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INJECTIVITY AND EQUATIONAL COMPACTNESS IN THE CLASS OF $\aleph_0$-SEMILATTICES

BY
EVELYN NELSON*

This note presents characterizations of the injective and of the equationally compact $\aleph_0$-semilattices, which are analogous, respectively, to the characterizations of the injective semilattices given by Bruns and Lakser [2] and of the equationally compact semilattices given by Grätzer and Lakser [3].

It is known (see Banaschewski [1]) that in an equational class of finitary algebras, the existence of enough injectives guarantees the existence of injective hulls; it will be seen here that the class of $\aleph_0$-semilattices has enough injectives, while not every $\aleph_0$-semilattice has an injective hull.

As a sideline, we will see that the class of $\aleph_0$-semilattices also provides an example of an equational class of infinitary algebras in which Birkhoff's Subdirect Representation Theorem holds, but in which the Pure Representation Theorem (proved by Walter Taylor [5] for finitary algebras) does not hold.

1. **Preliminaries.** An $\aleph_0$-semilattice is a semilattice S with the usual binary operation $\wedge$, and an additional $\aleph_0$-ary operation $\bigwedge$ which associates with each $\sigma \in S_\omega$ (ω the set of natural numbers) the meet, $\bigwedge\sigma$, and which satisfies the obvious identities. For $\sigma \in S^\omega$, we sometimes write $\bigwedge_{n\in\omega} \sigma(n)$ or $\bigwedge\{\sigma(n) \mid n \in \omega\}$ for $\bigwedge \sigma$.

S is the class of all $\aleph_0$-semilattices. By "homomorphism" we will always mean mappings which preserve countable meets.

Each $\aleph_0$-semilattice S has a natural partial ordering on it given by $a \leq b$ iff $a \wedge b = a$; with respect to this partial ordering $a \wedge b$ is the greatest lower bound of a and b, and $\bigwedge\sigma$ is the greatest lower bound of $\{\sigma(n) \mid n \in \omega\}$ for $\sigma \in S^\omega$. The symbol "$\bigvee$" is used to denote join (least upper bound) with respect to this partial ordering, and the symbol "$\bigwedge$" is used indiscriminately to denote meet (least upper bound) of arbitrary (not necessarily countable) subsets.

It is well-known that, for any semilattice S, the map which assigns to each $s \in S$, $\{t \in S \mid t \leq s\}$, is a semilattice embedding of S into $P(S)$, the set of all subsets of S, with the operation of set-intersection, and that moreover this embedding takes arbitrary (existing) meets to the corresponding set intersection in $P(S)$. In particular, it follows that every $\aleph_0$-semilattice has an $\aleph_0$-embedding into a power of **2**, the

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two-element $\aleph_0$-semilattice. Consequently, Birkhoff's Subdirect Representation Theorem holds in **S**, and **2** is the only subdirectly irreducible member of **S**.

Recall that the *characteristic* of a (possibly infinitary) algebra is the smallest infinite regular cardinal greater than the arities of all the fundamental operations. If A and B are algebras (of the same type) of characteristic m, then a homomorphism $f:A\to B$ is a *pure embedding* (see Nelson [4]) iff, for all X, every subset of $A[X]^2$ with fewer than m elements is contained in the kernel of a homomorphism $A[X]\to A$ over A whenever it is contained in the kernel of a homomorphism $A[X]\to B$ over f (where $A[X]$ is the free extension of A by the set X, in any equational class containing both A and B). Since $\aleph_0$-semilattices are of characteristic $\aleph_1$, a homomorphism $f:S\to T$ in S is a pure embedding iff every at most countable subset of $S[X]^2$ ($S[X]$ the free extension of S in **S** by the set X) is contained in the kernel of a homomorphism $S[X]\to S$ over S whenever it is contained in the kernel of a homomorphism $S[X]\to T$ over f.

