BULL. AUSTRAL. MATH. SOC. 49J50, 46B22, 46B20
VOL. 60 (1999) [109–118]

SMOOTH VARIATIONAL PRINCIPLES IN RADON-NIKODÝM SPACES

ROBERT DEVILLE AND ABDELHAKIM MAADEN

We prove that if f is a real valued lower semicontinuous function on a Banach space X, for which there exist $a > 0$ and $b \in \mathbb{R}$ such that $f(x) \geqslant 2a\|x\| + b, x \in X$, and if X has the Radon-Nikodým property, then for every $\varepsilon > 0$ there exists a real function φ on X such that φ is Fréchet differentiable, $\|\varphi\|_\infty < \varepsilon$, $\|\varphi'\|_\infty < \varepsilon$, φ' is weakly continuous and $f + \varphi$ attains a minimum on X. In addition, if we assume that the norm in X is β-smooth, we can take the function $\varphi = g_1 + g_2$ where g_1 is radial and β-smooth, g_2 is Fréchet differentiable, $\|g_1\|_\infty < \varepsilon$, $\|g_2\|_\infty < \varepsilon$, $\|g_1'\|_\infty < \varepsilon$, $\|g_2'\|_\infty < \varepsilon$, g_2' is weakly continuous and $f + g_1 + g_2$ attains a minimum on X.

1. INTRODUCTION

Let X be a Banach space. A function $f : X \longrightarrow \mathbb{R}$ is said to be Gâteaux differentiable at $x_0 \in X$ provided the limit $f'_G(x_0)(x) := \lim\limits_{t\to 0}\Big(f(x_0 + tx) - f(x_0)\Big)\Big/t$ exists for each $x \in X$, and the operator $f'_G(x_0)(.)$ is continuous and linear. The function $f'_G(x_0)$ is called the Gâteaux derivative (or Gâteaux differential) of f at x_0.

A bornology on X, denoted by β, will be any nonempty family of bounded sets whose union is all of X. Recall that the Gâteaux bornology consists of all singletons, and the Fréchet bornology consists of all bounded sets.

A function $f : X \longrightarrow \mathbb{R}$ is called β-smooth if it is Gâteaux differentiable and the defining limit exists uniformly on members of β.

A Banach space X has the Radon-Nikodým property if every bounded linear operator T from $L^1\big([0,1]\big)$ into X is representable, which means that there exists $g \in L^\infty\big([0,1], X\big)$ such that for $f \in L^1\big([0,1]\big), Tf = \int_0^1 f(t)\, g(t)\, dt$. The classical Radon-Nikodým theorem expresses the fact that $\mathbb{R}$ has the Radon-Nikodým property. Let us mention that every reflexive Banach space has the Radon-Nikodým property. On the other hand, $L^1(\mathbb{R}^n)$ and the space of bounded uniformly continuous functions on $\mathbb{R}^n$ fail the Radon-Nikodým property. This property has been extensively studied in recent years, and there is an equivalent geometrical definition of this property. Recall, first, the definition of a slice $S(x^*, A, \alpha)$ of a nonempty subset A of a Banach space X : For $\alpha > 0$ and $x^* \in X^*$, $S(x^*, A, \alpha) := \Big\{x \in A : \langle x^*, x\rangle > \sup\limits_A x^* - \alpha\Big\}$. We say that a nonempty subset A of X is

Received 8th February, 1999

dentable provided it admits slices of arbitrarily small diameter; that is, for every $\varepsilon > 0$ there exists $x^* \in X^*$ and $\alpha > 0$ such that diam $S(x^*, A, \alpha) < \varepsilon$. A subset A is said to have the Radon-Nikodým property if every nonempty bounded subset of A is dentable. For more details of the Radon-Nikodým property see [**4, 8**].

