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A CENTRAL LIMIT THEOREM FOR GENERAL STOCHASTIC PROCESSES

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ABSTRACT. We show that under mild conditions the Central Limit Theorem holds for general stochastic processes.

Let $M(\Omega, F)$ be the Banach space of all measures on the measurable space (Ω, F) of all functions ω mapping the nonnegative integers into a measurable space (S, Σ) where F is the σ-field generated by the events $X_n(\omega)=\omega(n)\in U\in \Sigma$. Let T be the linear operator on $M(\Omega, F)$ defined by

$$T\varphi(X_1\in U_1,\ldots,X_n\in U_n)=\varphi(X_2\in U_1,\ldots,X_{n+1}\in U_n),$$

let $E(U)$, $U\in\Sigma$ be the resolution of the identity

$$E(U)\varphi(\Lambda)=\varphi(X_0\in U,\Lambda),$$

and let p^* be the linear functional

$$p^*\varphi=\varphi(\Omega).$$

Suppose that Φ is a linear subspace of $M(\Omega, F)$ which is closed under the operators T, $E(U)$, $U\in\Sigma$; that Φ^* is the Banach space of all bounded linear functionals on Φ; and that T^* and $E(U)^*$, $U\in\Sigma$ are the adjoints of the operators T and $E(U)$, $U\in\Sigma$. Then, noting that the spectrum of T^* contains 1, that p^* is an eigenfunction corresponding to 1 of T^* and that $\int e^{iux}\,T^*E(dx)^*$ converges to T^* as $u\to 0$, we have the following Central Limit Theorem.

THEOREM. *If the operator $\int e^{iux}\,T^*E(dx)^*$ has an eigenvalue of the form $1+iau+bu^2+o(u^2)$ with a corresponding eigenfunction e_u^* which converges to p^* as $u\to 0$ in the norm topology of Φ^*, then for all $\varphi\in\Phi$ with $p^*\varphi=1$ we have*

$$\lim_{n\to\infty}\varphi\left[\frac{X_1+\cdots+X_n-na}{\sqrt{n(a^2+2b)}}<x\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-(u^2/2)}\,du$$

Proof. Noting that for each $\varphi\in\Phi$

$$\varphi[X_1\in U_1,\ldots,X_n\in U_n]=p^*E(U_n)T\cdots E(U_1)T\varphi,$$

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we have

$$\lim_{n\to\infty}\int \exp\left(iu\frac{X_1+\cdots+X_n-na}{\sqrt{n(a^2+2b)}}\right)d\varphi$$

$$=\lim_{n\to\infty}\int\ldots\int \exp\left(iu\frac{x_1+\cdots+x_n-na}{\sqrt{n(a^2+2b)}}\right)p^*E(dx_n)\,T\cdots E(dx_1)\,T\varphi$$

$$=\lim_{n\to\infty}\left[\int \exp\left(iu\frac{x-a}{\sqrt{n(a^2+2b)}}\right)T^*\,E(dx)^*\right]^n p^*\varphi$$

$$=\lim_{n\to\infty}\left[\exp\left(-iu\frac{a}{\sqrt{n(a^2+2b)}}\right)\right]^n\left[\int \exp\left(iu\frac{x}{\sqrt{n(a^2+2b)}}\right)T^*\,E(dx)^*\right]^n$$

$$\times e^*\frac{u}{\sqrt{n(a^2+2b)}}\,\varphi$$

$$=\lim_{n\to\infty}\left[1-iu\frac{a}{\sqrt{n(a^2+2b)}}-\frac{u^2a^2}{2n(a^2+2b)}+o\left(\frac{1}{n}\right)\right]^n$$

$$\times\left[1+ia\frac{u}{\sqrt{n(a^2+2b)}}+b\frac{u^2}{n(a^2+2b)}+o\left(\frac{1}{n}\right)\right]^n e^*\frac{u}{\sqrt{n(a^2+2b)}}\varphi$$

$$=\lim_{n\to\infty}\left[1-\frac{u^2}{2n}+o\left(\frac{1}{n}\right)\right]^n e^*_{u/\sqrt{n(a^2+2b)}}\varphi$$

$$=e^{-(u^2/2)}p^*\varphi$$

and the theorem is proved.

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