Let $\mathbf{N}\in\mathbf{S}$ have as underlying partially ordered set the natural numbers with the usual order; then the embedding $\phi:\mathbf{N}\to\mathbf{N}\times\mathbf{N}$ given by

$$\phi(0) = (0, 0)$$
$$\phi(1) = (1, 0)$$
$$\phi(n+1) = (n, n) \quad \text{for} \quad n\geq 1$$

has a retraction, namely $\psi:\mathbf{N}\times\mathbf{N}\to\mathbf{N}$ given by

$$\psi((0, n)) = 0 \quad \text{for all} \quad n$$
$$\psi((n, 0)) = 1 \quad \text{for all} \quad n\geq 1$$
$$\psi((n, m)) = (n\wedge m)+1 \quad \text{for} \quad n, m\geq 1,$$

and thus in particular ϕ is a pure embedding. However, $p_1\phi$ and $p_2\phi$ (p_1, p_2: $\mathbf{N}\times\mathbf{N}\to\mathbf{N}$ the projection maps) are not one-one, and hence **N** is pure-reducible.

Now, the homomorphic images of **N** are precisely those $\aleph_0$-semilattices which are isomorphic either to **N** or to a finite chain. Thus if $f:\mathbf{N}\to\Pi S_i$ $(i\in I)$ is an embedding of **N** into a product of pure-irreducibles S_i such that, for each i, the composite of f with the ith projection map is *onto* S_i, then each S_i is a finite chain, and hence ΠS_i $(i\in I)$ has a largest element, s, say. But then the homomorphism $\mathbf{N}[\{x\}]\to\Pi S_i$ over f which maps x to s contains $\{(n\wedge x, n) \mid n\in N\}$ in its kernel, whereas the existence of a homomorphism $\mathbf{N}[\{x\}]\to N$ over N containing $\{(n\wedge x, n) \mid n\in\mathbf{N}\}$ in its kernel would imply that **N** had a largest element. It follows that the Pure Representation Theorem (Taylor, [5, Theorem 3.6]) does not hold in S.

2. **Injectivity.** Recall that $S\in\mathbf{S}$ is *injective* (in **S**) iff every homomorphism from a sub-$\aleph_0$-semilattice R of an $\aleph_0$-semilattice T into S has an extension to a homomorphism $T\to S$. Also, an extension T of an $\aleph_0$-semilattice S is *essential* iff every homomorphism $f:T\to R$ in **S** whose restriction to S is one-to-one is itself one-to-one, and T is an *injective hull* of S iff it is an essential, injective extension of S.

Bruns and Lakser [2] showed that a semilattice C is injective (in the class of all semilattices) iff it is a complete lattice and satisfies $a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$ for all $a \in A$ and $X \subseteq A$. A simple modification of their proofs yields the following characterization of injective $\aleph_0$-semilattices.

PROPOSITION 1. *An $\aleph_0$-semilattice S is injective (in* **S**) *iff it is a complete lattice and satisfies*

(*) *for all countable families $(M_i)_{i\in\omega}$ of subsets of S,*

$$\bigwedge_{i\in\omega} \bigvee M_i = \bigvee_{\phi\in\Pi M_i} \bigwedge_{i\in\omega} \phi(i)$$

Proof. If S is injective in **S** then it is an absolute retract in **S**, and hence, by the remarks in §1, a retract of a power of **2**. Every power of **2** is a complete, completely distributive lattice and hence in particular satisfies (*); since **S**-homomorphisms preserve countable meets, it follows by an argument analogous to the proof of [2, Lemma 2] that every retract of a complete lattice satisfying (*) is also a complete lattice satisfying (*).

Conversely, if $S \in$ **S** satisfies the conditions of the Proposition, and if R is a sub-$\aleph_0$-semilattice of $T \in$ **S** and $f: R \to S$, then an argument analogous to the proof of [2, Lemma 1] shows that $g: T \to S$ given by $g(t) = \bigvee \{f(r) \mid r \leq t, r \in R\}$ is a homomorphism extending f.

COROLLARY. **S** *has enough injectives.*

In an equational class of finitary algebras which has enough injectives, every algebra which has no proper essential extensions is injective [1]. That this is not the case in the class of $\aleph_0$-semilattices is seen in the next proposition.