Let X be a Banach space. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function bounded from below. By a variational principle we mean the existence of a function $\varphi : X \longrightarrow \mathbb{R}$ belonging to a given class such that $f + \varphi$ attains its minimum on X. The first one was established by Ekeland [**9**], who showed that one can take φ to be a shift of the function $\varepsilon\|.\|$, where ε is an arbitrarily small positive number.

If we want φ to be smooth, then we speak about a smooth variational principle. The first result of this type was shown by Stegall [**12**] : Let X be a Banach space. Let C be a closed, convex and bounded subset of X with the Radon-Nikodým property. Let f be a lower semicontinuous function and bounded below on C. Then the set $\{x^* \in X^* : f + x^* \text{ attains a minimum on } C\}$ is residual in X^*.

Note that, there is an important class of Banach spaces satisfying the smooth variational principle but not having the Radon-Nikodým property. Indeed, c_0 does not have the Radon-Nikodým property and admits a C^∞-smooth norm [**2**], so, it satisfies the smooth variational principle. Borwein-Preiss [**3**] proved a smooth variational principal imposing no additional conditions on the space except the presence of some smooth norm.

The existence of a smooth norm implies the presence of a smooth bump function (a function $b : X \longrightarrow \mathbb{R}$ is bump if it has nonempty and bounded support); but the converse is not true in general. Haydon [**10**] gives an example of Banach space with smooth bump function but not having an equivalent smooth norm. Deville-Godefroy-Zizler [**5, 6**] extended the Borwein-Preiss principle to spaces admitting smooth bump functions. The proof of their principle uses the Baire category theorem.

The work presented in this paper is motivated by the Stegall variational principle [**12**]. A variant of Stegall's theorem due to Fabian [**11**] yields a minimum for a lower semicontinuos function f defined on a space with the Radon-Nikodým property. It replaces the condition that f be restricted to a bounded set with a strong lower-boundedness hypothesis on all of X : Suppose that the Banach space X has the Radon-Nikodým property and that $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function on X for which there exist $a > 0$ and $b \in \mathbb{R}$ such that $f(x) \geqslant 2a\|x\| + b$, $x \in X$. Then for any $\varepsilon > 0$ there exists $x^* \in X^*$ such that $\|x^*\|_{X^*} < \varepsilon$ and $f + x^*$ attains a minimum on X. This result leads to a natural question : what about the existence of a smooth function φ such that $\|\varphi\|_\infty < \varepsilon$, $\|\varphi'\|_\infty < \varepsilon$ and $f + \varphi$ attains a minimum on X?

We give a positive answer to this smooth Deville's type variational principle in Banach spaces with Radon-Nikodým property. We prove: Let X be a Banach space with the Radon-Nikodým property and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function on X for which there exist $a > 0$ and $b \in \mathbb{R}$ such that $f(x) \geqslant 2a\|x\| + b$, $x \in X$.

Then for every $\varepsilon > 0$ there exists a Fréchet differentiable function $\varphi : X \longrightarrow \mathbb{R}$ on X such that $\|\varphi\|_\infty < \varepsilon$, $\|\varphi'\|_\infty < \varepsilon$, φ' is weakly continuous and $f + \varphi$ attains a minimum on X.

On the other hand, we give another smooth variational principle, where we want the function φ to be of the form $g_1 + g_2$, where g_1 is radial, g_1 and g_2 are smooth with small norms.

2. Smooth variational principles

In this section we give two smooth variational principles in Banach spaces with the Radon-Nikodým property. More precisely we show the following :

Theorem 2.1. *Let X be a Banach space with the Radon-Nikodým property. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function on X for which there exist $a > 0$ and $b \in \mathbb{R}$ such that $f(x) \geqslant 2a\|x\| + b$, for all $x \in X$. Then for any $\varepsilon > 0$ and $x_1 \in X$ such that $f(x_1) < \inf_X f + \varepsilon^2/4$, there exists $\varphi \in C^1(X)$, such that :*

(1) $\|\varphi\|_\infty < \varepsilon$ *and* $\|\varphi'\|_\infty < \varepsilon$,
(2) φ' *is weakly continuous,*
(3) $f + \varphi$ *attains a minimum on X at some x_0,*
(4) $f(x_0) < \inf_X f + \varepsilon$.

