ON THE HOMOTOPY-COMMUTATIVITY OF LOOP-SPACES AND SUSPENSIONS

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Introduction. Let X be a space. We are interested in the homotopy-commutativity of the loop-space ΩX and the suspension ΣX, that is, in the question whether or not $\operatorname{nil} X \leqq 1$, $\operatorname{conil} X \leqq 1$, respectively. Let $c\colon \Omega X \times \Omega X \to \Omega X$, $c'\colon \Sigma X \to \Sigma X \vee \Sigma X$ be the commutator and co-commutator maps, respectively. Then $\operatorname{nil} X \leqq 1$ if and only if $c \simeq *$, and $\operatorname{conil} X \leqq 1$ if and only if $c' \simeq *$. Our aim in this paper is to obtain factorizations $c \simeq f_1 f_2 \ldots f_m$, $c' \simeq g_1 g_2 \ldots g_n$ of c, c' as compositions of various maps, or alternatively, factorizations of the adjoints of c, c'. This will then give us conditions for $\operatorname{nil} X \leqq 1$, $\operatorname{conil} X \leqq 1$, namely, whenever some combination of the factors in the compositions is null-homotopic. We take this idea and ring various changes on it. The maps in the compositions will be constructed from c, c' and various standard maps. We shall use the Hopf and co-Hopf constructions liberally, and they will be defined briefly below in order to make this paper relatively independent of others. This paper is motivated by Theorems 3.1 and 4.1 of (**3**), but we shall not be using any explicit results from that paper.

In Theorem 1 we give a factorization of c, while in Theorems 2 and 3 we give factorizations of the adjoint of c. In the dual situation, Theorem 4 gives a factorization of c', while Theorems 5 and 6 give factorizations of the adjoint of c'. We work in the category of spaces with base point and having the homotopy type of countable CW-complexes. For simplicity, we shall frequently use the same symbol for a map and its homotopy class. Part of this work was done while the author was a Fellow of the Summer Research Institute of the Canadian Mathematical Congress in 1967.

1. Let A and B be spaces. We can consider

$$A \flat B \xrightarrow{i} A \vee B \xrightarrow{j} A \times B$$

as a fibration, where j is the usual inclusion and $A \flat B$ is the flat product. Then we can find a map $\chi\colon \Omega(A \times B) \to \Omega(A \vee B)$ such that $(\Omega j)\chi \simeq 1_{\Omega(A\times B)}$. In fact, we can and shall take $\chi = \Omega(i_A p_A) + \Omega(i_B p_B)$, where p_A and p_B are the projections of $A \times B$ onto the factors, and $i_A\colon A \to A \vee B$, $i_B\colon B \to A \vee B$ are the obvious inclusions. The exact sequence of the fibration now shows that $(\Omega i)_*$ is a monomorphism, and that there exists a unique element $[g] \in [\Omega(A \vee B), \Omega(A \flat B)]$ such that $1_{\Omega(A\vee B)} = (\Omega i)g + \chi(\Omega j)$.

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Now, for any space X and a map $f: X \to A \vee B$, we can form the map $H(f) = g(\Omega f): \Omega X \to \Omega(A \flat B)$. We shall call this the co-Hopf construction. The element $[H(f)]$ is the unique element of $[\Omega X, \Omega(A \flat B)]$ satisfying $[\Omega f] = (\Omega i)_*[H(f)] + [\chi\Omega(jf)] = (\Omega i)_*[H(f)] + [\Omega(i_A\pi_A f)] + [\Omega(i_B\pi_B f)]$, where π_A: $A \vee B \to A$, π_B: $A \vee B \to B$ are induced by the projections.

