_j is M_2-saturated. Let $b_j \in V(G)$ be such that $a_jb_j \in M_2$, $j \in J = \{1, 2, \dots, n-k-3\}$. By the "Enlarge Matching Argument," $w_2b_j, w_3b_j \notin E(G_1)$ for all $j \in J$. Thus $n-k-1 \le d_{G_1}(w_i) \le n-(n-k-3+3) = k$, for each $i = 2, 3$. Since $n \ge 3k-4$ and $k \ge 5$, we now have $k = 5$ and $n = 3k-4 = 11$. Finally, for $n = 11$, $5 = d_{G_1}(w_i)$ $(i = 2, 3)$ implies that $w_i \in N(a_j)$ for all $i = 2, 3$, $j = 1, 2, 3$. If $b_ib_j \in E(G_1)$ for some $i \neq j$, then replacing $\{a_ib_i, a_jb_j\}$ in M_2 by $\{w_1a_i, b_ib_j, w_2a_j\}$ we obtain a larger matching M_2' satisfying (3) but contradicting (4). Hence $b_ib_j \notin E(G_1)$ for all i, j. Since $w_i, b_i \in V_2$ for all $i = 1, 2, 3$ $(b_i \in V_2$ because we can replace a_ib_i in M_2 by $w_1a_i)$, it follows that $w_i, b_i \in N(u)$, $N(v)$ for all $i = 1, 2, 3$. Hence $d_{G_1}(u)$, $d_{G_1}(v) \ge 6$ and so either $d(u) \ge 7$ or $d(v) \ge 7$ (because either u or v is a major vertex), which contradicts the fact that $\Delta(G) = 11-5 = 6$. Hence V_2 contains at most two M_2-unsaturated vertices, say w and w'.

Now let G^* be the graph obtained from $G_2 = G_1 - M_2$ by adding a new vertex c^* and adding an edge joining c^* to each vertex in $V(G)\backslash(T\cup\{u, v\})$. Then $\Delta(G^*) = n-k$ and G^* has at most four major vertices. By Theorem 2.6 and Theorem 2.8, $\chi_1(G^*) \le n-k$. Now any $(n-k)$-edge colouring π can be modified to yield a total colouring φ of G using $(n-k)+2$ colours. Hence $\chi_T(G) \le n-k+2$.

Theorem 3.2. *Let G be a graph of order n having $\Delta(G) = n-k$, where $k \ge 5$ and $n \ge \max(13, 3k-3)$. If G is maximal and $G \ge O_{k-3}$, then $\chi_T(G) \le \Delta(G)+2$.*

Proof. Let $\Delta = \Delta(G)$. By Lemma 3.1, we also assume that $G \not\ge O_{k-2}$. Let $T = \{v_1, v_2, \dots, v_{k-3}\}$ be a set of independent vertices in G. Let V_1 be the set of major vertices of G. Since $G \not\ge O_{k-2}$ and $\delta(\overline{G}) = n-1-\Delta = k-1$, we can find two pairs of nonadjacent vertices $\{x_i, y_i\}$, $i = 1, 2$, disjoint from T. Since G is maximal, we may assume that $x_1, x_2, v_1, \dots, v_{k-4}$ are major vertices. We first show that $H = G - T$ contains a matching M_1 satisfying

(5) all the major vertices in $\{x_1, y_1, x_2, y_2\}$ are M_1-saturated;

(6) $|V(M_1) \cap V_1|$ is maximum among all the matchings in H satisfying (5).

We know that each major vertex of G is of degree $n-k$. Hence each major vertex in $S = \{x_1, y_1, x_2, y_2\}$ is adjacent to at least $(n-k)-(k-3)-2 =$

$n-2k+1 \geq 4$ vertices in $V(G)\backslash(T \cup S)$. Hence, by Theorem 2.7, H contains a matching M_1 satisfying (5).

We now prove that $V_1\backslash T$ contains at most one M_1-unsaturated vertex. Suppose otherwise. Let $u_1, u_2 \in V_1\backslash T$ be two M_1-unsaturated vertices. As in the proof of Lemma 3.1, u_1 is adjacent to at least $n-2k+3$ vertices in H, whence $d(u_2) \leq n-(n-2k+3+2) = 2k-5$, a contradiction. Hence $V_1\backslash T$ contains at most one M_1-unsaturated vertex, say u, in H.