A mapping $g : X \longrightarrow \mathbb{R}$ is radial if it has the form $g(x) = h\big(\|x\|\big)$ for some $h : \mathbb{R} \longrightarrow \mathbb{R}$.

Theorem 2.2. *Let $\big(X, \|.\|\big)$ be a Banach space with the Radon-Nikodým property. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function bounded below on X. Then for any $\varepsilon > 0$ and $x_1 \in X$ such that $f(x_1) < \inf_X f + \varepsilon^2/128$, there exist two functions $g_1, g_2 : X \longrightarrow \mathbb{R}$ such that :*

(1) $f + g_1 + g_2$ *attains a minimum on X at some x_0,*
(2) $\|g_1\|_\infty < \varepsilon$ *and g_1 is radial,*
(3) g_2 *is Fréchet-continuously differentiable on X, $\|g_2\|_\infty < \varepsilon$ and $\|g_2'\|_\infty < \varepsilon$,*
(4) g_2' *is weakly continuous.*

We remark that in Theorem 2.1 we can not replace the hypothesis $f(x) \geqslant 2a\|x\| + b$ by the lesser condition that f is bounded below in X. Indeed, in this case Theorem 2.1 says: if $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and bounded below, and the space has the Radon-Nikodým property, then for every $\varepsilon > 0$ there exists $\varphi \in C^1(X)$ such that $f + \varphi$ attains its minimum on X and $\|\varphi\|_\infty < \varepsilon$, $\|\varphi'\|_\infty < \varepsilon$. But this means that X satisfies Deville's smooth variational principle [**5, 6**], and hence the space X admits a smooth bump function [**7**]. This contradicts the fact that not every Radon-Nikodým Banach space has a smooth bump function.

https://doi.org/10.1017/S0004972700033372 Published online by Cambridge University Press

We give now the proofs of our results. The tool used is the Stegall variational principle. Recall that the Stegall theorem [**12**] is motivated by the Bishop and Phelps theorem [**11**] : let C be a closed, bounded and convex subset of a Banach space X. Then the set of support functionals of C is a norm dense subset of X^*.

Phrasing this another way, if x^* is any element of X^* and ε any positive number, then there exists y^* in X^* with $\|y^*\|_{X^*} < \varepsilon$ such that $x^* + y^*$ attains its supremum on C.

Bollobás [**1**] has proved a stronger form of the Bishop-Phelps theorem, in the case of the localisation of the supremum : let C be a closed, bounded and convex subset of a Banach space X. Let x^* any element of X^* and ε any positive number. Let x_1 in C be such that $\sup_C x^* - \varepsilon^2/4 < \langle x^*, x_1 \rangle$. Then there exist y^* in X^* with $\|y^*\|_{X^*} < \varepsilon$ and x_0 in C such that $x^* + y^*$ attains its supremum on C at x_0 and $\|x_1 - x_0\| < \varepsilon$.

By copying the proof of Stegall [**12**], and using the Bollobás stronger form of the Bishop-Phelps theorem, one can prove the following:

Theorem 2.3. *Let C be a bounded, closed and convex subset of a Banach space X. Assume that C has the Radon-Nikodým property. Let $f : C \longrightarrow \mathbb{R}$ be lower semicontinuous and bounded below on C. Let $\varepsilon > 0$ and $x_1 \in C$ such that $f(x_1) < \inf_C f + \varepsilon^2/4$. Then there exist $x^* \in X^*$ and $x_0 \in C$ such that :*

(1) $\|x^*\|_{X^*} < \varepsilon$,

(2) *$f + x^*$ attains a minimum on C at some x_0,*

(3) $f(x_0) < \inf_C f + \varepsilon$,

(4) $\|x_0 - x_1\| < \varepsilon$.

