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MULTIPLICATIVE FUNCTIONS IN SHORT INTERVALS

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1. Introduction. A central problem in probabilistic number theory is to evaluate asymptotically the partial sums

$$M(f, x) = \sum_{n \leqq x} f(n)$$

of multiplicative functions f and, in particular, to find conditions for the existence of the "mean value"

$$\lim_{x \to \infty} M(f, x)/x. \tag{1.1}$$

In the last two decades considerable progress has been made on this problem, and the results obtained are very satisfactory. In particular, Halász [7] determined completely the asymptotic behavior of $M(f, x)$ for multiplicative functions of modulus $\leqq 1$ by proving

THEOREM 1 (Halász). *Let f be a multiplicative function of modulus $\leqq 1$. Then either the limit* (1.1) *exists and equals zero, or there exist constants $A \in \mathbf{C}\setminus\{0\}$, $\alpha \in R$ and a function $L(u)$ satisfying*

$$|L| = 1, \quad \sup_{1 \leqq t \leqq 2} |L(ut) - L(u)| \to 0 \quad (u \to \infty), \tag{1.2}$$

such that

$$M(f, x) = \frac{x^{1+i\alpha}}{1 + i\alpha} A L(\log x) + o(x) \quad (x \to \infty). \tag{1.3}$$

The first case occurs if and only if the Dirichlet series

$$F(s) = \sum_{n \geqq 1} \frac{f(n)}{n^s}$$

satisfies

$$\sup_{|t| \leqq T} |F(\sigma + it)| = o\left(\frac{1}{\sigma - 1}\right) \quad (\sigma \to 1+) \text{ for every } T > 0, \tag{1.4}$$

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and the second if and only if, in the notation of (1.3),

$$(1.5) \qquad F(\sigma - i\alpha) = \frac{A}{\sigma - 1} L\left(\frac{1}{\sigma - 1}\right) + o\left(\frac{1}{\sigma - 1}\right) \quad (\sigma \to 1+).$$

It is implicit in Theorem 1, and easy to prove directly, that for any multiplicative function f of modulus $\leqq 1$ the generating Dirichlet series $F(s)$ satisfies either (1.4) or (1.5).

For real-valued functions f, relation (1.3) (with $A \neq 0$) can only hold when $\alpha = 0$ and $L(u) = 1 + o(1)$ $(u \to \infty)$, and therefore reduces to

$$M(f, x) = Ax + o(x) \quad (x \to \infty).$$

Thus we obtain as a special case of Halász' theorem the following result, originally proved by Wirsing [**16**].

THEOREM 2 (Wirsing). *For any real-valued multiplicative function f of modulus $\leqq 1$ the mean value* (1.1) *exists.*

While Theorems 1 and 2 are of asymptotic nature, there exist also quantitative estimates for $M(f, x)$. We state here such a result, which implies Theorem 1 in the case (1.4) and is essentially due to Halász [**8**] (cf. [**5**, Lemma 6.10]).

THEOREM 3 (Halász). *Uniformly for all multiplicative functions f of modulus $\leqq 1$ and all x, $T \geqq 2$ we have*

$$M(f, x) \ll x\left\{\frac{1}{\log x} F^*\left(1 + \frac{1}{\log x}, T\right) + T^{-1}\right\}^{1/4},$$

where

$$F^*(\sigma, T) = \max_{|t| \leqq T} |F(\sigma + it)|.$$

Montgomery [**12**] recently obtained a sharper, but more complicated upper bound for $M(f, x)$, which implies the estimate of Theorem 3 with the exponent $1 - \epsilon$ instead of $1/4$.

These results show that the global behavior of multiplicative functions of modulus $\leqq 1$ is well understood. One may ask whether similar results hold for the "local" behavior of multiplicative functions. In particular, one may try to obtain asymptotic formulae like (1.3) for the short interval sums

$$(1.6) \qquad M(f, x) - M(f, x - \phi(x)) = \sum_{x - \phi(x) < n \leqq x} f(n)$$

under suitable growth conditions on the interval length $\phi(x)$.

In its general form, this problem has not been dealt with in the literature. The behavior of the quantities (1.6) has been determined only for a few special functions f or under rather strong assumptions on f ([**1**], [**2**], [**3**], [**13**], [**14**]). For example, Motohashi [**13**] and independently Ramachandra [**14**] proved that for any $\theta \in (7/12, 1]$

$$M(\mu, x) - M(\mu, x - x^{\theta}) = o(x^{\theta}) \quad (x \to \infty),$$

where μ is the Moebius function. Ramachandra [**13**] obtained similar results for functions, whose Dirichlet series are composed of L-functions. Babu [**3**] determined the limiting behavior of (1.6) in the case $\phi(x)$ satisfies $x^{\alpha} \leqq \phi(x) \leqq x$ for some $\alpha > 0$ for multiplicative functions f satisfying $|f| \leqq 1$ and

$$|\{p^m \leqq x: |f(p^m) - 1| > \epsilon\}| = o(\phi(x)) \quad (x \to \infty) \quad \text{for every } \epsilon > 0.$$

The purpose of this paper is to prove short interval analogues of the theorems of Halász and Wirsing. We shall get satisfactory results under the assumption

$$\phi(x) = x^{1+o(1)}.$$

This does not quite cover the above mentioned special cases, but in order to obtain results as general as Halász' or Wirsing's theorem, this restriction on $\phi(x)$ is essentially necessary, as we shall see.

2. Statement of results. Our main result exhibits, in a quantitative form, the connection between the "local" and the "global" behavior of f. It turns out that the local behavior of f is closely connected to the behavior of $M(f_{\alpha}, x)$, where

$$f_{\alpha}(n) = f(n)n^{i\alpha} \quad (n \geqq 1)$$

and $\alpha = \alpha(f, x)$ is such that

$$(2.1) \qquad |\alpha| \leqq x, \ |M(f_{\alpha}, x)| = \max_{|\alpha'| \leqq x} |M(f_{\alpha'}, x)|.$$

THEOREM 4. *Let f be a multiplicative function of modulus* $\leqq 1$ *and* $x \geqq y \geqq 3$. *Then we have*

$$(2.2) \qquad M(f, x) - M(f, x - y) = \frac{M(f_{\alpha}, x)}{x} \int_{x-y}^{x} t^{-i\alpha} dt + O(yR(x, y)),$$

where α is any real number satisfying (2.1) *and*

$$R(x, y) = \left(\log \frac{2 \log x}{\log(2x/y)}\right)^{-1/4}.$$

If in addition f is real-valued, then (2.2) *holds with* $\alpha = 0$.

