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DISCRETE COPIES OF RINGS OF SETS IN GROUPS AND ORLICZ-PETTIS THEOREMS

IWO LABUDA

1. Introduction. The Orlicz-Pettis type theorems are essentially statements about the equivalence of strong and weak subseries convergence of series in topological groups. The corresponding problem of the equivalence of subfamily summability for summable families was examined for instance in [**2**] and (**16**). We give here another uncountable generalization of the Orlicz-Pettis theorem in the sense that we assume less than subfamily summability and obtain straightforward generalizations of the classical case, which include or imply in a trivial manner—at least for polar topologies on groups—all known theorems "from weak to strong σ-additivity". This is attained by considering additive set functions acting from rings (generalizing $\mathscr{F}(\mathbf{N})$ and $\mathscr{P}(\mathbf{N})$) over an arbitrary set Γ, and by using concepts of a *discrete copy of a ring of sets in a group* and those of an *almost* $\mathfrak{N}$*-concentrated on* Γ or $\mathfrak{N}$*-exhaustive* additive set function.

In the last decade a huge number of papers on exhaustive (= strongly bounded) additive set functions appeared. The Orlicz-Pettis type theorems "from weak to strong exhaustivity" are available (see e.g. [**8**]). This phenomenon seems to be a consequence (comp. [**11**]) of a remarkable result of Drewnowski [**3**] linking exhaustivity and σ-additivity in metrizable groups. An uncountable analogue of Drewnowski's Theorem is also obtained in our approach (Theorem 3.2 below).

2. Preliminaries. X will be a Hausdorff topological commutative group throughout. Γ denotes an infinite set of arbitrary but fixed cardinality $\mathfrak{M}$, $\mathfrak{N}$ denotes an infinite cardinal number. $\mathscr{F}(\Gamma)$ is the ring of finite subsets of Γ, $\mathscr{P}(\Gamma)$—the power set of Γ. More generally, $\mathscr{O}_{\mathfrak{N}}(\Gamma) = \mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) = \{E \subset \Gamma: \text{card } E < \mathfrak{N}\}$ where card E denotes the cardinality of E. If f is a real function on Γ then $\text{supp} f = \{\gamma \in \Gamma: f(\gamma) \neq 0\}$. Cardinals are identified with initial ordinals.

Let $\mathscr{A}(\Gamma)$ be a ring of subsets of Γ *containing points*. We denote by $S(\Gamma)$ or more precisely $S(\Gamma, \mathscr{A})$ the set of $\mathscr{A}(\Gamma)$-simple functions with integer-valued coefficients, i.e. the set of functions of the form

$$\sum_{i=1}^{n} k_i \chi_{E_i}$$

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where $E_i \in \mathscr{A}(\Gamma)$, $E_i \cap E_j = \emptyset$ if $i \neq j$ and k_i are the integers. For instance, if $\mathscr{A}(\Gamma) = \mathscr{P}(\Gamma)$ then $S(\Gamma, \mathscr{P}) = S_\infty(\Gamma)$ is the space of all integer-valued bounded functions over Γ.

$\mathscr{A}(\Gamma)$ being fixed, for any $N \in \mathbf{N}$ we denote by $S_N(\Gamma) = S_N(\Gamma, \mathscr{A})$ the set of functions in $S(\Gamma)$ which are of the form

$$\sum_{i=1}^{n} k_i \chi_{E_i}, \quad |k_i| \leqq N, 0 \leqq n \leqq 2N.$$

We equip $S_\infty(\Gamma)$ with the supremum norm topology and consider all subsets of $S_\infty(\Gamma)$ with the induced topology. We note that for all $\mathscr{A}(\Gamma)$ we have $S(\Gamma, \mathscr{A}) \subset S_\infty(\Gamma)$ and that, in other terms, $S_\infty(\Gamma)$ has the discrete topology given by the $0 - 1$ metric.