We now define the Hopf construction. We consider

$$A \vee B \xrightarrow{j} A \times B \xrightarrow{q} A \wedge B$$

as a co-fibration, where $A \wedge B$ is the smash product. In a fashion dual to the above, we show that there exists a map $p: \Sigma(A \times B) \to \Sigma(A \vee B)$ such that $p(\Sigma j) \simeq 1_{\Sigma(A \vee B)}$. In fact, let $p_1, p_2: A \times B \to A \vee B$ be defined by $p_1 = i_A p_A$, $p_2 = i_B p_B$. Then we can and shall take $p = \nabla(\Sigma p_1 \vee \Sigma p_2)\phi'$, where ϕ': $\Sigma(A \times B) \to \Sigma(A \times B) \vee \Sigma(A \times B)$ is the usual suspension structure, and ∇ is the folding map. The exact sequence of the co-fibration now shows that $(\Sigma q)^*$ is a monomorphism. As above, we see that there exists a unique element $[d] \in [\Sigma(A \wedge B), \Sigma(A \times B)]$ satisfying $1_{\Sigma(A \times B)} = d(\Sigma q) + (\Sigma j)p = d(\Sigma q) + \Sigma(jp_1) + \Sigma(jp_2)$.

Given a space X and a map $f: A \times B \to X$, we can now define $J(f) = (\Sigma f)d: \Sigma(A \wedge B) \to \Sigma X$. We call $J(f)$ the map obtained from f by the Hopf construction. The element $[J(f)]$ is the unique element satisfying

$$[\Sigma f] = (\Sigma q)^*[J(f)] + [\Sigma(fj)p] = (\Sigma q)^*[J(f)] + [\Sigma(fjp_1)] + [\Sigma(fjp_2)].$$

Let us now establish some standard notation. Given spaces X, Y, we have a bijection $\tau: [\Sigma X, Y] \to [X, \Omega Y]$ given by $\tau(f)(x)(t) = fk_X(x, t)$, where k_X: $X \times I \to \Sigma X$ is the projection. For any space X, the maps $e: \Sigma\Omega X \to X$, $e': X \to \Omega\Sigma X$ shall be those given by $\tau(e) = 1_{\Omega X}$, $\tau(1_{\Sigma X}) = e'$.

Suppose that we are given spaces X_1, X_2. Consider the projections e: $\Sigma\Omega X_i \to X_i$ given by $\tau(e) = 1_{\Omega X_i}$. Let $e_1 = i_1 e$: $\Sigma\Omega X_1 \to X_1 \vee X_2$, $e_2 = i_2 e$: $\Sigma\Omega X_2 \to X_1 \vee X_2$, where i_1: $X_1 \to X_1 \vee X_2$, i_2: $X_1 \vee X_2$ are the inclusions. Let c: $\Omega(X_1 \vee X_2) \times \Omega(X_1 \vee X_2) \to \Omega(X_1 \vee X_2)$ be the commutator map. Then we can form $\bar{c} = \tau^{-1}\{c(\tau(e_1) \times \tau(e_2))\}$: $\Sigma(\Omega X_1 \times \Omega X_2) \to X_1 \vee X_2$. A simple check shows that if $X_1 = X_2 = X$ and ∇ is the folding map, then $\nabla\bar{c} = \tau^{-1}(c)$, where c: $\Omega X \times \Omega X \to \Omega X$ is the commutator map. If we apply the co-Hopf construction to $\bar{c}$: $\Sigma(\Omega X_1 \times \Omega X_2) \to X_1 \vee X_2$ in the general case, we obtain a map $H(\bar{c})$: $\Omega\Sigma(\Omega X_1 \times \Omega X_2) \to \Omega(X_1 \flat X_2)$. Let i: $X_1 \flat X_2 \to X_1 \vee X_2$ be the fibre of the inclusion j: $X_1 \vee X_2 \to X_1 \times X_2$. Then we have the following lemma.

LEMMA 1. $\Omega\bar{c} = (\Omega i)H(\bar{c})$: $\Omega\Sigma(\Omega X_1 \times \Omega X_2) \to \Omega(X_1 \vee X_2)$.

