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Centralizers involving Mathieu groups

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A simple group G cannot contain a central involution t with $C_G(t) = \langle t \rangle \times M$, where M is isomorphic to a simple Mathieu group.

There have been investigations of groups G which contain a central involution t such that $C(t)$ has the form $\langle t \rangle \times M$ where M is a simple non-abelian group ([1], [4], [5]). In this note, the case where M is isomorphic to a Mathieu group is considered.

THEOREM. *Let G be a finite group with a central involution t such that $C(t) = \langle t \rangle \times M$ where M is isomorphic to any one of the simple Mathieu groups. Then $G = O(G).C(t)$.*

Proof. Since t is central, $C(t)$ contains an S_2-subgroup S of G with $t \in Z(S)$. We show t is not conjugate in G to any other involution in S and the result then follows by Glauberman's Z^*-theorem ([2]).

(a) First suppose M is isomorphic to M_{11}, M_{22} , or M_{23} . Then M has only one class of involutions with representative z say. Since z is the square of an element of order 4 in M , it follows from the structure of $C(t)$ that t cannot be conjugate to z in G . If $t \sim tz$ in G , say $(tz)^a = t$ for some $a \in G$, then $t \in C(tz)$. So $t^a \in C(t)$ and without loss of generality we may suppose $t^a = tz$. Thus a

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normalizes $C(t, z)$ with $a^2 \in C(t, z)$, so $|\langle C(t, z), a\rangle| = 2|C(t, z)|$. This contradicts the fact that $C(t, z)$ contains a S_2-subgroup of G. Thus t is not conjugate to tz in G.

(b) Suppose M is isomorphic to M_{12} or M_{24}. Then M has two classes of involutions; a central class with representative z say (which is again the square of an element of order 4 in M), and a non-central class with representative y say. (When $M \approx M_{12}$ take $z = \pi$ and $y = \tau$ in [6], and when $M \approx M_{24}$ take $z = z_1$ and $y = z_3\pi$ in [3].)

(i) As in (a) above, t cannot be conjugate to z or tz.

(ii) Suppose $y \sim t$ in G, say $y^b = t$ for $b \in G$. Then $t^b \in C(t)$ and we may suppose $t^b = y$ or ty. In either case, b centralizes $C(t, y)$.

Now $C(t, y) = \langle t\rangle \times C_M(y)$ and $S = \langle t\rangle \times C_M(y, z)$ is an S_2-subgroup of $C(t, y)$ (see [3], [6]). Since S^b is also an S_2-subgroup of $C(t, y)$, $S^b = S^g$ for some $g \in C(t, y)$.

Thus $S^b = \langle t\rangle \times C_M(y, z^m)$ where $g = t^\alpha m$; $\alpha = 0$ or $\alpha = 1$ and $m \in M$. However $S' = \langle z\rangle$ when $M \approx M_{12}$, and $S'' = \langle z\rangle$ when $M \approx M_{24}$ (Lemma 1 in [6], Lemma 2.3 in [3]); so b conjugates $\langle z\rangle$ to $\langle z^m\rangle$. Replacing b by $c = bm^{-1}$ we have $y^c = t$ and $z^c = z$.

However a calculation shows $yz \sim y$ in M. Conjugating this relation by c we have $tz \sim t$ in G, which contradicts (i) above. Thus y is not conjugate to t in G.

(iii) Finally suppose $ty \sim t$ in G, say $(ty)^d = t$ for some $d \in G$. Then, as above, we may assume $t^d = ty$ and further, as in (ii), we may find an $e \in G$ such that $t^e = ty$ and $z^e = z$. Thus $(tz)^e = tyz \sim ty$ in G, again contradicting (i). So t is not conjugate to ty in G and the result now follows from Glauberman's Z^*-theorem.

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