Having an additive set function $x(:) : \mathscr{A}(\Gamma) \to X$, we will denote by I and call the *integral extension* (of $x(:)$ to $S(\Gamma)$) the homorphism defined by the usual formula

$$I\left(\sum_{i=1}^{n} k_i \chi_{E_i}\right) = \sum_{i=1}^{n} k_i x(E_i).$$

2.1. *Definition.* Let T be a subset of $S(\Gamma, \mathscr{A})$ and $H: S(\Gamma, \mathscr{A}) \to X$ a homomorphism. We say that $H|_T$ is an *isomorphism* (onto its image) if there exists a neighborhood W of zero in X such that

$$(\delta) \quad f, g \in T, f \neq g \Rightarrow H(f) - H(g) \notin W.$$

This denomination is justified by the fact that the condition (δ) means also that the topology of X induces on $H(T)$ the discrete topology and $H|_T$ is a homeomorphism (onto its image).

2.2. *Definition.* We say that X *contains an isomorphic copy* of T if there exist a homomorphism $H: S(\Gamma, \mathscr{A}) \to X$ such that $H|_T$ is an isomorphism.

An additive set function may always be treated as the restriction of its integral extension so, according to definitions 1 and 2, we have the following.

2.3. *Definition.* An additive set function $x(:): \mathscr{A}(\Gamma) \to X$ is called an *isomorphism* if

$$E, F \in \mathscr{A}(\Gamma), E \neq F \Rightarrow x(E) - x(F) \notin W$$

and we say that X *contains an isomorphic copy* or a *discrete copy of a ring* $\mathscr{A}(\Gamma)$ if there exists an isomorphism $x(:)$ transforming $\mathscr{A}(\Gamma)$ into X.

2.4 *Remark.* We note that the following conditions are equivalent:

(i) For any $f \in S_1(\Gamma), f \neq 0 \Rightarrow I(f) \notin W$.

(ii) $I: S_1(\Gamma) \to X$ is an isomorphism.

(iii) $x(:): \mathscr{A}(\Gamma) \to X$ is an isomorphism.

Indeed, (i) $\Leftrightarrow$ (ii) is obvious. Take $E, F \in \mathscr{A}(\Gamma)$, $E \neq F$. Then by (ii) $x(E) - x(F) = x(E \backslash F) - x(F \backslash E) = I(\chi_{E \backslash F} - \chi_{F \backslash E}) \notin W$ since $\chi_{E \backslash F} -$

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$\chi_{F\setminus E} \neq 0$. Hence (ii) $\Rightarrow$ (iii). Take $f \in S_1(\Gamma)$, $f \neq 0$. Then $f = \chi_E - \chi_F$ where $E, F \in \mathscr{A}(\Gamma)$ and are disjoint. Consequently, $I(f) = x(E) - x(F) \notin W$ by (iii). Hence (iii) $\Rightarrow$ (ii).

Let $\Delta \subset \Gamma$ be given and let $T = T(\Gamma)$ be a subset of $S_\infty(\Gamma)$. We denote

$$T(\Delta) = \{f \in T(\Gamma) : \operatorname{supp} f \subset \Delta\}.$$

2.5. *Definition*. Let $x(:) : \mathscr{A}(\Gamma) \to X$ be an additive set function. $x(:)$ is said to be *almost* $\mathfrak{N}$*-concentrated* on Γ if it satisfies the following condition:

$$\forall\ V \exists\ A \in \mathscr{O}_{\mathfrak{N}}(\mathfrak{M})\ \forall\ B \in \mathscr{A}(\Gamma),\ B \cap A = \emptyset \Rightarrow x(B) \in V$$

where V is a neighborhood of zero in X.

2.6. *Definition* [**13**]. Let $x(:) : \mathscr{A}(\Gamma) \to X$ be an additive set function. $x(:)$ is said to be $\mathfrak{N}$-exhaustive if

$$\forall\ V \forall\ (E_i)_{i \in I} \subset \mathscr{A}(\Gamma),\ E_i \cap E_j = \theta \text{ if } i \neq j,$$
$$\operatorname{card} \{i \in I : x(E_i) \notin V\} < \mathfrak{N}$$

where V is a neighborhood of zero in X.

Let $\mathscr{T}$ and $\mathscr{S}$ be two group topologies on X. We say that $\mathscr{T}$ is $\mathscr{S}$*-polar* if $\mathscr{T}$ has a base of $\mathscr{S}$-closed neighborhoods of zero.