Proof of Theorem 2.1: Let $g = f - b$. By hypothesis $f(x) \geqslant 2a\|x\| + b$, $x \in X$. So, for all $x_0^* \in X^*$ such that $\|x_0^*\|_{X^*} \leqslant a$, we have,

$$g(x) + \langle x_0^*, x \rangle \geqslant 2a\|x\| - a\|x\| = a\|x\|. \tag{1}$$

Observing that $2a\|x_1\| + b \leqslant f(x_1) < \inf_X f + \varepsilon^2/4$, then $\|x_1\| \leqslant \left(\inf_X f - b\right)/(2a) + \varepsilon^2/8a$. So, without loss of generality, we assume that $\|x_1\| < 1/16$ (replacing f by $f + \lambda$, for some $\lambda \in \mathbb{R}$ if it is necessary).

Let $B := B(0, r)$; $r := \max\left(\left(f(0) - b + 1\right)/a, \|x_1\|\right) > 0$. We have $g(x_1) < \inf_X g + \left(\varepsilon^2/4\right) \leqslant \inf_B g + \varepsilon^2/4$. By Theorem 2.3 (Stegall's theorem) there exists $x^* \in X^*$ such that $\|x^*\|_{X^*} < \varepsilon$ and $g + \langle x^*, . \rangle$ attains its minimum on B at some x_0 satisfying : $g(x_0) < \inf_B g + \varepsilon$ and $\|x_1 - x_0\| < \varepsilon$.

Claim 1. For any $x \in X$, $g(x_0) + \langle x^*, x_0 \rangle \leqslant g(x) + \langle x^*, x \rangle$ and $g(x_0) < \inf_X g + (5/4)\varepsilon^2$.

Let $x \in X$ be such that $g(x) + \langle x^*, x \rangle < g(x_0) + \langle x^*, x_0 \rangle$. So, by (1), assuming here that $0 < \varepsilon < a$:

$$a\|x\| \leqslant g(x) + \langle x^*, x \rangle$$

https://doi.org/10.1017/S0004972700033372 Published online by Cambridge University Press

$$
\begin{aligned}
&< g(x_0) + \langle x^*, x_0\rangle \\
&= \inf_B \big(g + \langle x^*, .\rangle\big) \\
&\leqslant g(0) + \langle x^*, 0\rangle = g(0) = f(0) - b,
\end{aligned}
$$

so that

$$\|x\| \leqslant \frac{f(0)-b}{a} < \frac{f(0)-b+1}{a} \leqslant r.$$

This shows that $x \in B$, contradicting the fact that $g + \langle x^*, .\rangle$ attains its minimum at x_0 on B.

We prove now that $g(x_0) < \inf_X g + (5/4)\varepsilon^2$. We have :

$$g(x_0) \leqslant g(x) + \langle x^*, x - x_0\rangle, \quad \forall x \in X.$$

In particular for $x = x_1$:

$$
\begin{aligned}
g(x_0) &\leqslant g(x_1) + \langle x^*, x_1 - x_0\rangle \\
&\leqslant \inf_X g + \frac{\varepsilon^2}{4} + \|x^*\|_{X^*}\|x_1 - x_0\| \\
&< \inf_X g + \frac{\varepsilon^2}{4} + \varepsilon^2 \\
&= \inf_X g + \frac{5}{4}\varepsilon^2
\end{aligned}
$$

and our claim is proved.

Let $\alpha_0 := f(x_0) - m + \langle x^*, x_0\rangle$, where $m := \inf_X f$, and in the first place we assume that $\alpha_0 > 0$.