Using the above quoted results to estimate $M(f_\alpha, x)$, we shall deduce from Theorem 4 a number of corollaries. In particular, we shall establish in Corollaries 1, 3 and 5 short interval analogues of Theorems 2, 3 and 1, respectively.

We first derive some consequences of Theorem 4 for real-valued functions f. In this case, (2.2) holds with $\alpha = 0$ and thus reduces to

$$(2.2)' \quad M(f, x) - M(f, x - y) = \frac{y}{x}M(f, x) + O(yR(x, y)).$$

Combining this with Theorem 2, we obtain immediately

COROLLARY 1. *Let f be a real-valued multiplicative function of modulus $\leqq 1$, and let $\phi(x)$ satisfy*

$$(2.3) \quad 3 \leqq \phi(x) \leqq x \quad (x \geqq 3), \quad \log \phi(x) \sim \log x \quad (x \to \infty).$$

Then the limit

$$(2.4) \quad \lim_{x \to \infty} \frac{1}{\phi(x)}(M(f, x) - M(f, x - \phi(x)))$$

exists and is equal to the mean value (1.1) *of f.*

Another obvious consequence of (2.2)′ is

COROLLARY 2. *For every $\epsilon > 0$ there exists a $\delta > 0$ such that for any real-valued multiplicative function f of modulus $\leqq 1$*

$$\limsup_{x \to \infty} |x^{\delta-1}(M(f, x) - M(f, x - x^{1-\delta})) - A(f)| \leqq \epsilon,$$

where

$$A(f) = \lim_{x \to \infty} M(f, x)/x.$$

These two results are essentially best-possible, since by an example of Erdös (cf. [**1**, p. 105] or [**2**, p. 102]) there exists, for every $\delta \in (0, 1)$, a multiplicative function f assuming only the values 0 and 1, such that in the case $\phi(x) = x^{1-\delta}$ the limit (2.4) does not exist.

We now turn to complex-valued multiplicative functions. Here the results will not be as neat as in the case of real-valued functions. The reason for this is that the behavior of the normalized short interval sums

$$(2.5) \quad \frac{1}{y}(M(f, x) - M(f, x - y))$$

in general depends on the interval length y, a phenomenon which does not occur when f is real-valued. In fact, from (2.2) we see that the quantity (2.5) is, as a function of y, approximately proportional to

$$\frac{1}{y}\int_{x-y}^{x} t^{-i\alpha}dt = x^{-i\alpha}\frac{1-\left(1-\frac{y}{x}\right)^{1-i\alpha}}{(1-i\alpha)y/x}.$$

This expression is close to $x^{-i\alpha}$, if y is small compared to $x/|\alpha|$ but for $y \gg x/|\alpha|$ it behaves like an oscillatory function with decreasing amplitude.

We first give a O-estimate for $M(f, x) - M(f, x - y)$ analogous to that of Theorem 3. If we bound $M(f_\alpha, x)$ by means of Theorem 3 and estimate the integral in (2.2) trivially by

$$\int_{x-y}^{x} t^{-i\alpha}dt \ll \begin{cases} y & (|\alpha| \leqq x/y) \\ x/|\alpha| & (|\alpha| > x/y)' \end{cases}$$

we obtain from Theorem 4

COROLLARY 3. *Uniformly for $x \geqq y \geqq 3$ and all multiplicative functions f of modulus $\leqq 1$ we have*

$$\frac{1}{y}|M(f, x) - M(f, x - y)|$$
$$\ll \max_{\alpha \geqq x/y} \frac{x}{\alpha y} \min_{T \geqq 2\alpha} \left(\frac{1}{\log x}F^*\left(1 + \frac{1}{\log x}, T\right) + \frac{1}{T}\right)^{1/4} + R(x, y),$$

where $F^*(\sigma, T)$ is defined as in Theorem 3.

This yields a sufficient condition for the existence and vanishing of the limit (2.4), namely

COROLLARY 4. *Let f be a multiplicative function of modulus $\leqq 1$ and let $\phi(x)$ satisfy* (2.3). *If*

$$(2.6) \qquad F^*\left(1 + \frac{1}{\log x}, Kx/\phi(x)\right) = o(\log x) \quad (x \to \infty) \text{ for every } K > 0,$$

then the limit (2.4) *exists and equals zero.*

Condition (2.6) is stronger than (1.4) in general, and it is equivalent to (1.4) if $\phi(x) = x$. In the latter case, the condition is actually equivalent to the existence and vanishing of the mean value (2.4). However, this is not true in general, and it seems difficult to derive a reasonably simple necessary and sufficient condition for the vanishing of the limit (2.4) without imposing some regularity conditions on $\phi(x)$.

We shall use Theorem 4 in combination with Theorem 1 and Corollary 3 to derive the following short interval analogue of Halász' theorem.

COROLLARY 5. *Let f be a multiplicative function of modulus $\leqq 1$, and let $\phi(x)$ satisfy* (2.3) *and*

(2.7) $\phi(x) = o(x) \quad (x \to \infty)$.

Then either

$$\liminf_{x\to\infty} \frac{1}{\phi(x)} |M(f, x) - M(f, x - \phi(x))| = 0, \tag{2.8}$$

or there exist constants $A \in \mathbf{C}\backslash\{0\}$, $\alpha \in \mathbf{R}$ *and a function* $L(u)$ *satisfying* (1.2) *such that*

$$\begin{aligned} &M(f, x) - M(f, x - \phi(x)) \\ &= \phi(x)\left\{\frac{A}{1 + i\alpha} x^{i\alpha} L(\log x) + o(1)\right\}. \end{aligned} \tag{2.9}$$

The first alternative occurs if and only if (1.4) *holds, and the second if and only if* (1.5) *is satisfied with* A, α *and* $L(u)$ *as in* (2.9).

In the first case (i.e., when (1.4) holds), the result may seem unsatisfactory, and one would like to describe the behavior of f more precisely than by (2.8). However, it seems unlikely that this can be done without additional hypotheses on f and $\phi(x)$. In particular, the limit in (2.8) need not exist, and the behavior of the quantities $M(f, x) - M(f, x - \phi(x))$ may be quite irregular, even when $\phi(x)$ is a fairly "regular" function. We shall give a counterexample in the last section.