Let $G_1 = G - M_1$, $H_1 = G_1 - \{x_1, y_1\}$ and $V_2 = \{h \in V(H_1) | d_{G_1}(h) \geq \Delta - 1\}$. We now show that H_1 contains a matching M_2 satisfying

(7) u and all the major vertices in $T \cup \{x_2, y_2\}$ are M_2-saturated;

(8) $|V(M_2) \cap V_2|$ is maximum among all the matchings in H_1 satisfying (7).

We know that $G[v_1, v_2, \dots, v_{k-3}, x_2, y_2] \not\supseteq O_{k-2}$. Hence there exist integers $r \neq s$ such that x_2v_r, $y_2v_s \in E(G)$. Now each major vertex in $T\backslash\{v_r, v_s\}$ is adjacent to at least $n-k-4 \geq 2k-7 \geq k-2$ vertices in $V(G)\backslash(T \cup S)$. By Theorem 2.7, there exists a matching M_2' that saturates all the major vertices in $T\backslash\{v_r, v_s\}$ in the bipartite graph having bipartition $T\backslash\{v_r, v_s\}$ and $V(G)\backslash(T \cup S)$. If u is adjacent to some $v_t \in T\backslash\{v_r, v_s\}$ and v_t is incident with an edge e of M_2', we put $M_2 = (M_2'\backslash\{e\}) \cup \{uv_t\}$. Otherwise since $d(u) = \Delta$ and $\Delta - (k-5) - 6 \geq k-4 \geq 1$, u is adjacent to some $u' \in V(G)\backslash(V(M_2') \cup S \cup \{v_r, v_s\})$ and we can take $M_2 = M_2' \cup \{uu'\}$.

We now show that V_2 contains at most one M_2-unsaturated vertex. Suppose otherwise. Let $z_1, z_2 \in V_2$ be M_2-unsaturated. Now z_1 is adjacent to at least $\Delta-3$ vertices in H_1. Thus $\Delta - 1 \leq d_{G_1}(z_2) \leq n-(\Delta-3+2) = k+1$, from which it follows that $n \leq 2k+2$, a contradiction. Hence V_2 contains at most one M_2-unsaturated vertex, say z.

Let $G_2 = G_1 - M_2$, $H_2 = G_2 - \{x_2, y_2\}$ and $V_3 = \{h \in V(H_2) | d_{G_2}(h) \geq \Delta - 2\}$. We shall show that H_2 contains a matching M_3 satisfying

(9) every vertex in $\{v_1, v_2, \dots, v_{k-3}, u, z, x_1, y_1\}$ whose degree in G_2 is $\Delta - 1$ is M_3-saturated;

(10) $|V(M_3) \cap V_3|$ is maximum among all the matchings in H_2 satisfying (9).

We know that $G[v_1, v_2, \dots, v_{k-3}, x_1, y_1] \not\supseteq O_{k-2}$. Hence there exist integers $r' \neq s'$ such that $x_1v_{r'}$, $y_1v_{s'} \in E(G)$. Now, each major vertex in $T\backslash\{v_{r'}, v_{s'}\}$ is adjacent to at least $(n-k-1)-4 \geq 2k-8 \geq k-3$ vertices in $V(G)\backslash(T \cup S)$. Note that $d_{G_2}(u) = \Delta - 1$ and $d_{G_2}(z) = \Delta - 1$. Hence, as before and by Theorem 2.7, H_2 contains a matching M_3 satisfying (9).

We next show that V_3 contains at most one M_3-unsaturated vertex. Suppose otherwise. Let $w_1, w_2 \in V_3$ be M_3-unsaturated. As before, w_1 is adjacent to at least $\Delta - 4$ vertices $a_1, a_2, \dots, a_{\Delta-4}$ in H_2 and each a_i is M_3-saturated. Let $b_i \in V(H_2)$ be such that $a_ib_i \in M_3$, $i = 1, 2, \dots, \Delta-4$. By the "Enlarge Matching Argument," $b_iw_2 \notin E(G_2)$ for all $i = 1, 2, \ldots, \Delta - 4$. Thus $d_{G_2}(w_2) \leq n - (\Delta - 4) - 2 = k + 2$. Now $2k - 5 \leq n - k - 2 = \Delta - 2 \leq d_{G_2}(w_2) \leq k + 2$ implies that $k \leq 7$ and $n \leq 2k + 4$.