2.7. LEMMA. *Let $\mathscr{T}$, $\mathscr{S}$ be two Hausdorff group topologies on X such that $\mathscr{T}$ is $\mathscr{S}$-polar. Let $x(:) : \mathscr{A}(\Gamma) \to (X, \mathscr{S})$ be an additive set function almost $\mathfrak{M}$-concentrated on Γ. Assume that for some $\mathscr{T}$-neighborhood U of zero in X the set $\Delta = \{\gamma \in \Gamma : x(\{\gamma\}) \notin U\}$ is of the same cardinality as Γ. Then there exists $\Gamma_1 \subset \Delta$ with* card $\Gamma_1 =$ card Γ *such that*

$$x(:) : \mathscr{A}(\Gamma_1) \to (X, \mathscr{T})$$

is an isomorphism [i.e. $(X, \mathscr{T})$ contains a discrete copy of $\mathscr{A}(\Gamma_1)$].

Proof. If α is an ordinal number, then P_α will denote the set of all ordinals less than α. Let μ be the (least) ordinal number α with card $P_\alpha = \mathfrak{M} =$ card Γ. For each $\alpha < \mu$ we denote $F_\alpha = \{\beta : \alpha \leqq \beta < \mu\}$.

The fact that $x(:) : \mathscr{A}(\Gamma) \to (X, \mathscr{S})$ is almost $\mathfrak{M}$-concentrated on Γ implies in the present situation:

$$(1)\quad \forall\ V \exists\ \alpha < \mu\ \forall\ E \in \mathscr{A}(\Gamma),\ E \in F_\alpha \Rightarrow x(E) \in V$$

where V denotes an $\mathscr{S}$-neighborhood of zero in X. It is clear, in view of the very definition of $S_1(\Gamma)$, that (1) implies

$$(2)\quad \forall\ V \exists\ \alpha < \mu\ \forall f \in S_1(F_\alpha),\ I(f) \in V.$$

Since $\mathscr{T}$ is $\mathscr{S}$-polar we can choose an $\mathscr{S}$-closed balanced $\mathscr{T}$-neighborhood W of zero in W such that $W \subset$ U. Hence

$$(3)\quad x(\{\gamma\}) \notin W \quad \text{for } \gamma \in \Gamma.$$

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We claim that the following holds:

(4) For any $\sigma \in \Gamma$ there exists $\alpha(\sigma)$, $\sigma < \alpha(\sigma) < \mu$

such that $I(\{\sigma\} + f) \notin W$ for any $f \in S_1(F_{\alpha(\sigma)})$.

Indeed, assume that (4) does not hold for some $\sigma \in \Gamma$. Then, given any $\mathscr{S}$-neighborhood of zero in G, V say, we can find an α so large that for some $f \in S_1(F_\alpha)$

(5) $I(\{\sigma\} + f) \in W$

(6) $I(f) \in V$.

This means that $I(\{\sigma\})$ is in the $\mathscr{S}$-closure of W, hence in W—a contradiction with (3).

W is balanced, hence having $I(\{\sigma\} + f) \notin W$ as in (4), by the definition of I, $I(-\{\sigma\} + f) = -I(\{\sigma\} - f) \notin W$. Thus

(7) For any $\sigma \in \Gamma$ there exists $\alpha(\sigma)$, $\sigma < \alpha(\sigma) < \mu$, such that for any

$f \in S_1(F_{\alpha(\sigma)})$ $I(\pm\{\sigma\} + f) \notin W$.

We shall define an increasing transfinite sequence $(\eta(\alpha))$ on Γ (with terms in Γ) such that

$$I(\pm\eta(\alpha) + f) \notin W \quad \text{if} \quad f \in S_1(\{\eta(\gamma): \alpha < \gamma < \mu\}).$$

We put $\eta(0) = 0$. Suppose that α has no predecessor, $\alpha < \mu$, and that we have already defined all the terms $\eta(\gamma)$, where $\gamma < \alpha$, $\eta(\gamma_1) < \eta(\gamma_2)$ if $\gamma_1 < \gamma_2 < \alpha$ and $I(\pm\eta(\gamma) + f) \notin W$ for all $f \in S_1(F_{\eta(\alpha)})$. Then choose τ for which (7) holds (with $\sigma = \alpha$) such that $\eta(\gamma) < \tau$ for all $\gamma < \alpha$. If α has the predecessor $\alpha - 1$ we choose τ according to (7) with $\sigma = \eta(\alpha - 1)$. Then we set $\tau = \eta(\alpha)$.