Delange **[4]** has characterized those multiplicative functions of modulus $\leqq 1$, which have a non-zero mean value (1.1). By Theorem 1, this is the case if and only if (1.5) holds with $\alpha = 0$, $L(u) = 1 + o(1)$ and some $A \in \mathbf{C}\backslash\{0\}$. Corollary 5 shows that this condition is also equivalent to the existence and non-vanishing of the short interval mean value (2.4). Thus we have

COROLLARY 6. *Under the assumptions of Corollary* 5 *the limit* (2.4) *exists and is non-zero if and only if the limit* (1.1) *exists and is non-zero, and both limits are equal if they exist.*

It is well-known that mean value theorems for multiplicative functions can be used to obtain information about the distribution of additive functions. Delange **[4]** and Halász **[8]** applied their mean value theorems to derive limit theorems for additive functions. We shall give here an application of this type for the mean value theorem of Corollary 6.

Let g be a real-valued additive function, and for integers $N \geqq M \geqq 1$ define the distribution functions

$$F_{N,M}(z) = \frac{1}{M} \sum_{\substack{N-M<n\leqq N \\ g(n)\leqq z}} 1, \quad F_N(z) = F_{N,N}(z).$$

The well-known Erdös-Wintner Theorem (cf. [**5**, Theorem 5.1]) gives necessary and sufficient conditions on the function g in order that the distributions $F_N(z)$ converge weakly to a limit distribution $F(z)$. By Levy's continuity theorem this is the case if and only if, in some neighborhood of $t = 0$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leqq N} f_t(n) = \varphi(t),$$

where

$$f_t(n) = \exp(itg(n))$$

and $\varphi(t)$ is the characteristic function of $F(z)$. An analogous criterion characterizes the convergence of $F_{N,M(N)}(z)$ to $F(z)$. Since, for every $t \in \mathbf{R}$, f_t is a multiplicative function of modulus 1 and $\varphi(t)$ as a characteristic function is non-zero for sufficiently small $|t|$, we obtain, by applying Corollary 6 to f_t for every sufficiently small t,

COROLLARY 7. *Let $M(N)$, $N \geqq 1$, be a sequence of integers satisfying*

$$M(N) = o(N), \quad \log M(N) \sim \log N \quad (N \to \infty).$$

Then the distributions $F_{N,M(N)}(z)$ converge weakly to a limit distribution $F(z)$ if and only if the distributions $F_N(z)$ converge to $F(z)$.

Previously, Babu [**3**] had shown that under the condition

$$(2.10) \quad |\{p^m \leqq N: |g(p^m)| \geqq \epsilon\}| = o(M(N)) \quad (N \to \infty) \quad \text{for every } \epsilon > 0$$

the conclusion of Corollary 7 holds, whenever $N^\alpha \leqq M(N) \leqq N$ for some fixed $\alpha > 0$. Babu also observed that in this result the rather strong condition (2.10) cannot be completely deleted. This shows that in Corollary 7, which holds unconditionally, the range for $M(N)$ is essentially best-possible. A counterexample can be derived from the above-mentioned example of a multiplicative function assuming only the values 0 and 1, for which the mean value (1.1), but not the short interval mean value (2.4) with $\phi(x) = x^\alpha$ exists (cf. [**1**, p. 105] or [**2**, p. 102]).

To prove Theorem 4, we shall use an essentially elementary method, based on the large sieve. This method was used in [**10**] to give a new proof for Wirsing's mean value theorem, i.e., Theorem 2. To obtain Theorem 4 for real-valued functions f would require only minor changes in the arguments of [**10**]. However, for complex-valued functions the method does not apply as easily, and several new ideas are needed to obtain Theorem 4 in its full strength.

We shall prove Theorem 4 in Section 3 assuming a general result (Lemma 2) on certain approximate functional equations. This result will be proved in Sections 4 and 5 by first reducing the problem to a similar one involving Cauchy's equation, and then, in Section 5, solving the latter one. In Section 6 we shall prove Corollary 5. (The other corollaries are obvious and do not require a proof.) In Section 7 we shall construct the counterexample mentioned in the remarks following Corollary 5.

In deriving the corollaries, we appealed to the mean value theorems of Wirsing and Halász. It is therefore of some interest to note that these results can also be proved by our method. In the case of Wirsing's theorem, this has been done in [**10**], while Halász' theorem (Theorem 1) can be obtained by a simple modification of our proof for Theorem 4. We shall give the details of this deduction in the appendix.

3. Proof of theorem 4. It will be convenient to assume that $|f(p)|$ is, in some average sense, close to 1. If this is not the case, we can argue crudely using the following Brun-Titchmarsh type estimate due to Shiu [**15**].

LEMMA 1. *The estimate*

$$M(f, x) - M(f, x - y) \ll_\epsilon y \exp\left(\sum_{p \leqq x} \frac{f(p) - 1}{p}\right)$$

holds uniformly for all multiplicative functions f satisfying $0 \leqq f \leqq 1$ and $x \geqq y \geqq x^\epsilon \geqq 1$, where ϵ is any fixed positive number.

This has been proved, in a more general form, in [**15**, Theorem 1]. (In the quoted result, the implied constant may depend on f. However, an inspection of the proof shows that the estimate actually holds uniformly for all multiplicative functions f satisfying $0 \leqq f \leqq 1$.)

Applying Lemma 1 to $|f|$, we see that under the hypotheses of Theorem 4 the main terms in (2.2) are of order

$$\ll y \exp\left(\sum_{p \leqq x} \frac{|f(p)| - 1}{p}\right)$$

provided $y \geqq x^{1/2}$, say, as we may assume (since otherwise $R(x, y) \gg 1$). Thus (2.2) holds trivially, unless

$$\sum_{p \leqq x} \frac{1 - |f(p)|}{p} \leqq R(x, y)^{-2} \tag{3.1}$$

is satisfied, and we may henceforth assume (3.1).

The main tool in the proof of Theorem 4 is the large sieve in its arithmetic form (see, e.g., [**6**, p. 105])

$$\sum_{p \leqq Q} \frac{1}{p} \sum_{r=1}^{p} \left| \frac{p}{N} \sum_{\substack{M-N<n\leqq M \\ n \equiv r \bmod p}} a_n - \frac{1}{N} \sum_{M-N<n\leqq M} a_n \right|^2$$