We consider three cases separately.

Case 1. $b_i \notin A = \{a_1, a_2, \ldots, a_{\Delta-4}\}$ for $i = 1, 2, \ldots, \Delta - 4$.

We first note that if $k = 7$, then from the inequality $2k - 5 \leq \Delta - 2 \leq d_{G_2}(w_2) \leq k + 2$, it follows that $d_{G_2}(w_2) = \Delta - 2$ and thus w_2 is adjacent to every vertex in A; also if $k = 6$ or $k = 5$ (and thus $n \geq 13$), then w_2 is not adjacent to at most one vertex in A. Hence if $b_ib_j \in E(G_2)$ for some $1 \leq i < j \leq \Delta - 4$, then replacing $\{a_ib_i, a_jb_j\}$ in M_3 by $\{w_2a_i, w_1a_j, b_ib_j\}$ (or $\{w_2a_j, w_1a_i, b_ib_j\}$), we obtain a larger matching M_3' satisfying (9) but contradicting (10). Hence $G_2[w_1, w_2, b_1, b_2, \ldots, b_{\Delta-4}] = O_{\Delta-2}$. Since $(M_1 \cup M_2) \cap E(G[w_1, w_2, b_1, b_2, \ldots, b_{\Delta-4}])$ is a union of paths and even cycles, and $\Delta - 2 = n - k - 2 \geq 2k - 5$, we have $G[w_1, w_2, b_1, b_2, \ldots, b_{\Delta-4}] \geq O_{k-2}$, a contradiction to the assumption.

Case 2. $b_i \in A$ and $b_j \notin A$ for some $1 \leq i, j \leq \Delta - 4$.

Suppose $b_1, b_2 \in A$, that is, $b_1 = a_2$ and $b_2 = a_1$, and suppose $b_{\Delta-4} \notin A$. By symmetry, we also have $C = \{c_1, c_2, \ldots, c_{\Delta-r}\} \subseteq N_{H_2}(w_2)$, $\{c_1c_2, c_3d_3, \ldots, c_{\Delta-4}d_{\Delta-4}\} \subseteq M_3$ and $d_{\Delta-4} \notin C$. First suppose $c_{\Delta-4} = a_{\Delta-4}$, say. Then $d_{\Delta-4} = b_{\Delta-4}$ and by the "Enlarge Matching Argument," $w_1, w_2, a_1, a_2, c_1, c_2, b_3, \ldots, b_{\Delta-5}, d_3, \ldots, d_{\Delta-5} \notin N_{H_2}(b_{\Delta-4})$. (Note that b_i may be equal to d_j for some i and j such that $3 \leq i \leq j \leq \Delta - 5$.) Hence

$$2k - 5 \leq n - k - 2 = \Delta - 2 \leq d_{G_2}(b_{\Delta-4}) \leq n - (\Delta - 5) - 5 = k,$$

from which it follows that $k = 5$ and $n \leq 12$, a contradiction. Next, suppose $C \cap A = \varnothing$. Then the minimum order of G is attained when $b_{\Delta-4}$ is the only vertex in $B = \{b_1, b_2, \ldots, b_{\Delta-4}\}$ such that $b_{\Delta-4} \notin A$. Thus $2(\Delta-5)+8 \leq n$ (note that $|\{x_2, y_2, w_1, w_2, a_{\Delta-4}, b_{\Delta-4}, c_{\Delta-4}, d_{\Delta-4}\}| = 8$), from which it follows that $k = 5$ and $n = 12$, again a contradiction.

Case 3. $b_j \in A$ for $j = 1, 2, \ldots, \Delta - 4$.

In this case $\Delta - 4$ is even. We consider the following subcases separately.