By the definition for $(\eta(\alpha)) = \Gamma_1$ we have card Γ_1 = card Γ (since Γ_1 is increasing) and

(8) For every $\sigma \in \Gamma_1$, $I(\pm\{\sigma\} + f) \notin W$ if $f \in S_1(\Gamma_1 \cap F_{\sigma+1})$

which means obviously that taking any $f \in S_1(\Gamma_1)$, $f \neq 0$, we have $I(f) \notin W$. This implies the result in view of Remark 1.

Consider the following property of $\mathscr{A}(\Gamma)$:

(κ) For each family $(B_\alpha)_{\alpha\in\Gamma}$ of disjoint elements of $\mathscr{A}(\Gamma)$ the set

$(\bigcup_{\alpha\in E} B_\alpha) \in \mathscr{A}(\Gamma)$ for any $E \in \mathscr{A}(\Gamma)$.

Denote $\mathfrak{N} = \sup\{\text{card } E: E \in \mathscr{A}(\Gamma)\}$ i.e., the least cardinal number $\mathfrak{N}$ such that for each $E \in \mathscr{A}(\Gamma)$ card $E < \mathfrak{N}$. A moment's reflection shows that if (κ) is satisfied, $\mathscr{A}(\Gamma)$ must be $\mathfrak{N}$-complete for disjoint elements as a subring of $\mathscr{P}(\Gamma)$. The latter implies clearly that $\mathscr{A}(\Gamma)$ contains all sets $E \subset \Gamma$ with card $E < \mathfrak{N}$ as we have assumed that $\mathscr{A}(\Gamma)$ contains the points. Consequently,

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$\mathscr{A}(\Gamma) = \mathscr{O}_{\mathfrak{N}}(\mathfrak{M})$ which justifies the restriction to the rings $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M})$ in the following lemma.

2.8. LEMMA. *Let $\mathscr{T}$, $\mathscr{S}$ be two Hausdorff group topologies on X such that $\mathscr{T}$ is $\mathscr{S}$-polar. Let $x(:)$ $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \rightarrow (X, \mathscr{S})$ be an additive set function almost $\mathfrak{M}$-concentrated on Γ. If $(X, \mathscr{T})$ contains no discrete copy of $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M})$ then $x(:)$: $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \rightarrow (X, \mathscr{T})$ is $\mathfrak{M}$-exhaustive.*

Proof. If $x(:)$ is not $\mathfrak{M}$-exhaustive we can find $\mathfrak{M}$ disjoint sets $(B_\gamma)_{\gamma \in \Gamma}$ such that $B_\gamma \in \mathscr{O}_{\mathfrak{N}}(\mathfrak{M})$ and $x(B_\gamma) \notin V$. Identifying each B_γ with the point γ in Γ and putting

$$y(E) = x\left(\bigcup_{\gamma \in E} B_\gamma\right), \quad E \in \mathscr{O}_{\mathfrak{N}}(\Gamma),$$

we define an additive set function $y(:)$: $\mathscr{O}_{\mathfrak{N}}(\Gamma) \rightarrow X$ such that $y(\{\gamma\}) \notin V$ for each $\gamma \in \Gamma$.

It is easy to see that $y(:)$: $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \rightarrow (X, \mathscr{S})$ is almost $\mathfrak{M}$-concentrated on Γ. Hence by Lemma 2.7 $(X, \mathscr{T})$ contains a discrete copy of $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M})$. This contradicts the assumption of the lemma.

2.9. *Remark.* The proofs of Lemmas 2.7 and 2.8 involve no essentially new techniques. Similar methods were already employed in [**4**; **5**; **6**; **9**; and **13**].

2.10. PROPOSITION. *An $\mathfrak{M}$-exhaustive additive set function $x(:)$: $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \rightarrow X$, $\mathfrak{N} \leqq \mathfrak{M}$, is almost $\mathfrak{M}$-concentrated on Γ.*

Proof. If $x(:)$ is not almost $\mathfrak{M}$-concentrated on Γ then we can find inductively $\mathfrak{M}$ disjoint sets E_i in $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M})$ such that $x(E_i) \notin V$ for some V, thus contradicting the $\mathfrak{M}$-exhaustivity of $x(:)$.