We choose an even function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$
t \longmapsto \varphi(t) = \begin{cases} \alpha_0 & \text{if} \quad |t| \geqslant \alpha_0 \\ \varphi\big(\langle x^*, x_0\rangle\big) = \langle x^*, x_0\rangle & \\ |t| \leqslant \varphi(t) \leqslant \alpha_0 & \text{if} \quad |t| \leqslant \alpha_0 \end{cases}
$$

and φ is in $C^1(X)$ and $\|\varphi'\|_\infty \leqslant 8$.

CLAIM 2. The function $f + \varphi \circ x^*$ attains its minimum on X at x_0.

Indeed, let $x \in X$:

CASE 1. x is such that $\big|\langle x^*, x\rangle\big| \geqslant \alpha_0$. Then

$$
\begin{aligned}
f(x) + (\varphi \circ x^*)(x) &= f(x) + \varphi\big(\langle x^*, x\rangle\big) \\
&= f(x) + \alpha_0 \\
&= f(x) + f(x_0) - m + \langle x^*, x_0\rangle \\
&\geqslant m + f(x_0) - m + \langle x^*, x_0\rangle \\
&= f(x_0) + \langle x^*, x_0\rangle \\
&= f(x_0) + \varphi\big(\langle x^*, x_0\rangle\big).
\end{aligned}
$$

https://doi.org/10.1017/S0004972700033372 Published online by Cambridge University Press

CASE 2. x is such that $\left|\langle x^*, x\rangle\right| \leqslant \alpha_0$. Then,

$$\begin{aligned} f(x) + \varphi\left(\langle x^*, x\rangle\right) &\geqslant f(x) + \langle x^*, x\rangle \\ &\geqslant f(x_0) + \langle x^*, x_0\rangle \\ &= f(x_0) + \varphi\left(\langle x^*, x_0\rangle\right). \end{aligned}$$

Therefore, the proof of Claim 2 is complete.

CLAIM 3. $\|\varphi \circ x^*\|_\infty < \varepsilon$ and $\left\|(\varphi \circ x^*)'\right\|_\infty < \varepsilon$.

It is clear that:

$$\begin{aligned} \|\varphi \circ x^*\|_\infty &\leqslant f(x_0) - m + \left|\langle x^*, x_0\rangle\right| \\ &\leqslant f(x_0) - m + \|x^*\|_{X^*} \|x_0\| \\ &\leqslant \frac{5}{4}\varepsilon^2 + \varepsilon\|x_0\| \\ &\leqslant \frac{5}{4}\varepsilon^2 + \varepsilon\left(\varepsilon + \|x_1\|\right) \\ &\leqslant \frac{9}{4}\varepsilon^2 + \frac{\varepsilon}{16}. \end{aligned}$$

So, $\|\varphi \circ x^*\|_\infty < \varepsilon$ (for ε small).

We now prove the second inequality. Let $x \in X$.

CASE 1. $\left|\langle x^*, x\rangle\right| \geqslant \alpha_0$. Then, by definition of φ, we have $\varphi\left(\langle x^*, x\rangle\right) = \alpha_0$ and φ is in $C^1(X)$. Hence, $(\varphi \circ x^*)'(x) = 0$.

CASE 2. $\left|\langle x^*, x\rangle\right| < \alpha_0$. Then we have $(\varphi \circ x^*)'(x) = \varphi'\left(\langle x^*, x\rangle\right)\langle x^*, x\rangle$. Hence

$$\begin{aligned} \left|(\varphi \circ x^*)'(x)\right| &\leqslant \|\varphi'\|_\infty \left|\langle x^*, x\rangle\right| \\ &\leqslant \|\varphi'\|_\infty \left[f(x_0) - m + \left|\langle x^*, x_0\rangle\right|\right] \\ &\leqslant \|\varphi'\|_\infty \left[\frac{5}{4}\varepsilon^2 + \varepsilon\left(\varepsilon + \|x_1\|\right)\right] \\ &< 8\left[\frac{9}{4}\varepsilon^2 + \frac{\varepsilon}{16}\right] \end{aligned}$$

Case 1 and Case 2 prove that (for ε small) :

$$\left\|(\varphi \circ x^*)'\right\|_\infty \leqslant \varepsilon.$$

Therefore, Claim 3 is proved.