PROPOSITION 2. *For an $\aleph_0$-semilattice S, the following are equivalent;*

(1) *S has no proper essential extensions (in* **S**).
(2) *The underlying semilattice of S is an injective semilattice.*
(3) *S is a retract of every extension which is singly generated over S.*

Proof. (1)⇒(2): For any $\aleph_0$-semilattice S, the canonical semilattice embedding of S into its semilattice-injective hull given in Bruns-Lakser [2] preserves arbitrary (existing) meets, and hence is an $\aleph_0$-embedding; since it is essential as a semilattice homomorphism it is essential as an $\aleph_0$-semilattice homomorphism. Thus if S has no proper essential extensions then it is equal to its semilattice-injective hull.

(2)⇒(3): Suppose that S satisfies (2), and that $T \supseteq S$ is an $\aleph_0$-semilattice extension generated over S by $\{a\}$. Then $T = S \cup \{u \wedge a \mid u \in S\}$. Let $X = \{s \in S \mid s \leq a\}$ and let $b = \bigvee_S X$, where $\bigvee_S$ denotes join in S. If, for $u, v \in S$, $u \wedge a = v \wedge a$ then for all $s \in X$, $u \wedge s = u \wedge s \wedge a = v \wedge s \wedge a = v \wedge s$, and consequently $u \wedge b = u \wedge \bigvee_S X = \bigvee_S \{u \wedge s \mid s \in X\} = \bigvee_S \{v \wedge s \mid s \in X\} = v \wedge \bigvee_S X = v \wedge b$. In particular, for $u \in S$, if $u \wedge a \in S$ then $u \wedge a = (u \wedge a) \wedge b = u \wedge b$. Thus we can define a function $f: T \to S$

by $f(u)=u$ for $u \in S$ and $f(u \wedge a)=u \wedge b$ for $u \in S$. It is easy to check that f preserves all at most countable meets, and hence is a retraction of T to S.

(3)⇒(1): Suppose S is a retract of each of its extensions which is singly generated over S. Let T be a proper extension of S, and let $R \subseteq T$ be a proper extension of S which is singly generated over S. Let V be an injective extension of S. The retraction of R onto S followed by the embedding $S \to V$ is a homomorphism $f: R \to V$ which is one-one on S but not one-one on R. By the injectivity of V, f has an extension $\bar{f}: T \to V$, and then $\bar{f}$ is one-one on S, and not one-one on T, which implies that T is not an essential extension of S. Thus S has no proper essential extensions.

COROLLARY. *Not every* $\aleph_0$*-semilattice has an injective hull* (*in* **S**).

Proof.. Any $\aleph_0$-semilattice which is not injective in **S**, but whose underlying semilattice is injective as a semilattice, has no injective hull in **S**. An example of such an $\aleph_0$-semilattice is the MacNeille completion M of the free Boolean algebra on countably many generators; every complete Boolean algebra is injective as a semilattice but it is easy to see that M does not satisfy the distributivity condition of Proposition 1.

3. **Equational Compactness.** An algebra A of characteristic m is *equationally compact* (see Nelson [4]) iff every subset $\Sigma \subseteq A[X]^2$ is contained in the kernel of a homomorphism $A[X] \to A$ over A whenever every subset of Σ with fewer than m elements already has this property (where $A[X]$ is the free extension of A by the set X in any equational class containing A). In particular, $S \in \mathbf{S}$ is equationally compact iff every subset $\Sigma \subseteq S[X]^2$ ($S[X]$ the free extension of S in **S** by the set X) is contained in the kernel of a homomorphism $S[X] \to S$ over S whenever every at most countable subset of Σ has this property.