(i) $k = 7$. From $18 = 3k - 3 \leq n \leq 2k + 4 = 18$ we have $n = 18$. However, we now also have $\Delta - 4 = n - k - 4 = n - 11 = 7$ is odd, which is a contradiction.

(ii) $5 \leq k \leq 6$. By the "Enlarge Matching Argument," $d_{G_2}(a_i) \geq n - k - 2$, $i = 1, 2, \ldots, \Delta - 4$, and every a_i is not adjacent to any $v \in V(G_2) \backslash (\{w_1, x_2, y_2\} \cup A)$. Hence $G_2[w_1, a_2, a_2, \ldots, a_{\Delta-4}] = K_{\Delta-3}$. Similarly, we have $G_2[w_2, c_1, c_2, \ldots, c_{\Delta-4}] = K_{\Delta-3}$. Thus each vertex in $W = \{w_1, w_2\} \cup A \cup C$ must be adjacent to both x_2 and y_2, and consequently

$2\Delta - 6 \leq d_{G_2}(x_2) \leq \Delta - 2$ from which it follows that $\Delta \leq 4$, a contradiction. Hence V_3 has at most one M_3-unsaturated vertex.

Finally, let G^* be obtained from $G_3 = G_2 - M_3$ by adding a new vertex c^* and adding an edge joining c^* to each vertex in $V(G_3)\backslash(T \cup \{x_1, x_2, y_1, y_2\})$. Then $\Delta(G^*) = \Delta - 1$, and by the choices of M_1, M_2 and M_3, G^* has at most four major vertices. Hence, by Theorem 2.6 and Theorem 2.8, $\chi_1(G^*) = \Delta - 1$ and if π is a $(\Delta - 1)$-edge colouring of G^*, then we can modify π to a total colouring φ of G using $(\Delta - 1) + 3$ colours. Hence $\chi_T(G) \leq (\Delta - 1) + 3 = \Delta + 2$.

The proof of Theorem 3.2 is complete.

We shall also require the following lemmas.

LEMMA 3.3. *Let G be a graph of order* 10 *having* $\Delta(G) = 5$. *If G is maximal and $G \geq O_3$, then $\chi_T(G) \leq \Delta(G) + 2$.*

PROOF. The proof is almost identical to the proof of Lemma 3.1. Hence we need only to modify some parts of the proof of Lemma 3.1.

In the case of showing that V_1 contains at most one M_1-unsaturated vertex, we let $\{x\} = V(G)\backslash(T \cup \{z_1, z_2, a_1, a_2, b_1, b_2\})$ if $b_i \notin A = \{a_1, a_2\}$. Then clearly $xz_2 \notin E(G)$. Hence $a_1, a_2 \in N(z_2)$. Also since $G[b_1, b_2, z_1, z_2] \neq O_4$, we have $b_1b_2 \in E(G)$. Consequently, by the "Enlarge Matching Argument," $a_1, a_2 \notin N(z_2)$, a contradiction. On the other hand, if $b_i \in A$, we let $c_1, c_2 \in N_H(z_2)$. Then by the "Enlarge Matching Argument," a_1, a_2, c_1, c_2, z_1 and z_2 are all adjacent to each vertex in T, which is false. Hence V_1 contains at most one M_1-unsaturated vertex, z say.

In the case of showing that V_2 contains at most two M_2-unsaturated vertices, we now have $n - k - 1 \leq d_{G_1}(w_i) \leq k$ for each $i = 2, 3$.

Again, we let $\{x\} = V(G)\backslash\{u, v, w_1, w_2, w_3, a_1, a_2, b_1, b_2\}$. Then x is M_2-unsaturated and $xw_i \notin E(G_1)$ for all $i = 1, 2, 3$. By the "Enlarge Matching Argument," $b_1, b_2 \in V_2$, and $a_1, a_2 \in N_H(w_j)$, $j = 2, 3$. Hence $xb_1, xb_2, b_1b_2 \notin E(G_1)$. Thus $b_1, b_2, w_1, w_2, w_3 \in N(u) \cap N(v)$ and consequently either $d(u) \geq 6$ or $d(v) \geq 6$, a contradiction.