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$$\ll (N + Q^2)\frac{1}{N^2} \sum_{M-N<n\leqq M} |a_n|^2,$$

which is valid uniformly for Q, M, $N \geqq 1$ and arbitrary complex numbers a_n, $M - N < n \leqq M$. As a special case of this inequality we obtain that

$$\sum_{p\leqq\sqrt{y}} \frac{1}{p}\left|\frac{p}{y} \sum_{\substack{x-y<n\leqq x \\ p|n}} f(n) - \frac{1}{y} \sum_{x-y<n\leqq x} f(n)\right|^2 \ll 1$$

holds uniformly for all $x \geqq y \geqq 1$ and all arithmetic functions f of modulus $\leqq 1$. Specializing further to multiplicative functions f, we have

$$\sum_{\substack{x-y<n\leqq x \\ p|n}} f(n) = f(p) \sum_{\frac{x-y}{p}<n\leqq\frac{x}{p}} f(n) + O\left(\frac{y}{p^2}\right)$$

for every $p \leqq \sqrt{y}$, and therefore get

$$\sum_{p\leqq\sqrt{y}} \frac{1}{p}\left|f(p)m\left(\frac{x}{p}, \frac{y}{p}\right) - m(x, y)\right|^2 \ll 1,$$

where

$$m(x, y) = m(f; x, y) = \frac{1}{y} \sum_{x-y<n\leqq x} f(n).$$

By the Cauchy-Schwarz inequality it follows that

$$\sum_{y_1<p\leqq y_2} \frac{1}{p}\left|f(p)m\left(\frac{x}{p}, \frac{y}{p}\right) - m\left(x, y\right)\right| \ll \left(\sum_{y_1<p\leqq y_2} \frac{1}{p}\right)^{1/2}$$

holds uniformly for $x \geqq y \geqq 3$ and $1 \leqq y_1 \leqq y_2 \leqq \sqrt{y}$. If in addition $y_2 \leqq y^{1/4}$, then this estimate remains valid with $(x/t, y/t)$ in place of (x, y), for $1 \leqq t \leqq y^{1/4}$, i.e., we have

$$\sum_{y_1<p\leqq y_2} \frac{1}{p}\left|f(p)m\left(\frac{x}{tp}, \frac{y}{tp}\right) - m\left(\frac{x}{t}, \frac{y}{t}\right)\right| \ll \left(\sum_{y_1<p\leqq y_2} \frac{1}{p}\right)^{1/2} \tag{3.2}$$

$$(x \geqq y \geqq 3,\ 1 \leqq y_1 \leqq y_2 \leqq y^{1/4},\ 1 \leqq t \leqq y^{1/4}).$$

Our proof rests on the inequality (3.2). For fixed x, y, y_1 and y_2, (3.2) can be regarded as an approximate functional equation for the function

$$\varphi(t) = m(x/t, y/t)$$

and determines the behavior of $\varphi(t)$ to a large degree. An additive analogue of this equation, with the product $f(p)\varphi(tp)$ replaced by the sum $f(p) + \varphi(tp)$, has been investigated by Elliott [**6**, Chapter 9], and plays a key role in the proof of some recent deep results in the theory of additive functions (cf. [**6**]). To "solve" (3.2), we shall use the following general lemma, which will be proved in the next two sections.

LEMMA 2. *Let* $Q \geqq 2$, $\kappa \geqq 10$ *and* a_p, $Q < p \leqq Q^{\kappa}$, *be complex numbers of modulus* 1. *Let* $\varphi(t)$ *be a real- or complex-valued function, defined for* $1 \leqq t \leqq Q^{2\kappa}$, *and suppose that, for some* $\epsilon > 0$, *the inequalities*

$$(3.3) \qquad |\varphi(t') - \varphi(t)| \leqq \epsilon \quad (1 \leqq t \leqq t' \leqq t(1 + Q^{-1/3}) \leqq Q^{2\kappa}),$$

and

$$(3.4) \qquad \sum_{Q<p\leqq Q^{\kappa}} \frac{1}{p}|a_p\varphi(tp) - \varphi(t)| \leqq \epsilon \log \kappa \quad (1 \leqq t \leqq Q^{\kappa})$$

hold. Then there exists a real number α *satisfying* $|\alpha| \leqq Q$ *and such that*

$$(3.5) \qquad \varphi(t) = \varphi(1)t^{i\alpha} + O(\epsilon) \quad (1 \leqq t \leqq Q),$$

where the O-constant is absolute. Moreover, if $\varphi(t)$ *is real-valued, then* (3.5) *holds with* $\alpha = 0$.

We now complete the proof of Theorem 4 assuming the validity of the lemma. With f, x, and y being fixed as in the theorem, we put

$$\delta = \max\left(\frac{\log(x/y)}{\log x}, (\log x)^{-1/2}\right),$$

so that

$$R(x, y) \asymp (\log(2/\delta))^{-1/4}.$$

Clearly, we may assume

$$(3.6) \qquad 0 < \delta < 1/200.$$

We apply the lemma to the function $\varphi(t) = m(x/t, y/t)$ with the parameters

$$Q = x^{4\delta}, \quad \kappa = (20\delta)^{-1}, \quad a_p = f(p)/|f(p)|$$

(with the convention $0/0 := 1$) and $\epsilon = c_1R^2(x, y)$, where c_1 is a large absolute constant to be specified presently. By the definition of δ and our assumption (3.6) we have

$$(3.7) \qquad Q \geqq \exp(4\sqrt{\log x}) \geqq e^4, \quad Q^{\kappa} = x^{1/5} \leqq y^{1/4}$$

and $\kappa = 1/(20\delta) \geqq 10$, as required in the lemma. The trivial estimate

$$\varphi(t') - \varphi(t) = \frac{t}{y}\left(\sum_{\frac{x-y}{t'}<n\leqq\frac{x-y}{t}} f(n) - \sum_{\frac{x}{t'}<n\leqq\frac{x}{t}} f(n)\right) + O\left(\frac{t'-t}{t}\right)$$

$$= O\left(\frac{(t'-t)x}{t'y} + \frac{t}{y}\right) = O\left(\frac{t'-t}{t'}Q^{1/4} + \frac{t}{y}\right)$$

$$(1 \leqq t \leqq t' \leqq 2t \leqq 2y)$$

shows that (3.3) holds with the above choice of ϵ, provided c_1 is sufficiently large. Moreover, (3.2) and the estimate

$$\left(\sum_{Q<p\leqq Q^\kappa} \frac{1}{p}\right)^{1/2} \asymp (\log \kappa)^{1/2} \asymp (\log(2/\delta))^{-1/2} \log \kappa$$

$$\asymp R^2(x, y) \log \kappa$$

yield (3.4) with $f(p)$ in place of $a_p = f(p)/|f(p)|$. Since, by our initial assumption (3.1),

$$\sum_{Q<p\leqq Q^\kappa} \frac{1}{p}|a_p - f(p)| \leqq \sum_{p\leqq x} \frac{1}{p}(1 - |f(p)|)$$

$$\leqq R(x, y)^{-2} \asymp R^2(x, y) \log \kappa,$$

we obtain (3.4) as stated, after adjusting the constant c_1, if necessary.