Grätzer and Lakser showed that a semilattice A is equationally compact iff every non-empty subset of A has a meet, every up-directed subset of A has a join, and for every up-directed subset $X \subseteq A$ and all $a \in A$, $a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$. A modification of their techniques yields the following characterization of equationally compact $\aleph_0$-semilattices (where a partially ordered set is called $\aleph_1$-up-directed iff every at most countable subset has an upper bound):

PROPOSITION 3. $S \in \mathbf{S}$ *is equationally compact iff*

(1) *every non-empty subset of* S *has a meet in* S

(2) *every* $\aleph_1$*-up-directed subset of* S *has a join in* S

(3) *for all countable families* $(M_i)_{i\in\omega}$ *of* $\aleph_1$*-up-directed subsets of* S,

$$\bigwedge_{i\in\omega} \bigvee M_i = \bigvee_{\phi\in\Pi M_i} \bigwedge_{i\in\omega} \phi(i).$$

Proof. Suppose $S \in \mathbf{S}$ is equationally compact, and $T \subseteq S$ is a non-empty subset. Then every at most countable subset of

$$\Sigma = \{(t \wedge x, x) \mid t \in T\} \cup \{(u \wedge x, u) \mid u \leq t \quad \text{for all} \quad t \in T\} \subseteq S[\{x\}]^2$$

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is contained in one of the form

$$\Sigma_{T'} = \{(t \wedge x, x) \mid t \in T'\} \cup \{(u \wedge x, u) \mid u \leq t \quad \text{for all} \quad t \in T\}$$

where T' is a non-empty, at most countable, subset of T. Since the homomorphism $S[\{x\}] \to S$ over S which maps x to $\bigwedge T'$ contains $\Sigma_{T'}$ in its kernel, it follows that there exists a homomorphism $f: S[\{x\}] \to S$ over S with $\Sigma \subseteq \operatorname{Ker} f$, and then $f(x) = \bigwedge T$.

A similar argument shows that every $\aleph_1$-up-directed subset of S has a join.

If $(M_i)_{i \in \omega}$ is a countable family of $\aleph_1$-up-directed subsets of S then $\{\bigwedge_{i \in \omega} \phi(i) \mid \phi \in \Pi M_i\}$ is $\aleph_1$-up-directed and hence has a join, a, say. Let

$$\Sigma = \bigcup_{i \in \omega} \{(x_i \wedge m, m) \mid m \in M_i\} \cup \{(a \wedge \bigwedge_{i \in \omega} x_i, \bigwedge_{i \in \omega} x_i)\} \subseteq S[X]^2$$

where $X = \{x_i \mid i \in \omega\}$ is a countable set disjoint from S.

If, for each i, M_i' is a countable subset of M_i, then in view of the $\aleph_1$-up-directedness of M_i there exists $m_i \in M_i$ with $m_i \geq m$ for each $m \in M_i'$. Also $\bigwedge_{i \in \omega} m_i \leq a$. Thus the homomorphism $f: S[X] \to S$ over S with $f(x_i) = m_i$ contains $\bigwedge_{i \in \omega}\{(x_i \wedge m, m) \mid m \in M_i'\} \cup \{(a \wedge \bigwedge_{i \in \omega} x_i, \bigwedge_{i \in \omega} x_i)\}$ in its kernel. Thus every at most countable subset of Σ is contained in the kernel of a homomorphism $S[X] \to S$ over S. By the equational compactness of S, there exists a homomorphism $h: S[X] \to S$ over S with $\Sigma \subseteq \operatorname{Ker} h$, and then $\bigwedge_{i \in \omega} \bigvee M_i \leq \bigwedge_{i \in \omega} h(x_i) \leq a$; the reverse inequality is trivial, and thus S satisfies (3).

For the converse, suppose $S \in \mathbf{S}$ satisfies (1)–(3) and that $\Sigma \subseteq S[X]^2$ such that every at most countable subset of Σ is contained in the kernel of a homomorphism $S[X] \to S$ over S. Then the $\aleph_0$-semilattice S^X also satisfies (1)–(3). For each $\phi \in S^X$, let $\bar{\phi}: S[X] \to S$ be the homomorphism over S extending ϕ. Then for any $\aleph_1$-up-directed subset $K \subseteq S^X$, the mapping $\phi_K: S[X] \to S$ given by $\phi_K(p) = \bigvee_{\phi \in K} \bar{\phi}(p)$ is a homomorphism (this follows from the distributivity condition (3)) over S and hence, since its restriction to X is $\bigvee K$, $\phi_K = \overline{\bigvee K}$. Consequently, for any $(p, q) \in S[X]^2$, if $K \subseteq S^X$ is an $\aleph_1$-up-directed subset such that $(p, q) \in \operatorname{Ker}(\bar{\phi})$ for each $\phi \in K$ then $(p, q) \in \operatorname{Ker}(\overline{\bigvee K})$.