LEMMA 3.4. *Let G be a graph of order $n = 10, 11$ or 12. Suppose $\Delta(G) = n - 5$, G is maximal and $G \not\geq O_3$. Then $\chi_T(G) \leq \Delta(G) + 2$.*

PROOF. We shall apply the technique used in the proof of Theorem 3.2 by explicitly constructing out the three matchings M_1, M_2 and M_3.

Let $u_1 \in V(G)$ be such that $d(u_1) = \delta = \delta(G)$ and let $v \in V(G)$ be such that $u_1 v \notin E(G)$. Thus v is a major vertex. Let $N_{\overline{G}}(v) = \{u_1, u_2, u_3, u_4\}$ and $N_G(v) = \{v_1, v_2, \dots, v_{n-5}\}$. Since $G \geq O_3$, $G[u_1, u_2, u_3, u_4] = K_4$. If $d(u_1) = 3$, then $N_{\overline{G}}(u_1) = \{v, v_1, v_2, \dots, v_{n-5}\}$ forms a clique in G, that is, $G = K_4 \cup K_{n-4}$. Thus $\chi_T(G) \leq \Delta(G) + 2$. Hence we assume that $d(u_1) = \delta \geq 4$. Let $v_1 \in N(u_1)$ and let v_2 be such that $v_1 v_2 \notin E(G)$. Since $\delta \geq 4$, we may also assume that $v_{n-6}, v_{n-5} \notin N(u_1)$ (note that there is also another vertex v_j, $2 \leq j \leq n-7$, such that $v_j \notin N(u_1)$ but we shall not specifically assume which one.) Thus $G[v, v_{n-6}, v_{n-5}] = K_3$.

We now consider the following cases separately.

Case 1. $n = 10$ or 11.

(i) Suppose $v_3 \notin N(v_{n-6}) \cap N(v_{n-5})$, say $v_3 \notin N(v_{n-6})$.

For $n = 10$, $G - \{v_1, v_2\}$ has a 1-factor $M_1 = \{u_1u_2, u_3u_4, v_3v, v_4v_5\}$, $G - M_1 - \{v_3, v_4\}$ has a 1-factor $M_2 = \{u_1u_4, u_2u_3\} \cup FG[v_1, v_2, v, v_5]$ where $FG[v_1, v_2, v, v_5]$ denotes a 1-factor in $G[v_1, v_2, v, v_5]$, and we can choose M_3 to be any maximum matching in $G - (M_1 \cup M_2) - \{u_1, v\}$.

For $n = 11$, we have $G[v_4, v_1, v_2] \neq O_3$, $G[v_4, v_3, v_5] \neq O_3$ and $G[v_4, u_1, v_6] \neq O_3$. Hence by symmetry, we can assume that $v_4u_4 \notin E(G)$. Now $G - \{v_1, v_2\}$ has a matching $M_1 = \{v_3u_1, v_4v, v_5v_6, u_3u_4\}$ which misses u_2, $G - M_1 - \{v_3, v_5\}$ has a matching $M_2 = \{u_1u_4, u_2u_3\} \cup FG[v, v_1, v_2, v_4]$ which misses v_6, and $G - (M_1 \cup M_2) - \{u_4, v_4\}$ has a matching $M_3 = \{u_1u_2, vv_6\} \cup FG[v_1, v_2, v_3, v_5]$ which misses u_3.

(ii) $v_3v_{n-6}, v_3v_{n-5} \in E(G)$.

By symmetry, we can also assume that $v_4v_{n-6}, v_4v_{n-5} \in E(G)$ if $n = 11$. Thus, by symmetry again, we may further assume that $v_{n-6}u_3, v_{n-5}u_4 \notin E(G)$.

For $n = 10$, $G - \{v_1, v_2\}$ has a 1-factor $M_1 = \{u_1u_2, u_3u_4, v_3v, v_4v_5\}$, $G - M_1 - \{u_1, v_5\}$ has a 1-factor $M_2 = \{v_3v_4, u_2u_4\} \cup FG[v_1, v_2, v, u_3]$, and we can choose M_3 to be any maximum matching in $G - (M_1 \cup M_2) - \{u_4, v\}$.