Proof. $H(\bar{c})$ satisfies $\Omega\bar{c} = (\Omega i)H(\bar{c}) + \Omega(i_1\pi_1\bar{c}) + \Omega(i_2\pi_2\bar{c})$, where π_1: $X_1 \vee X_2 \to X_1$, π_2: $X_1 \vee X_2 \to X_2$ are induced by the projections and i_1: $X_1 \to X_1 \vee X_2$, i_2: $X_2 \to X_1 \vee X_2$ are the inclusions. Let us consider $\tau(i_1\pi_1\bar{c})$: $\Omega X_1 \times \Omega X_2 \to \Omega(X_1 \vee X_2)$. Let ϕ: $\Omega X_1 \times \Omega X_1 \to \Omega X_1$ be the loop multiplication and μ: $\Omega X_1 \to \Omega X_1$ the loop inverse. Let γ_1: $\Omega X_1 \times \Omega X_2 \to \Omega X_1$ be the projection. Then, a simple check shows that $\tau(i_1\pi_1\bar{c}) = (\Omega i_1)\phi\{\phi(1 \times *)\Delta \times \phi(1 \times *)\Delta\mu\}\Delta\gamma_1$. Since $\phi(1 \times *)\Delta \simeq 1$ and $\phi(1 \times \mu)\Delta \simeq *$, it follows that $\tau(i_1\pi_1\bar{c}) = 0$. Hence, $i_1\pi_1\bar{c} = 0$. Similarly, $i_2\pi_2\bar{c} = 0$. Hence, $\Omega\bar{c} = (\Omega i)H(\bar{c})$.

LEMMA 2. *There exists a map* b: $\Sigma(\Omega X_1 \times \Omega X_2) \to X_1 \flat X_2$ *such that* $ib = \bar{c}$ *and* $\Omega b = H(\bar{c})$, *where* i: $X_1 \flat X_2 \to X_1 \vee X_2$ *is the inclusion.*

Proof. Let j: $X_1 \vee X_2 \to X_1 \times X_2$ be the inclusion. Then we have that $\tau(j\bar{c})$: $\Omega X_1 \times \Omega X_2 \to \Omega(X_1 \times X_2)$. Let K: $\Omega(X_1 \times X_2) \to \Omega X_1 \times \Omega X_2$ be the homeomorphism given by $K(l) = (p_1 l, p_2 l)$, where p_1 and p_2 are the projections. Then $K\tau(j\bar{c})(l_1, l_2) = (p_1\tau(j\bar{c})(l_1, l_2), p_2\tau(j\bar{c})(l_1, l_2))$. A simple check shows that $p_1\tau(j\bar{c})(l_1, l_2)(t) = \phi\{\phi(1 \times *)\Delta \times \phi(1 \times *)\Delta\mu\}\gamma_1(l_1, l_2)(t)$, where γ_1: $\Omega X_1 \times \Omega X_2 \to \Omega X_1$ is the projection and ϕ and μ give the loop structure on ΩX. Hence, as above, $p_1\tau(j\bar{c}) \simeq *$. Similarly, $p_2\tau(j\bar{c}) \simeq *$. Since K is a homeomorphism, it follows that $j\bar{c} = 0$. Hence, from the fibration

$$X_1 \flat X_2 \xrightarrow{i} X_1 \vee X_2 \xrightarrow{j} X_1 \times X_2,$$

it follows that there exists a map b with $ib = \bar{c}$. Thus, we have that $(\Omega i)(\Omega b) = \Omega\bar{c}$. But by Lemma 1, $(\Omega i)H(\bar{c}) = \Omega\bar{c}$. Since $(\Omega i)_*$ is a monomorphism, it follows that $H(\bar{c}) = \Omega b$.

THEOREM 1. $c = \Omega(\nabla i)(H\bar{c})e'$: $\Omega X \times \Omega X \to \Omega X$, *where* c *is the commutator map.*

Proof. We apply Lemma 1 with $X_1 = X_2 = X$. We have that $\Omega\bar{c} = (\Omega i)H(\bar{c})$. Hence, $\Omega(\nabla\bar{c}) = \Omega(\nabla i)H(\bar{c})$, where ∇ is the folding map. Since $\nabla\bar{c} = \tau^{-1}(c)$ and $\Omega(\tau^{-1}(c))e' = c$, we have that $c = (\nabla\bar{c})e' = \Omega(\nabla i)H(\bar{c})e'$.