The assumptions of Lemma 2 are therefore all satisfied, and we conclude that for some $\alpha_1 \in \mathbf{R}$, $|\alpha_1| \leqq Q$, (3.5) holds, i.e., we have

$$(3.8) \qquad m(x, y) = t^{-i\alpha_1} m\left(\frac{x}{t}, \frac{y}{t}\right) + O(R^2(x, y)) \quad (1 \leqq t \leqq Q).$$

Multiplying both sides of this equation by $t^{-i(\alpha-\alpha_1)-2}$ and integrating over $1 \leqq t \leqq Q$, we obtain, for any $\alpha \in \mathbf{R}$,

$$(3.9) \qquad \frac{m(x, y)}{1 + i(\alpha - \alpha_1)}$$

$$= \int_1^Q m\left(\frac{x}{t}, \frac{y}{t}\right) t^{-i\alpha-2} dt + O\left(\frac{1}{Q} + R^2(x, y)\right)$$

$$= \frac{1}{y} \sum_{n\leqq x} f(n) \int_{\max(1,(x-y)/n)}^{\min(Q,x/n)} t^{-i\alpha-1} dt + O(R^2(x, y))$$

$$= \frac{1}{y} m_\alpha(x) I_\alpha(x, y) + O\left(\frac{y}{x} + R^2(x, y)\right),$$

where

$$m_\alpha(x) = \frac{1}{x} \sum_{n\leqq x} f(n) n^{i\alpha}, \quad I_\alpha(x, y) = \int_{x-y}^x t^{-i\alpha} dt.$$

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This yields the desired estimate (2.2) with $\alpha = \alpha_1$ for the range

(3.10) $y \leqq x/\log x$.

We have yet to show that (2.2) remains valid for any real number α satisfying (2.1). To this end we fix such a number, α_0 say, so that in the above notation

$$|\alpha_0| \leqq x, \quad |m_{\alpha_0}(x)| = \max_{|\alpha| \leqq x} |m_\alpha(x)|.$$

Put

$$y_1 = x((|\alpha_0| + 1)\log x)^{-1},$$

and suppose for the moment $y \leqq y_1$ (so that, in particular, (3.10) holds). Then we have

$$(3.11) \quad I_{\alpha_0}(x, y) = yx^{-i\alpha_0}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Thus, applying (3.9) with $\alpha = \alpha_1$ and $\alpha = \alpha_0$, we get

$$\begin{aligned} |m(x, y)| &= \frac{1}{y}|m_{\alpha_1}(x)I_{\alpha_1}(x, y)| + O(R^2(x, y)) \\ &\leqq |m_{\alpha_0}(x)| + O(R^2(x, y)) \\ &= \frac{1}{y}|m_{\alpha_0}(x)I_{\alpha_0}(x, y)| + O(R^2(x, y)) \\ &= \left|\frac{m(x, y)}{1 + i(\alpha_0 - \alpha_1)}\right| + O(R^2(x, y)). \end{aligned}$$

This implies

$$(3.12) \quad |m(x, y)| \min(1, |\alpha_0 - \alpha_1|^2) \ll |m(x, y)|\left(1 - \frac{1}{|1 + i(\alpha_0 - \alpha_1)|}\right) \ll R^2(x, y)$$

and hence

$$\begin{aligned} m(x, y) &= \frac{m(x, y)}{1 + i(\alpha_0 - \alpha_1)} + O(|m(x, y)| \min(1, |\alpha_0 - \alpha_1|)) \\ &= \frac{m(x, y)}{1 + i(\alpha_0 - \alpha_1)} + O(R(x, y)). \end{aligned}$$

A further application of (3.9) then shows that (in the case $y \leqq y_1$) we can replace $\alpha = \alpha_1$ by $\alpha = \alpha_0$ in (2.2) without introducing new error terms.

Next, let for $y \leqq \min(y_1, x/\log^2 x)$,

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$$y' = y \log^2 x.$$

We then have

$$m(x, y') = \frac{x}{y'} \int_1^{x/(x-y')} m\left(\frac{x}{t}, \frac{y}{t}\right) t^{-2} dt + O\left(\frac{1}{\log^2 x}\right),$$

as can be easily checked using the definition of $m(x/t, y/t)$ and the bound $|f| \leqq 1$. By (3.8) it follows that

$$\text{(3.13)} \quad m(x, y') = \frac{x}{y'} m(x, y) \int_1^{x/(x-y')} t^{i\alpha_1 - 2} dt + O(R^2(x, y))$$

$$= \frac{1}{y'} m(x, y) x^{i\alpha_1} I_{\alpha_1}(x, y') + O(R^2(x, y)).$$

Now,

$$|x^{i\alpha_1} I_{\alpha_1}(x, y') - x^{i\alpha_0} I_{\alpha_0}(x, y')|$$

$$= \left| \int_{x-y'}^{x} \left\{ \left(\frac{t}{x}\right)^{-i\alpha_1} - \left(\frac{t}{x}\right)^{-i\alpha_0} \right\} dt \right|$$

$$\ll y' \min(1, |\alpha_0 - \alpha_1|) \ll y' \frac{R(x, y)}{|m(x, y)|}$$

by (3.12). Therefore we obtain, using (3.11) and (2.2) with $\alpha = \alpha_0$,

$$m(x, y') = \frac{1}{y'} m(x, y) x^{i\alpha_0} I_{\alpha_0}(x, y') + O(R(x, y))$$

$$= \frac{1}{y'} m_{\alpha_0}(x) I_{\alpha_0}(x, y') + O(R(x, y))$$

$$= \frac{1}{y'} m_{\alpha_0}(x) I_{\alpha_0}(x, y') + O(R(x, y')),$$

which is the estimate (2.2) with $\alpha = \alpha_0$ and y' in place of y. This shows that (2.2) remains valid with $\alpha = \alpha_0$ in the range $y_1 < y \leqq y_2$, where

$$y_2 = \min(x, y_1 \log^2 x) = x \min(1, (|\alpha_0| + 1)^{-1} \log x).$$

It remains to establish (2.2) (with $\alpha = \alpha_0$), when $y_2 < y \leqq x$ (so that $y_2 = x(|\alpha_0| + 1)^{-1} \log x < x$). In this case we have

$$|I_{\alpha_0}(x, y)| \ll \frac{x}{|\alpha_0| + 1} = \frac{y_2}{\log x} < \frac{y}{\log x}.$$