For $n = 11$, $G - \{v_1, v_2\}$ has a matching $M_1 = \{u_1u_4, u_2u_3, v_3v, v_5v_6\}$ which misses v_4, $G - M_1 = \{u_3, v_5\}$ has a matching $M_2 = \{u_2u_4, v_3v_6\} \cup FG[v_1, v_2, v_4, v]$ which misses u_1, and $G - (M_1 \cup M_2) - \{u_4, v_6\}$ has a matching $M_3 = \{v_4v, u_1u_2\} \cup FG[v_1, v_2, v_5, u_3]$ which misses v_3.

Case 2. $n = 12$.

This case is very similar to Case 1. Although the length of proof of Case 2 is slightly shorter than that of Case 1, the proof is very technical and thus we omit it.

Finally, from the above results we deduce

Theorem 3.5. *For any graph G of order n having $\Delta(G) = n - 5$,*

$$\chi_T(G) \leq \Delta(G) + 2.$$

PROOF. By Lemma 2.2, we can assume that G is maximal. By Lemma 3.1, if $n \geq 11$ and $G \geq O_3$, then this theorem is true. By Theorem 3.2, if $n \geq 13$, then this theorem is also true. By Theorem 2.1, this theorem is true for $n \leq 9$. Hence we need only to consider the following remaining cases:

(i) $n = 10$ and $G \geq O_3$;

(ii) $n = 10, 11$ or 12 and $G \not\geq O_3$.

However, these two cases have been settled by Lemma 3.3 and Lemma 3.4.

REMARK. From the proofs of Theorem 3.2 and Theorem 3.5, we see that the technique we apply will be extremely difficult to prove that the TCC also holds for graphs of order n having $\Delta(G) = n - 6$.

ACKNOWLEDGMENT. We thank the referee for valuable comments.

References

[1] M. Behzad, *Graphs and their chromatic numbers* (Doctoral Thesis Michigan State Univ., 1965).

[2] M. Behzad, G. Chartrand and J. K. Cooper, 'The colour numbers of complete graphs', *J. London Math. Soc.* **42** (1967), 226–228.

[3] A. G. Chetwynd and A. J. W. Hilton, 'Regular graphs of high degree are 1-factorizable', *Proc. London Math. Soc.* **50** (3) (1985), 193–206.

[4] A. G. Chetwynd and A. J. W. Hilton, 'The chromatic index of graphs with at most four vertices of maximum degree' in: *Proc. of the Fifteenth Southeastern Conf. on Combinatorics, Graph Theory and Computing* (Baton Rouge, La., 1984) *Congr. Numer.* **43** (1984), 221–248.

[5] A. J. W. Hilton, 'Recent results on edge-colouring graphs, with applications to the total-chromatic number', *J. Combin. Math. and Combin. Computing* **3** (1988), 121–134.

[6] A. V. Kostochka, 'The total coloring of a multigraph with maximal degree 4', *Discrete Math.* **17** (1977), 161–163.

[7] M. Rosenfeld, 'On the total colouring of certain graphs', *Israel J. Math.* **9** (3) (1971), 396–402.

[8] N. Vijayaditya, 'On total chromatic number of a graph', *J. London Math. Soc.* (2) **3** (1971), 405–408.

[9] V. G. Vizing, 'On an estimate of the chromatic class of a p-graph' (Russian), *Diskret. Analiz* **3** (1964), 25–30.

[10] V. G. Vizing, 'Critical graphs with a given chromatic class' (Russian), *Diskret. Analiz* **5** (1965), 9–17.

[11] V. G. Vizing, 'Some unsolved problems in graph theory', *Uspehi Mat. Nauk* **23** (1968), 117–134 (Russian Math. Surveys **23** (1968), 125–142).

[12] H. P. Yap, *Some topics in graph theory* (Cambridge Univ. Press, 1986).

[13] H. P. Yap, 'Total colourings of graphs', *Bull. London Math. Soc.* **21** (1989), 159–163.

[14] H. P. Yap, Wang Jian-Fang and Zhang Zhongfu, 'Total chromatic number of graphs of high degree', *J. Austral. Math. Soc. Ser.* A **47** (1989), 445–452.

National University of Singapore
10 Kent Ridge Crescent
Singapore 0511