Let us now again consider $\bar{c}$: $\Sigma(\Omega X_1 \times \Omega X_2) \to X_1 \vee X_2$ in the general case. We have that $\tau(\bar{c})$: $\Omega X_1 \times \Omega X_2 \to \Omega(X_1 \vee X_2)$. The Hopf construction now yields $J(\tau(\bar{c}))$: $\Sigma(\Omega X_1 \wedge \Omega X_2) \to \Sigma\Omega(X_1 \vee X_2)$. Let q: $\Omega X_1 \times \Omega X_2 \to \Omega X_1 \wedge \Omega X_2$ be the projection. Then we have the following lemma.

LEMMA 3. $\Sigma(\tau(\bar{c})) = J(\tau(\bar{c}))(\Sigma q)$ *and hence,* $\bar{c} = eJ(\tau(\bar{c}))(\Sigma q)$.

Proof. The element $J(\tau(\bar{c}))$ satisfies the relation $\Sigma(\tau(\bar{c})) = J(\tau(\bar{c}))(\Sigma q) + \Sigma(\tau(\bar{c})jp_1) + \Sigma(\tau(\bar{c})kp_2)$. A simple check shows that $\tau(\bar{c})jp_1 = (\Omega i_1)\phi\{\phi(1 \times *)\Delta \times \phi(1 \times *)\Delta\mu\}\Delta\gamma_1 \simeq *$, where ϕ and μ give the loop structure on ΩX, γ_1: $\Omega X_1 \times \Omega X_2 \to \Omega X_1$ is the projection and i_1: $X_1 \to X_1 \vee X_2$ is the inclusion. Similarly, $\tau(\bar{c})jp_2 = 0$. Hence, $\Sigma(\tau(\bar{c})) = J(\tau(\bar{c}))(\Sigma q)$. Since $e\Sigma(\tau(\bar{c})) = \bar{c}$, the second part of the lemma follows easily.

THEOREM 2. $\tau^{-1}(c) = \nabla eJ(\tau(\bar{c}))(\Sigma q)$, *where* c: $\Omega X \times \Omega X \to \Omega X$ *is the commutator map and* e: $\Sigma\Omega(X \vee X) \to X \vee X$ *is the standard map.*

Proof. We apply Lemma 3 with $X_1 = X_2 = X$ to obtain $\bar{c} = eJ(\tau(\bar{c}))(\Sigma q)$. Since $\tau^{-1}(c) = \nabla\bar{c}$, the theorem follows.

Now recall that by Lemma 2, we have a map $b: \Sigma(\Omega X_1 \times \Omega X_2) \to X_1 \flat X_2$ such that $ib = \bar{c}$. Then $(\Omega b)e' = \tau(b): \Omega X_1 \times \Omega X_2 \to \Omega(X_1 \flat X_2)$. The Hopf construction yields $J(\tau(b)): \Sigma(\Omega X_1 \wedge \Omega X_2) \to \Sigma\Omega(X_1 \flat X_2)$.

LEMMA 4. *$\Sigma(\tau(b)) = J(\tau(b))(\Sigma q)$ and hence, $b = eJ(\tau(b))(\Sigma q)$.*