Thus it suffices to show

$$\text{(3.14)} \quad |m(x, y)| \ll R(x, y).$$

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By (3.9) we have

$$(3.15)\quad |m(x, y)| \leqq \frac{1}{y}|m_{\alpha_1}(x)I_{\alpha_1}(x, y)| + O(R^2(x, y))$$

for $y \leqq x/\log x$. In view of (3.13), this estimate remains valid in the range $x/\log x < y \leqq x$. If now

$$|\alpha_1| + 1 > \frac{1}{100}(|\alpha_0| + 1)(\log x)^{-1/2},$$

then

$$|I_{\alpha_1}(x, y)| \ll \frac{x(\log x)^{1/2}}{|\alpha_0| + 1} \leqq \frac{y}{(\log x)^{1/2}},$$

and (3.14) follows from (3.15). We may therefore suppose

$$|\alpha_1| + 1 \leqq \frac{1}{100}(|\alpha_0| + 1)(\log x)^{-1/2},$$

so that

$$(3.16)\quad |\alpha_0 - \alpha_1| \geqq (|\alpha_0| + 1)\left(1 - \frac{1}{100}(\log x)^{-1/2}\right) \geqq \frac{1}{2}\log x,$$

since $x \geqq 3$ and $(|\alpha_0| + 1) \geqq \log x$ by our assumption $y_2 < x$. We shall show that (3.16) together with the upper bound $|\alpha_j| \leqq x$, $j = 0, 1$, implies

$$(3.17)\quad \min_{j=0,1} |m_{\alpha_j}(x)| \ll (\log x)^{-1/100}.$$

Since, by the definition of α_0, the minimum in (3.17) equals $m_{\alpha_1}(x)$, this yields (3.14).

To prove (3.17), we apply Theorem 3 with $T = (\log x)/10$ to the functions

$$f_{\alpha_j}(n) = f(n)n^{i\alpha_j}, \quad j = 0, 1.$$

This yields the bound

$$|m_{\alpha_j}(x)| \ll \left\{\frac{1}{\log x}\left|F\left(1 + \frac{1}{\log x} - it_j\right)\right| + 1\right\}^{1/4} \quad (j = 0, 1)$$

for suitable real numbers t_0 and t_1 satisfying

$$|t_j - \alpha_j| \leqq T = \frac{1}{10}\log x \quad (j = 0, 1).$$

By (3.16) and the bound $|\alpha_j| \leqq x, j = 0, 1$, we have

$$(3.18)\quad \frac{3}{10}\log x \leqq |t_0 - t_1| \leqq 2x.$$

A standard argument shows that

$$\frac{1}{\log x}\left|F\left(1 + \frac{1}{\log x} - it_j\right)\right| \ll e^{-S_j} \quad (j = 0, 1),$$

where

$$S_j = \operatorname{Re} \sum_{p \leqq x} \frac{1 - f(p)p^{it_j}}{p}.$$

If now $S_j \geqq (\operatorname{loglog} x)/25$ for $j = 0$ or 1, then (3.17) follows. In the remaining case

$$\max_{j=0,1} S_j \leqq (\operatorname{loglog} x)/25),$$

we get, using Cauchy's inequality along with the elementary inequality

$$|1 - z|^2 \leqq 2(1 - \operatorname{Re} z) \quad (|z| \leqq 1),$$

$$\operatorname{Re} \sum_{p \leqq x} \frac{1 - p^{i(t_1 - t_0)}}{p} \leqq S_1 + \sum_{p \leqq x} \frac{|1 - f(p)p^{it_0}|}{p}$$

$$\leqq S_1 + \left(2S_0 \sum_{p \leqq x} \frac{1}{p}\right)^{1/2} \leqq \left(\frac{1}{25} + \sqrt{\frac{2}{25}}\right) \operatorname{loglog} x + O(1).$$

But this inequality is impossible for sufficiently large x, since the left-hand side equals

$$\log \left| \frac{\zeta\left(1 + \frac{1}{\log x}\right)}{\zeta\left(1 + \frac{1}{\log x} - i(t_1 - t_0)\right)} \right| + O(1),$$

and hence is

$$\geqq \frac{1}{3} \operatorname{loglog} x + O(1)$$

by (3.18) and the well-known estimate

$$(3.19)\quad \zeta(\sigma + it) = \begin{cases} \dfrac{1}{\sigma - 1 + it} + O(1) & (\sigma > 1, |t| \leqq 2) \\ O((\log t)^{2/3}) & (\sigma > 1, |t| > 2). \end{cases}$$

Thus (3.16) implies (3.17), and our proof of (2.2) with $\alpha = \alpha_0$ is

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complete, apart from the proof of Lemma 2. It remains to verify that for real-valued functions f, (2.2) holds also with $\alpha = 0$. From Lemma 2 we see that in this case we can take $\alpha_1 = 0$ in (3.8) and hence also in (3.9). We therefore get (2.2) with $\alpha = 0$ for the range (3.10). Using (3.13), we obtain the same estimate for all $y \leqq x$.

4. Proof of lemma 2. We first assume $a_p = 1$, $Q < p \leqq Q^\kappa$. By the hypothesis (3.3) and a weak form of Huxley's prime number theorem [**11**] we have, for $1 \leqq t \leqq Q^\kappa$ and every interval $I \subset (Q, Q^\kappa]$ of the form $I = (x, x(1 + Q^{-1/3})]$,

$$\int_I |\varphi(t) - \varphi(ts)| \frac{ds}{s \log s} \leqq (|\varphi(t) - \varphi(tx)| + \epsilon) \int_I \frac{ds}{s \log s}$$

$$\ll |\varphi(t) - \varphi(tx)| \sum_{p \in I} \frac{1}{p} + \epsilon \int_I \frac{ds}{s \log s}$$

$$\ll \sum_{p \in I} \frac{1}{p} |\varphi(t) - \varphi(tp)| + \epsilon \int_I \frac{ds}{s \log s}.$$