Similarly if $K \subseteq S^X$ is a non-empty subset with $(p,q) \in \operatorname{Ker}(\bar{\phi})$ for each $\phi \in K$ then $(p, q) \in \operatorname{Ker}(\overline{\bigwedge K})$.

Now, by assumption, for each at most countable $T \subseteq \Sigma$, $K_T = \{\phi \in S^X \mid T \subseteq \operatorname{Ker}(\bar{\phi})\} \neq \varnothing$. It follows from the above remark that $T \subseteq \operatorname{Ker}(\overline{\bigwedge K_T})$. Let $\phi_T = \bigwedge K_T$. Then, for each $(p, q) \in \Sigma$, $\{\phi_T \mid (p, q) \in T \subseteq \Sigma, |T| \leq \aleph_0\}$ is an $\aleph_1$-up-directed subset of S^X and $(p, q) \in \operatorname{Ker}(\bar{\phi}_T)$ for each at most countable $T \subseteq \Sigma$ with $(p, q) \in T$, and hence, by the above remarks, $(p, q) \in \operatorname{Ker}(\bar{\phi})$ where $\phi = \bigvee\{\phi_T \mid (p, q) \in T \subseteq \Sigma, |T| \leq \aleph_0\}$. However, for each $(p, q) \in \Sigma$,

$$\bigvee \{\phi_T \mid (p, q) \in T \subseteq \Sigma, |T| \leq \aleph_0\} = \bigvee \{\phi_T \mid T \subseteq \Sigma, |T| \leq \aleph_0\}$$

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(this follows essentially from the fact that if T is at most countable so is $T \cup \{(p, q)\}$) and hence $\Sigma \subseteq \operatorname{Ker} \psi$ where $\psi: S[X] \to S$ is the homomorphism over S whose restriction to X is $\bigvee\{\phi_T \mid T \subseteq \Sigma, |T| \leq \aleph_0\}$. Thus S is equationally compact.

NOTE. The above methods can be used to show that $S \in \mathbf{S}$ is (in the terminology of Grätzer-Lakser [3]), 1-equationally compact (i.e. every subset $\Sigma \subseteq S[\{x\}]^2$ is contained in the kernel of a homomorphism $S[\{x\}] \to S$ over S whenever every at most countable subset has this property) iff it satisfies (1) and (2) of Proposition 2, and $s \wedge \bigvee T = \bigvee \{s \wedge t \mid t \in T\}$ for all $s \in S$, all $\aleph_1$-up-directed $T \subseteq S$. Thus, in contrast to the situation for semilattices, 1-equationally compact is not equivalent to equationally compact in $\mathbf{S}$.

COROLLARY 1. *Every at most countable $\aleph_0$-semilattice is equationally compact.*

COROLLARY 2. *Every $\aleph_0$-semilattice of at most countable height in which every non-empty subset has a meet is equationally compact.*

Note added in proof: All of the foregoing works just as well with $\aleph_0$ replaced by *any* infinite cardinal $\mathscr{M}$, giving characterizations of the injective and the equationally compact $\mathscr{M}$-semilattices. By combining these results, or by analagous arguments, one concludes that the injectives in the category of all complete semilattices are precisely the complete, completely distributive semilattices, and hence (since the latter condition is self-dual) these also coincide with the projective complete semilattices. These results on *complete* semilattices were presented by A. Waterman in a seminar at McMaster in 1966, and also appeared in Crown [Math. Annalen. **187** (1970) 295–299].

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MCMASTER UNIVERSITY
HAMILTON, ONTARIO