2.11. *Remark.* If $\mathfrak{N} > \mathfrak{M}$ i.e., if $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) = \mathscr{P}(\Gamma)$, an $\mathfrak{M}$-exhaustive additive set function need not be almost $\mathfrak{M}$-concentrated. Indeed, it is well known that for $\mathfrak{M} = \aleph_0$ $x(:)$: $\mathscr{P}(\mathbf{N}) \rightarrow \mathbf{R}$ may be bounded ($= \aleph_0$-exhaustive) and not σ-additive ($=$ almost $\aleph_0$-concentrated on $\mathbf{N}$).

2.12. PROPOSITION. *An almost $\mathfrak{M}$-concentrated additive set function $x(:)$: $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \rightarrow X$ is $\mathfrak{M}$-exhaustive.*

Proof. Let V and A as appearing in Definition 2.5 be given. Take a family $\mathscr{E}$ of $\mathfrak{M}$-disjoint elements of $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M})$. Since card $A < \mathfrak{M}$, card $\{E \in \mathscr{E} : E \cap A \neq \phi\} < \mathfrak{M}$. This implies the result.

2.13. PROPOSITION. *Let $x(:)$: $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \rightarrow X$ be an almost $\mathfrak{N}$-concentrated additive set function, $\mathfrak{N} \leqq \mathfrak{M}$. Let $\Delta \subset \Gamma$, card $\Delta = \mathfrak{N}$ and $\Delta \cap \mathscr{O}_{\mathfrak{N}}(\mathfrak{M})$ be directed by inclusion. Then $\{x(E)$: $E \in \Delta \cap \mathscr{O}_{\mathfrak{N}}(\mathfrak{M})\}$ is a Cauchy net.*

Proof. By assumption on $x(:)$ $\forall$ $V \exists$ $A \in \mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \cap \Delta \, \forall B \in \mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \cap \Delta$, $A \cap B = \emptyset \Rightarrow x(B) \in V$. Hence, taking $F, G \supset A$ we have $x(F) - x(G) = x(F \backslash G) - x(G \backslash F) \in V - V$.

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With the assumption of the lemma, let $\mathfrak{N}^+ = \text{card}\,\eta$ where $\eta = \inf\{\gamma: \gamma$ is an ordinal such that $\text{card}\,\gamma > \mathfrak{N}\}$. Assume X is complete and define $\tilde{x}(:)$: $\mathscr{O}_{\mathfrak{N}^+}(\mathfrak{M}) \to X$ by

$$\tilde{x}(:)|\ \mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \equiv x(:)$$

and, if $\Delta \in \mathscr{O}_{\mathfrak{N}^+}(\mathfrak{M}) \backslash \mathscr{O}_{\mathfrak{N}}(\mathfrak{M})$ then $\tilde{x}(\Delta) = \lim\{x(E): E \in \Delta \cap \mathscr{O}_{\mathfrak{N}}(\mathfrak{M})\}$.

By the definition, (1) $\tilde{x}(:)$ is additive, and (2) $\tilde{x}(:)$ is almost $\mathfrak{N}$-concentrated on Γ.

Now, let $x(:)$: $\mathscr{P}(\Gamma) \to X$ be an $\mathfrak{M}$-exhaustive additive set function. The restriction $x(:)|\ \mathscr{O}_{\mathfrak{M}}(\mathfrak{M})$ is almost $\mathfrak{M}$-concentrated on Γ by 2.10. We can therefore define

$$x'(:): \mathscr{P}(\Gamma) \to X$$

by putting $x'(:) \equiv \tilde{x}(:)$. We note that (3) $z(:) = (x - x')(:)$ is $\mathfrak{M}$-exhaustive, and (4) $z'(:) \equiv 0$ [since $z(:) \equiv 0$ on $\mathscr{O}_{\mathfrak{M}}(\mathfrak{M})$].

An additive $\mathfrak{M}$-exhaustive set function having property (4) will be called *purely $\mathfrak{M}$-exhaustive.*

2.14. PROPOSITION. *Let X be complete. An $\mathfrak{M}$-exhaustive additive set function $x(:)$: $\mathscr{P}(\Gamma) \to X$ may be uniquely represented in the form*

$$x(:) = y(:) + z(:)$$

where $y(:)$ is almost $\mathfrak{M}$-concentrated and $z(:)$ is purely $\mathfrak{M}$-exhaustive. Consesequently $y(:) \equiv x'(:)$.