Decomposing the interval $(Q, Q^\kappa]$ into intervals of type I, we obtain, using (3.4) with $a_p = 1$,

$$\int_Q^{Q^\kappa} |\varphi(t) - \varphi(ts)| \frac{ds}{s \log s}$$

$$\ll \sum_{Q < p \leqq Q^\kappa} \frac{1}{p} |\varphi(t) - \varphi(tp)| + \epsilon \int_Q^{Q^\kappa} \frac{ds}{s \log s}$$

$$\leqq 2\epsilon \log \kappa \quad (1 \leqq t \leqq Q^\kappa).$$

It follows that

$$|\varphi(t) - \varphi(1)| \log \frac{\log Q^\kappa}{\log(tQ)}$$

$$\leqq \int_{tQ}^{Q^\kappa} \{ |\varphi(t) - \varphi(s)| + |\varphi(1) - \varphi(s)| \} \frac{ds}{s \log s}$$

$$\leqq \int_Q^{Q^\kappa/t} |\varphi(t) - \varphi(ts)| \frac{ds}{s \log s} + \int_Q^{Q^\kappa} |\varphi(1) - \varphi(s)| \frac{ds}{s \log s}$$

$$\ll \epsilon \log \kappa \quad (1 \leqq t \leqq Q^{\kappa - 1})$$

and therefore

$$\varphi(t) = \varphi(1) + O\left(\frac{\epsilon}{1 - \eta}\right) \quad (1 \leqq t \leqq Q^{\kappa^\eta - 1})$$

for any $\eta \in (0, 1)$. This proves (3.5) in the case $a_p = 1$, $Q < p \leqq Q^{\kappa}$, with $\alpha = 0$ and for the larger range

$$1 \leqq t \leqq Q^{\kappa^{\eta} - 1}.$$

Notice that here the hypothesis $\kappa \geqq 10$ of the lemma is not needed in its full strength; to obtain (3.5) for the range $1 \leqq t \leqq Q$, it suffices that $\kappa \geqq 2 + \delta$ for some fixed $\delta > 0$.

We now turn to the general case. We first note that if $\varphi(t)$ satisfies (3.3) and (3.4) with arbitrary coefficients a_p of modulus 1, then $\varphi_0(t) = |\varphi(t)|$ satisfies the same hypotheses with $a_p \equiv 1$. By the above proved case of the lemma, we conclude

$$(4.1) \qquad |\varphi(t)| = |\varphi(1)| + O(\epsilon) \quad (1 \leqq t \leqq Q^{\kappa_0}),$$

where

$$(4.2) \qquad \kappa_0 = \kappa^{0.99} - 1 \geqq 10^{0.99} - 1 > 8$$

by the assumption $\kappa \geqq 10$ of the lemma. This implies that either

$$(4.3) \qquad 0 < \frac{1}{2}|\varphi(1)| \leqq |\varphi(t)| \leqq 2|\varphi(1)| \quad (1 \leqq t \leqq Q^{\kappa_0})$$

or

$$\varphi(t) = O(\epsilon) \quad (1 \leqq t \leqq Q^{\kappa_0})$$

holds with an absolute O-constant. Since in the latter case the conclusion of the lemma is trivially valid, we may assume (4.3).

Next, let $\kappa_1 = \kappa_0/3$ and $1 \leqq t_0, t \leqq Q^{\kappa_1}$. Noting that by (4.3) for every $p \leqq Q^{\kappa_1}$,

$$\begin{aligned}\left|\frac{\varphi(tt_0)}{\varphi(t)} - \frac{\varphi(tt_0p)}{\varphi(tp)}\right| &\leqq 4\left|\frac{\varphi(tt_0)}{\varphi(tt_0p)} - \frac{\varphi(t)}{\varphi(tp)}\right| \\ &\leqq 4\left\{\left|a_p - \frac{\varphi(tt_0)}{\varphi(tt_0p)}\right| + \left|a_p - \frac{\varphi(t)}{\varphi(tp)}\right|\right\} \\ &\leqq \frac{8}{|\varphi(1)|}\{|a_p\varphi(tt_0p) - \varphi(tt_0)| \\ &\quad + |a_p\varphi(tp) - \varphi(t)|\},\end{aligned}$$

we obtain, using the assumption (3.4),

$$\sum_{Q<p\leqq Q^{\kappa_1}} \frac{1}{p}\left|\frac{\varphi(tt_0)}{\varphi(t)} - \frac{\varphi(tt_0p)}{\varphi(tp)}\right| \leqq \frac{16}{|\varphi(1)|}\epsilon \log \kappa \leqq \epsilon_1 \log \kappa_1,$$

where

$$\epsilon_1 = \frac{16\epsilon}{|\varphi(1)|} \max_{u \geqq 10} \frac{\log u}{\log(u^{0.99} - 1)/3} = \frac{c_2}{|\varphi(1)|}\epsilon,$$

say. Thus, for every fixed $t_0 \in [1, Q^{\kappa_1}]$, the function

$$\varphi_1(t) = \varphi(tt_0)/\varphi(t)$$

satisfies (3.4) with κ_1 and ϵ_1 in place of κ and ϵ, and the coefficients a_p replaced by 1. Moreover, in view of the inequality

$$\left| \frac{\varphi(t't_0)}{\varphi(t')} - \frac{\varphi(tt_0)}{\varphi(t)} \right|$$

$$\leqq \frac{1}{|\varphi(t')|} \left\{ |\varphi(t't_0) - \varphi(tt_0)| + \frac{|\varphi(tt_0)|}{|\varphi(t)|} |\varphi(t') - \varphi(t)| \right\}$$

$$\leqq \frac{10}{|\varphi(1)|}\epsilon < \epsilon_1 \quad (1 \leqq t \leqq t' \leqq t(1 + Q^{-1/3}) \leqq Q^{2\kappa_1}),$$

(3.3) remains valid for $\varphi_1(t)$ with the parameters κ_1 and ϵ_1. Hence we can again apply the above proved case and obtain

$$\text{(4.4)} \quad \frac{\varphi(tt_0)}{\varphi(t)} = \frac{\varphi(t_0)}{\varphi(1)} + O(\epsilon_1) \quad (1 \leqq t, t_0 \leqq Q^{\kappa_2}),$$

where

$$\kappa_2 = \kappa_1^{0.99} - 1 \geqq \left(\frac{8}{3}\right)^{0.99} - 1 > 1$$

by (4.2).

(4.4) can be regarded as a multiplicative form of Cauchy's functional equation with error term, and we shall show that, under some mild regularity condition, it has only trivial solutions. To this end, we use the following general result, which will be proved in the next section.

LEMMA 3. *Let $T \geqq 2$ and $q(t)$ be a complex-valued function, defined for $1 \leqq t \leqq T$ and satisfying, for some $\delta \in (0, 1/2)$,*

$$\text{(4.5)} \quad |q(t') - q(t)| \leqq \delta |q(t)| \quad (1 \leqq t \leqq t' \leqq t(1 + T^{-1/2}) \leqq T)$$

and

$$\text{(4.6)} \quad |q(tt') - q(t)q(t')| \leqq \delta |q(tt')| \quad (t, t' \geqq 1, 1 \leqq tt' \leqq T).$$

Then there exists a complex number z, $|z| \leqq T$, such that

$$\text{(4.7)} \quad q(t) = t^z \exp\{O(\delta)\} \quad (1 \leqq t \leqq T),$$

where the implied constant is absolute.