In the second place assume $\alpha_0 \leqslant 0$. Let $\lambda > 0$ be such that $\beta_0 := f(x_0) - m + \langle x^*, x_0\rangle + \lambda > 0$. In this case, we repeat the same proof as before just replacing α_0 by β_0 and $\langle x^*, x_0\rangle$ by $\langle x^*, x_0\rangle + \lambda$.

Set $\psi := \varphi \circ x^*$. Therefore, the function ψ satisfies the requirements of the theorem and the proof of our result is complete. ∎

https://doi.org/10.1017/S0004972700033372 Published online by Cambridge University Press

Proof of Theorem 2.2: Let $\varepsilon > 0$ and $x_1 \in X$ be such that $f(x_1) < \inf_X f + \varepsilon^2/128$. Let $\varepsilon > 32\varepsilon_1 > 0$ be such that $\varepsilon_1\|x_1\| < \varepsilon^2/128$. Let $g := f + \varepsilon_1\|.\|$ and let $m := \inf_X f$. Then

$$\begin{aligned} g(x_1) &\leqslant \inf_X f + \frac{\varepsilon^2}{128} + \varepsilon_1\|x_1\| \\ &\leqslant \inf_X g + \frac{\varepsilon^2}{128} + \frac{\varepsilon^2}{128} \\ &= \inf_X g + \frac{\varepsilon^2}{64}. \end{aligned}$$

We have $g(x) = f(x) + \varepsilon_1\|x\|$. Then, $g(x) \geqslant m + \varepsilon_1\|x\|$ for all $x \in X$ and g is a lower semicontinuous function on X. Applying Theorem 2.1, there exists $g_2 : X \longrightarrow \mathbb{R}$ a C^1-function on X with $\|g_2\|_\infty < \varepsilon/4$, $\|g_2'\|_\infty < \varepsilon/4$, g_2' weakly continuous and $g + g_2$ attains its minimum at some x_0 on X, $g(x_0) < \inf_X g + \varepsilon/4$ and $\|x_0 - x_1\| \leqslant \varepsilon/4$. Therefore, for all $x \in X, f(x) + \varepsilon_1\|x\| + g_2(x) \geqslant f(x_0) + \varepsilon_1\|x_0\| + g_2(x_0)$.

To finish the proof, we adopt the construction of the function φ as in the proof of Theorem 2.1.

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be an even C^1-function with $\|\varphi'\|_\infty < 8$ such that :

$$\varphi(t) = \begin{cases} \alpha_0 & \text{if } |t| \geqslant \alpha_0 \\ \varphi\big(\varepsilon_1\|x_0\|\big) = \varepsilon_1\|x_0\| & \\ |t| \leqslant \varphi(t) < \alpha_0 & \text{if } |t| < \alpha_0 \end{cases}$$

where $\alpha_0 := f(x_0) + g_2(x_0) - M + \varepsilon_1\|x_0\|$, $M := \inf_X (f + g_2)$ and we assume that φ is constant in a neighbourhood of 0 (so $\varphi\big(\varepsilon_1\|.\|\big)$ is smooth in X whenever the norm is assumed to be smooth in $X \setminus \{0\}$)

Claim 1. $f + \varphi\big(\varepsilon_1\|.\|\big) + g_2$ attains its minimum at x_0 on X.

Indeed, let $x \in X$.