Proof. $J(\tau(b))$ satisfies $\Sigma(\tau(b)) = J(\tau(b))(\Sigma q) + \Sigma(\tau(b)jp_1) + \Sigma(\tau(b)jp_2)$. Let us consider the map $\tau(b)jp_1: \Omega X_1 \times \Omega X_2 \to \Omega(X_1 \flat X_2)$. We have that $(\Omega i)\tau(b)jp_1: \Omega X_1 \times \Omega X_2 \to \Omega(X_1 \vee X_2)$. Again, a simple check shows that $(\Omega i)\tau(b)jp_1 = (\Omega i_1)\phi\{\phi(1 \times *)\Delta \times \phi(1 \times *)\Delta\mu\}\Delta\gamma_1$, where ϕ and μ give the loop structure on ΩX_1 and $\gamma_1: \Omega X_1 \times \Omega X_2 \to \Omega X_1$ is the projection. Hence, $\tau^{-1}\{(\Omega i)\tau(b)jp_1\} = 0$. Since $(\Omega i)_*$ is a monomorphism, we have that $\tau(b)jp_1 = 0$. Similarly, $\tau(b)jp_2 = 0$. Hence, $\Sigma(\tau(b)) = J(\tau(b))(\Sigma q)$. The second part of the lemma follows from the fact that $e\Sigma(\tau(b)) = b$.

THEOREM 3. *$\tau^{-1}(c) = \nabla ie(\Sigma H(\bar{c}))J(e')(\Sigma q)$, where $c: \Omega X \times \Omega X \to \Omega X$ is the commutator map, $e': \Omega X \times \Omega X \to \Omega\Sigma(\Omega X \times \Omega X)$, $e: \Sigma\Omega(X \flat X) \to X \flat X$ are the standard maps, $i: X \flat X \to X \vee X$ is the inclusion, and $\nabla: X \vee X \to X$ is the folding map.*

Proof. We apply Lemma 4 with $X_1 = X_2 = X$ and obtain $b = eJ(\tau(b))(\Sigma q)$. Hence, $\bar{c} = ib = ieJ(\tau(b))(\Sigma q)$. But $\nabla\bar{c} = \tau^{-1}(c)$, and hence we have that $\tau^{-1}(c) = \nabla\bar{c} = \nabla ieJ(\tau(b))(\Sigma q)$. But since $\Omega b = H(\bar{c})$ by Lemma 2, we have that $\tau(b) = H(\bar{c})e'$. Clearly, $J(\tau(b)) = J(H(\bar{c})e') = (\Sigma H(\bar{c}))J(e')$. This proves the theorem.

Remark 1. Let $e': X_1 \times X_2 \to \Omega\Sigma(X_1 \times X_2)$, $e: \Sigma\Omega\Sigma(X_1 \times X_2) \to \Sigma(X_1 \times X_2)$ be the usual maps. Then $e(\Sigma e') = 1_{\Sigma(X_1 \times X_2)}$. This means that if $f: X_1 \times X_2 \to Y$ is a map, then the Hopf construction yields $J(f) = (\Sigma f)eJ(e')$: $\Sigma(X_1 \wedge X_2) \to \Sigma Y$. It is amusing to note the relation $(\Sigma q)eJ(e') = 1_{\Sigma(X_1 \wedge X_2)}$. Using this we can "solve" the equations in Lemmas 3 and 4 to obtain $J(\tau(\bar{c})) = \Sigma(\tau(\bar{c}))eJ(e')$, $J(\tau(b)) = \Sigma(\tau(b))eJ(e')$.

2. We now dualize. Since many of the proofs of the results in this section are exact duals of those in §1, we shall omit most of the details. Suppose that X_1 and X_2 are given spaces. Let $e': X_i \to \Omega\Sigma X_i$ be the standard maps. Let $e_1' = e'p_1: X_1 \times X_2 \to \Omega\Sigma X_1$, $e_2' = e'p_2: X_1 \times X_2 \to \Omega\Sigma X_2$, where p_1 and p_2 are the projections. Let $c': \Sigma(X_1 \times X_2) \to \Sigma(X_1 \times X_2) \vee \Sigma(X_1 \times X_2)$ be the co-commutator map. Then, we have a map $\bar{c}' = \tau\{(\tau^{-1}(e_1') \vee \tau^{-1}(e_2'))c'\}$: $X_1 \times X_2 \to \Omega(\Sigma X_1 \vee \Sigma X_2)$. A simple check shows that if $X_1 = X_2 = X$, then $\bar{c}'\Delta = \tau(c'): X \to \Omega(\Sigma X \vee \Sigma X)$, where Δ is the diagonal map and c': $\Sigma X \to \Sigma X \vee \Sigma X$ is the co-commutator map. The Hopf construction yields a map $J(\bar{c}'): \Sigma(X_1 \wedge X_2) \to \Sigma\Omega(\Sigma X_1 \vee \Sigma X_2)$. Dual to Lemma 1, we have the following lemma.