Proof. Let $x = u + w$ be another such representation. Then $(z - w)(:) \equiv (u - y)(:)$ and $(z - w)'(:) \equiv 0$. Hence $(u - y)'(:) \equiv 0$ but $(u - y)'(:) \equiv (u - y)(:)$, thus $u(:) = y(:)$.

3. Main results.

3.1. THEOREM (Orlicz-Pettis Theorem). *Let $\mathscr{T}$, $\mathscr{S}$ be two Hausdorff group topologies on X such that either*

A. *$\mathscr{T}$ is $\mathscr{S}$-polar, or*

B. *$(X, \mathscr{T})$ is complete, metrizable, and $\mathscr{S} \subset \mathscr{T}$.*

Let $x(:)$: $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \to (X, \mathscr{S})$ be an additive set function almost $\mathfrak{M}$-concentrated on Γ. If $(X, \mathscr{T})$ contains no discrete copy of $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M})$ then $x(:)$: $\mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \to (X, \mathscr{T})$ is almost $\mathfrak{M}$-concentrated on Γ.

Proof. A. $\mathscr{T}$ is $\mathscr{S}$-polar. Using [**1**, II § 3, no 3, Prop. 7], and [**12**, Lemma 2.3] we may assume that $\mathscr{S} \subset \mathscr{T}$ and $(X, \mathscr{T})$ is complete. By 2.10 only the case of $x(:)$ on $\mathscr{P}(\Gamma)$ needs proof. Applying 2.14 it is sufficient to show that the purely $\mathfrak{M}$-exhaustive part $z(:)$ of $x(:)$ is identically zero. This follows from the fact that $z(:)$: $\mathscr{P}(\Gamma) \to (X, \mathscr{S})$ is simultaneously purely $\mathfrak{M}$-exhaustive and almost $\mathfrak{M}$-concentrated on Γ.

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B. $(X, \mathscr{T})$ is complete, metrizable and $\mathscr{S} \subset \mathscr{T}$. Let $\mathscr{W}$ be the strongest group topology on X such that $\mathscr{W} \subset \mathscr{T}$ and $x(:)\colon \mathscr{O}_{\mathfrak{N}}(\mathfrak{M}) \to (X, \mathscr{W})$ is almost $\mathfrak{M}$-concentrated on Γ. Let $\overline{\mathscr{W}}$ be the topology with a base of neighborhoods of zero formed by $\mathscr{W}$-closures of $\mathscr{T}$-neighborhoods of zero. Suppose that $\overline{\mathscr{W}}$ is strictly stronger than $\mathscr{W}$. Then $x(:)$ into $(X, \overline{\mathscr{W}})$ cannot be almost $\mathfrak{M}$-concentrated on Γ. As $\overline{\mathscr{W}}$ is $\mathscr{W}$-polar we can apply part A to get an isomorphism into $(X, \overline{\mathscr{W}})$ hence into $(X, \mathscr{T})$ thus contradicting the assumption of the theorem. Consequently $\overline{\mathscr{W}} = \mathscr{W}$. This means also that the identity operator $j\colon (X, \mathscr{W}) \to (X, \mathscr{T})$ is almost continuous. Hence $\mathscr{W} = \mathscr{T}$ by the Closed Graph Theorem [**10**, p. 213] which proves the result.

3.2. Theorem (comp. Drewnowski [**3**]). *Assume X has a base $\mathscr{V} = (V_\alpha)_{\alpha \in A}$ of neighborhoods of* 0 *with* card $A \leqq \mathfrak{M}$. *Let $x(:)\colon \mathscr{P}(\Gamma) \to X$ be an $\mathfrak{M}$-exhaustive additive set function. Then there exists Δ with* card $\Delta =$ card Γ *such that $x(:)|\mathscr{P}(\Delta) \equiv x'(:)|\mathscr{P}(\Delta)$. In particular, $x(:)\colon \mathscr{P}(\Delta) \to X$ is almost $\mathfrak{M}$-concentrated on Δ.*

Proof. I) Assume that $x(:)$ is purely $\mathfrak{M}$-exhaustive. By a theorem or Sierpinski and Tarski [**15**, p. 448] there is a family $\mathscr{G}$ of subsets in Γ such that card $\mathscr{G} > \mathfrak{M}$, for any $\Gamma_\alpha \in \mathscr{G}$, card $\Gamma_\alpha = \mathfrak{M}$, and card $(\Gamma_\alpha \cap \Gamma_\beta) < \mathfrak{M}$ if $\Gamma_\alpha, \Gamma_\beta \in \mathscr{G}$, $\Gamma_\alpha \neq \Gamma_\beta$.