Case 1. $\varepsilon_1\|x\| \geqslant \alpha_0$. Then

$$\begin{aligned} f(x) + g_2(x) + \varphi\big(\varepsilon_1\|x\|\big) &= f(x) + g_2(x) + f(x_0) + g_2(x_0) - M + \varepsilon_1\|x_0\| \\ &\geqslant M + f(x_0) + g_2(x_0) - M + \varepsilon_1\|x_0\| \\ &= f(x_0) + g_2(x_0) + \varepsilon_1\|x_0\| \\ &= f(x_0) + g_2(x_0) + \varphi\big(\varepsilon_1\|x_0\|\big). \end{aligned}$$

Case 2. $\varepsilon_1\|x\| \leqslant \alpha_0$. Then

$$\begin{aligned} f(x) + g_2(x) + \varphi\big(\varepsilon_1\|x\|\big) &\geqslant f(x) + g_2(x) + \varepsilon_1\|x\| \\ &\geqslant f(x_0) + g_2(x_0) + \varepsilon_1\|x_0\| \\ &= f(x_0) + g_2(x_0) + \varphi\big(\varepsilon_1\|x_0\|\big). \end{aligned}$$

Therefore we have proved Claim 1.

CLAIM 2. $\left\|\varphi\left(\varepsilon_1\|.\|\right)\right\|_\infty < \varepsilon$.

It is clear that :

$$\|\varphi\left(\varepsilon_1\|.\|\right)\|_\infty \leqslant \alpha_0 = f(x_0) + g_2(x_0) - M + \varepsilon_1\|x_0\|.$$

On the other hand, we have :

$$\begin{aligned} f(x_0) + g_2(x_0) &\leqslant f(x) + g_2(x) + \varepsilon_1\left(\|x\| - \|x_0\|\right) \\ &\leqslant f(x) + g_2(x) + \varepsilon_1\|x - x_0\| \qquad \forall x \in X. \end{aligned}$$

In particular for $x = x_1$:

$$\begin{aligned} f(x_0) + g_2(x_0) &\leqslant f(x_1) + g_2(x_1) + \varepsilon_1\left(\|x_1 - x_0\|\right) \\ &\leqslant \inf_X f + \frac{\varepsilon^2}{128} - \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \varepsilon_1\|x_1 - x_0\| \\ &\leqslant \inf_X f + \frac{\varepsilon^2}{128} + \inf_X g_2 + \frac{\varepsilon}{2} + \varepsilon_1\|x_1 - x_0\| \\ &\leqslant \inf_X (f + g_2) + \frac{\varepsilon^2}{128} + \frac{\varepsilon}{2} + \varepsilon_1\frac{\varepsilon}{4} \\ &\leqslant M + \frac{\varepsilon^2}{64} + \frac{\varepsilon}{2}. \end{aligned}$$

Hence, $f(x_0) + g_2(x_0) - M \leqslant (2\varepsilon)/3$, (for ε small). Thus,

$$\begin{aligned} \left\|\left(\varphi \circ \varepsilon_1\|.\|\right)\right\|_\infty &\leqslant \frac{2\varepsilon}{3} + \varepsilon_1\|x_0\| \\ &\leqslant \frac{2\varepsilon}{3} + \varepsilon_1\left[\|x_0 - x_1\| + \|x_1\|\right] \\ &\leqslant \frac{2\varepsilon}{3} + \varepsilon_1\left[\varepsilon + \|x_1\|\right] \\ &\leqslant \frac{2\varepsilon}{3} + \frac{5\varepsilon^2}{128} < \varepsilon. \end{aligned}$$

CLAIM 3. $\left\|\left(\varphi \circ \varepsilon_1\|.\|\right)'\right\|_\infty \leqslant \varepsilon$.

Let $x \in X$.

CASE 1. $\varepsilon_1\|x\| \geqslant \alpha_0$. Then, by definition of the function φ, we have $\left(\varphi \circ \varepsilon_1\|.\|\right)'(x) = 0$.