LEMMA 5. *$\Sigma(\bar{c}') = J(\bar{c}')(\Sigma q): \Sigma(X_1 \times X_2) \to \Sigma\Omega(\Sigma X_1 \vee \Sigma X_2)$.*

LEMMA 6. *There exist maps*

$$a_1'\colon X_1 \wedge X_2 \to \Omega(\Sigma X_1 \vee \Sigma X_2), \qquad a'\colon \Omega\Sigma(X_1 \wedge X_2) \to \Omega(\Sigma X_1 \vee \Sigma X_2)$$

such that $a_1' = a'e'$, $a_1'q = \bar{c}'$ *and* $\Sigma a_1' = J(\bar{c}')$.

Proof. Consider the co-fibration

$$X_1 \vee X_2 \xrightarrow{j} X_1 \times X_2 \xrightarrow{q} X_1 \wedge X_2.$$

Let $K\colon \Sigma X_1 \vee \Sigma X_2 \to \Sigma(X_1 \vee X_2)$ be the obvious homeomorphism. We have that $\tau^{-1}(\bar{c}'j)K\colon \Sigma X_1 \vee \Sigma X_2 \to \Sigma X_1 \vee \Sigma X_2$. Let $\phi_1'\colon \Sigma X_1 \to \Sigma X_1 \vee \Sigma X_1$, $\mu_1'\colon \Sigma X_1 \to \Sigma X_1$ and $\phi_2'\colon \Sigma X_2 \to \Sigma X_2 \vee \Sigma X_2$, $\mu_2'\colon \Sigma X_2 \to \Sigma X_2$ be the suspension structures. Let $f_1 = \nabla\{\nabla(1 \vee *)\phi_1' \vee \mu_1' \nabla(1 \vee *)\phi_1'\}\phi_1'\colon \Sigma X_1 \to \Sigma X_1$, $f_2 = \nabla\{\nabla(* \vee 1)\phi_2' \vee \mu_2'\nabla(* \vee 1)\phi_2'\}\phi_2'\colon \Sigma X_2 \to \Sigma X_2$. Then $f_1 \simeq * \simeq f_2$. A simple check shows that $f_1 \vee f_2 = \tau^{-1}(\bar{c}'j)K$. Since K is a homeomorphism, it follows that $\bar{c}'j = 0$. From the co-fibration, it follows that there exists a map $a_1'\colon X_1 \wedge X_2 \to \Omega(\Sigma X_1 \vee \Sigma X_2)$ such that $\bar{c}' = a_1'q$. The map a' can be taken as $\Omega(\tau^{-1}(a_1'))$. Then, clearly, $a'e' = \Omega(\tau^{-1}(a_1'))e' = a_1'$. Since $a_1'q = \bar{c}'$, we have that $(\Sigma a_1')(\Sigma q) = \Sigma\bar{c}'$. Since $\Sigma\bar{c}' = J(\bar{c}')(\Sigma q)$ by Lemma 5, and since $(\Sigma q)^*$ is a monomorphism, it follows that $\Sigma a_1' = J(\bar{c}')$.

Dual to Theorem 1, we have the following theorem.

THEOREM 4. $c' = eJ(\bar{c}')\Sigma(q\Delta)\colon \Sigma X \to \Sigma X \vee \Sigma X$, *where* c' *is the co-commutator map*.