Given any $V_\alpha \in \mathscr{V}$ we claim that

$$(*) \quad \text{card}\left\{\Delta \in \mathscr{G} : \underset{\Delta' \subset \Delta}{\exists}\ x(\Delta') \notin V_\alpha\right\} < \mathfrak{M}.$$

Otherwise we can produce $\mathfrak{M}$ disjoint sets Δ_γ for which $x(\Delta_\gamma) \notin V_\alpha$, thus obtaining a contradiction to the $\mathfrak{M}$-exhaustivity of $x(:)$. Indeed, let Δ_γ, $\gamma < \delta < \mathfrak{M}$ be already defined. Since $\delta < \mathfrak{M}$ we find $\Delta_\beta \in \mathscr{G}$ $\Delta_\beta \neq \Delta_\gamma$ for $\gamma < \delta$ such that $x(\Delta_\beta') \notin V_\alpha$ for some $\Delta_\beta' \subset \Delta_\beta$. As $x(:)$: is purely $\mathfrak{M}$-exhaustive, card $\Delta_\beta' = \mathfrak{M}$. Furthermore, card $\{\bigcup_{\gamma<\delta} \Delta_\gamma \cap \Delta_\beta'\} < \delta$ card $(\Delta_\beta \cap \Delta_\gamma) < \mathfrak{M}$. Define $\Delta_\delta = \Delta_\beta' \backslash \{\bigcup_{\gamma<\delta} \Delta_\gamma \cap \Delta_\beta'\}$. Applying once more the fact that $x(:)$ is purely $\mathfrak{M}$-exhaustive, $x(\Delta_\delta) = x(\Delta_\beta') \notin V_\alpha$. This finishes the inductive definition of (Δ_γ) and proves $(*)$.

Now, as card $\mathscr{G} > \mathfrak{M}$, card $A \leqq \mathfrak{M}$ and $(*)$ holds, there must be (at least $\mathfrak{M}^+$ by the way) sets $\Delta \in \mathscr{G}$ such that $x(\mathscr{P}(\Delta\ ) = 0$.

II) If $x(:)$ is not purely $\mathfrak{M}$-exhaustive then by 2.14 $x(:) = x'(:) + z(:)$. Applying the reasoning above we find $\Delta \in \mathscr{G}$ such that $z(\Delta) \equiv 0$. Hence $x(:) = x'(:)$ on $\mathscr{P}(\Delta)$. This finishes the proof.

3.3. *Remarks.* 1) The proof above is completely different from the original one of Drewnowski. I am indebted to Z. Lipecki for discussions concerning the relevance of Sierpinski-Tarski Theorem for the proof of Drewnowski's results.

2) The countable case of 3.1 is implicitly contained already in [**5**] and was applied in [**14**]. The typical application of 3.1B is as follows.

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Assume that the character of density of X is at most $\mathfrak{M}$. Then X cannot contain any discrete copy of $\mathscr{P}(\mathfrak{M})$ and therefore any $x(:): \mathscr{P}(\Gamma) \rightarrow X$ which is almost $\mathfrak{M}$-concentrated on Γ for some Hausdorff group topology weaker than the original one on X is almost $\mathfrak{M}$-concentrated on Γ (for the original topology on X). In particular, if X is separable the subseries convergence coincide for all Hausdorff topologies weaker than the original one on X (Kalton [**7**]); if $\mathfrak{M}$ is uncountable, then, since the topology of X has a countable base of neighborhoods of zero, there is $A \subset \Gamma$ with card $A < \mathfrak{M}$ such that $x(:)|\mathscr{P}(\Gamma \backslash A) \equiv 0$, i.e., $x(:)$ is concentrated on $\mathscr{P}(A)$.

For further results in the same direction, see the supplement to this paper which will appear in Bull. Acad. Polon. Sci.

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University of Florida,
Gainesville, Florida;
Mathematical Institute of The Polish Academy of Sciences,
ul. Mielzynskiego 27/29, 61725 Poznan, Poland

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