CASE 2. $\varepsilon_1\|x\| < \alpha_0$. Then

$$\begin{aligned} \sup\left\{\left|\left(\varphi \circ \varepsilon_1\|.\|\right)'(x)\right| ; x \in X, \varepsilon_1\|x\| < \alpha_0\right\} &\leqslant \|\varphi'\|_\infty\varepsilon \\ &\leqslant 8\varepsilon_1 < \varepsilon. \end{aligned}$$

Therefore, we have $\left\|\left(\varphi \circ \varepsilon_1\|.\|\right)'\right\|_\infty < \varepsilon$.

https://doi.org/10.1017/S0004972700033372 Published online by Cambridge University Press

□

If we take the function φ to be constant in a neighbourhood of 0, then the function $\varphi\big(\varepsilon_1\|.\|\big)$ is β-smooth whenever the norm is assumed to be β-smooth in $X \setminus \{0\}$. Recall that a mapping $g : X \longrightarrow \mathbb{R}$ is radial if it has the form $g(x) = h\big(\|x\|\big)$ for some $h : \mathbb{R} \longrightarrow \mathbb{R}$. We have the following :

Corollary 2.4. *Let $\big(X, \|.\|\big)$ be a Banach space with the Radon-Nikodým property and assume that the norm $\|.\|$ is β-smooth on $X \setminus \{0\}$. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function bounded below on X. Then for all $\varepsilon > 0$, there exist two functions $g_1, g_2 : X \longrightarrow \mathbb{R}$ such that :*

(1) *$f + g_1 + g_2$ attains a minimum on X at some x_0,*

(2) *g_1 is β-$C^1(X)$, $\|g_1\|_\infty < \varepsilon$ and $\|g'_{1,\beta}\|_\infty < \varepsilon$ where $g'_{1,\beta}(x)$ is the β-derivative of g_1 at x,*

(3) *g_1 is radial,*

(4) *g_2 is Fréchet-$C^1(X)$, $\|g_2\|_\infty < \varepsilon$ and $\|g'_2\|_\infty < \varepsilon$ where $g'_2(x)$ is the Fréchet-derivative of g_2 at x*

(5) *g'_2 is weakly continuous.*

References

[1] B. Bollobás, 'An extension to the theorem of Bishop and Phelps', *Bull. London Math. Soc. (2)* (1970), 181-182.

[2] N. Bonic and J. Frampton, 'Smooth functions on Banach manifolds', *J. Math. Mech.* **15** (1966), 877–898.

[3] J. Borwein and D. Preiss, 'A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions', *Trans. Amer. Math. Soc.* **303** (1987), 517–527.

[4] R. Bourgin, *Geometrical aspects of convex sets with the Radon-Nikodým property*, Lecture notes in Math. **993** (Springer-Verlag, Berlin, Heidelberg, New York, 1983).

[5] R. Deville, G. Godefroy and V. Zizler, 'A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions', *J. Func. Anal.* **111** (1993), 197–212.

[6] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics **64** (Longman Scientific and Technical, 1993).

[7] R. Deville, 'Stability of subdifferentials of nonconvex functions in Banach spaces', *Set-Valued Anal.* **2** (1994), 141–157.

[8] J. Diestel and J. Uhl, *Vector measures* (Amer. Math. Soc., Providence, RI, 1977).

[9] I. Ekeland, 'On the variational principle', *J. Math. Anal. Appl.* **47** (1974), 324–353.

[10] R. Haydon, 'A counterexample to several questions about scattered compact spaces', *Bull. Lond. Math. Soc.* **22** (1990), 261–268.

[11] R.R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture notes in Math. **1364** (Springer-Verlag, Berlin, Heidelberg, New York, 1991).

[12] C. Stegall, 'Optimization of functions on certain subsets of Banach spaces', *Math. Ann.* **236** (1978), 171–176.

Université Bordeaux 1
Faculté des Sciences
Laboratoire de Mathématiques pures
351 cours de la Libération
33400 Talence
France

Université Cadi-Ayyad
Faculté des Sciences et Techniques
Departement de Mathematiques
B.P. 523 Béni-Mellal
Maroc