Let us now again consider $\bar{c}'\colon X_1 \times X_2 \to \Omega(\Sigma X_1 \vee \Sigma X_2)$ in the general case. We have that $\tau^{-1}(\bar{c}')\colon \Sigma(X_1 \times X_2) \to \Sigma X_1 \vee \Sigma X_2$. The co-Hopf construction yields a map $H(\tau^{-1}(\bar{c}'))\colon \Omega\Sigma(X_1 \times X_2) \to \Omega(\Sigma X_1 \flat \Sigma X_2)$. Hence, we have that $(\Omega i)H(\tau^{-1}(\bar{c}'))\colon \Omega\Sigma(X_1 \times X_2) \to \Omega(\Sigma X_1 \vee \Sigma X_2)$.

LEMMA 7. $\Omega(\tau^{-1}(\bar{c}')) = (\Omega i)H(\tau^{-1}(\bar{c}'))$ *and hence*, $\bar{c}' = (\Omega i)H(\tau^{-1}(\bar{c}'))e'$.

Dual to Theorem 2, we have the following theorem.

THEOREM 5. $\tau(c') = (\Omega i)H(\tau^{-1}(\bar{c}'))e'\Delta$, *where* $c'\colon \Sigma X \to \Sigma X \vee \Sigma X$ *is the co-commutator map*.

Let us now consider the maps a' and a_1' defined above. We have the following lemma.

LEMMA 8. $a' = (\Omega i)H(\tau^{-1}(a_1'))$ *and hence*, $a_1' = (\Omega i)H(\tau^{-1}(a_1'))e'$.

THEOREM 6. $\tau(c') = (\Omega i)H(e)\Omega(J(\bar{c}'))e'q\Delta$, *where* $c'\colon \Sigma X \to \Sigma X \vee \Sigma X$ *is the co-commutator map*,

$$e'\colon X \wedge X \to \Omega\Sigma(X \wedge X), \qquad e\colon \Sigma\Omega(\Sigma X \vee \Sigma X) \to \Sigma X \vee \Sigma X$$

are the standard maps, and $\Delta\colon X \to X \times X$ *is the diagonal map*.

Proof. We apply Lemma 8 with $X_1 = X_2 = X$ and obtain

$$a_1' = (\Omega i)H(\tau^{-1}(a_1'))e'.$$

Hence, $\bar{c}' = a_1'q = (\Omega i)H(\tau^{-1}(a_1'))e'q$. Since $\bar{c}'\Delta = \tau(c')$, we have that $\tau(c') = (\Omega i)H(\tau^{-1}(a_1'))e'q\Delta$. Since $\Sigma a_1' = J(\bar{c}')$, we have that $eJ(\bar{c}') = e(\Sigma a_1') = \tau^{-1}(a_1')$. Hence, $\tau(c') = (\Omega i)H(e)\Omega(J(\bar{c}'))e'q\Delta$.

Remark 2. Let e: $\Sigma\Omega(X_1 \vee X_2) \to X_1 \vee X_2$,

$$e': \Omega(X_1 \vee X_2) \to \Omega\Sigma\Omega(X_1 \vee X_2)$$

be the standard maps. Then we have that $H(e)e'$: $\Omega(X_1 \vee X_2) \to \Omega(X_1 \flat X_2)$. Since $(\Omega e)e' = 1_{\Omega(X_1 \vee X_2)}$, it follows that if f: $Y \to X_1 \vee X_2$ is any map, then the co-Hopf construction yields $H(f) = H(e)e'(\Omega f)$. We note that we then have the relation $H(e)e'(\Omega i) = 1_{\Omega(X_1 \flat X_2)}$. Using this, we can "solve" the equations in Lemmas 7 and 8 to obtain $H(\tau^{-1}(\bar{c}')) = H(e)e'\Omega(\tau^{-1}(\bar{c}'))$, $H(\tau^{-1}(a_1')) = H(e)e'a'$ and $H(\tau^{-1}(a_1'))e' = H(e)e'a_1'$.

Remark 3. Our factorizations of c' reflect the well-known result that conil$X \leqq$ w cat$X$, where w cat denotes "weak category".

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