We apply Lemma 3 with $q(t) = \varphi(t)/\varphi(1)$ and $T = Q$. From (3.3), (4.1), (4.3) and (4.4) we see that (4.5) and (4.6) are satisfied with

$$\delta = c_3\epsilon/|\varphi(1)|$$

for some absolute constant c_3. We may assume $0 < \delta < 1/2$, as required in the lemma, since otherwise by (4.1) the desired relation (3.5) holds trivially for any $\alpha \in \mathbf{R}$. By Lemma 3 we therefore conclude that for some $z \in \mathbf{C}$, $|z| \leqq T = Q$,

$$(4.8) \quad \varphi(t) = q(t)\varphi(1) = t^z\varphi(1)\exp\{O(\delta)\} = t^z(\varphi(1) + O(\epsilon)) \quad (1 \leqq t \leqq Q).$$

This implies, in particular,

$$|\varphi(Q)| = Q^{\mathrm{Re} z}(|\varphi(1)| + O(\epsilon)),$$

whence, by (4.1),

$$Q^{\mathrm{Re} z}\varphi(1) = \varphi(1) + O(|(Q^{\mathrm{Re} z} - 1)\varphi(1)|) = \varphi(1) + O(\epsilon).$$

Therefore we can replace z by i Im z in (4.8) and obtain the desired relation (3.5) with $\alpha = \mathrm{Im}\, z$. Since $|z| \leqq T = Q$, we have $|\alpha| \leqq Q$, as required in Lemma 2. For real-valued functions $\varphi(t)$, the relation (3.5) with $\alpha = 0$ follows already from (4.1) and the hypothesis (3.3); alternatively, it can easily be seen from (3.5), that if this relation holds with some $\alpha \in \mathbf{R}$ for a real-valued function $\varphi(t)$, then it also holds with $\alpha = 0$. Hence all the assertions of Lemma 2 are proved.

5. Proof of lemma 3. We first remark that since by hypothesis $0 < \delta \leqq 1/2$ in (4.5) and (4.6), we can write these relations in the more convenient form

$$(4.5)' \quad q(t') = q(t)\exp\{0(\delta)\} \quad (1 \leqq t \leqq t' \leqq t(1 + T^{-1/2}) \leqq T),$$

$$(4.6)' \quad q(tt') = q(t)q(t')\exp\{0(\delta)\} \quad (t, t' \geqq 1, 1 \leqq tt' \leqq T).$$

Let now $q(t)$ satisfy (4.5)′ and (4.6)′, put $\overline{q}(1) = 1$ and for $1 < t \leqq T$ define

$$\overline{q}(t) = q(t^{2^k})^{2^{-k}},$$

where $k = k(t)$ is defined by

$$t^{2^k} \leqq T < t^{2^{k+1}}.$$

Since by (4.5)′ $q(t)$ is bounded on the interval $[1, T]$, we have

$$(5.1) \quad \lim_{t \to 1+} \overline{q}(t) = 1 = \overline{q}(1).$$

Also, since by (4.6)′

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$$q(t^{2^i})^{2^{-i}} = q(t^{2^{i-1}})^{2^{-(i-1)}} \exp\{O(2^{-i}\delta)\}$$

for $i \leqq k(t)$, we get inductively

(5.2) $\quad \bar{q}(t) = q(t^{2^i})^{2^{-i}} \exp\{O(2^{-i}\delta)\} \quad (1 < t \leqq T, 0 < i \leqq k(t)),$

and, in particular,

(5.3) $\quad \bar{q}(t) = q(t) \exp\{0(\delta)\} \quad (1 \leqq t \leqq T),$

where the case $t = 1$ follows from (5.1) and (4.5)′. (4.6)′ and (5.2) yield

(5.4)
$$\begin{aligned} \bar{q}(t_1t_2) &= q((t_1t_2)^{2^k})^{2^{-k}} = q(t_1^{2^k})^{2^{-k}} q(t_2^{2^k})^{2^{-k}} \exp\{O(2^{-k}\delta)\} \\ &= \bar{q}(t_1)\bar{q}(t_2) \exp\{O(2^{-k}\delta)\} \\ &= \bar{q}(t_1)\bar{q}(t_2) \exp\left\{O\left(\frac{\log(t_1t_2)}{\log T}\delta\right)\right\} \end{aligned}$$

$$(t_1, t_2 \geqq 1, 1 < t_1t_2 \leqq T, k = k(t_1t_2)).$$

Put

$$\lambda = \left(1 + \frac{1}{\sqrt{T}}\right), \quad z = \frac{\log \bar{q}(\lambda)}{\log \lambda},$$

where the determination of $\log \bar{q}(\lambda)$ is such that

$$|\mathrm{Im} \log \bar{q}(\lambda)| \leqq \pi.$$

Since by (5.3) and (4.5)′

$$\begin{aligned} \bar{q}(\lambda) &= q(\lambda) \exp\{O(\delta)\} = q(1) \exp\{O(\delta)\} \\ &= \bar{q}(1) \exp\{O(\delta)\} = \exp\{O(\delta)\}, \end{aligned}$$

we have

$$|\log \bar{q}(\lambda)| \leqq c_4\delta$$

for some absolute constant c_4 and hence

(5.5) $\quad |z| \leqq c_4\delta(\log \lambda)^{-1} \leqq 2c_4\delta\sqrt{T}.$

We shall prove (4.7) with this number z.

By (4.5)′, (5.3) and (5.5) it is enough to show

$$\bar{q}(\lambda^n) = \lambda^{nz} \exp\{O(\delta)\} \quad (1 \leqq n \leqq \log T/\log \lambda).$$

This holds trivially, if n is of the form $n = 2^i$, since, by the definition of the function $\bar{q}$,

$$\bar{q}(t^{2^i}) = \bar{q}(t)^{2^i} \text{ for any } t \in [1, T^{2^{-i}}].$$

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Otherwise, we write $n \leqq \log T/\log \lambda$ in the form

$$n = \sum_{i \in I} 2^i, \quad I \subset \mathbf{N} \cup \{0\},$$

and obtain, by a repeated application of (5.4),

$$\bar{q}(\lambda^n) = \prod_{i \in I} \bar{q}(\lambda^{2^i}) \exp\left\{ O\left(\delta \sum_{i \in I} \frac{2^i \log \lambda}{\log T}\right)\right\}$$

$$= \bar{q}(\lambda)^n \exp\left\{ O\left(\delta \frac{\log(\lambda^n)}{\log T}\right)\right\} = \lambda^{zn} \exp\{O(\delta)\},$$

as wanted.

The required bound $|z| \leqq T$ follows from (5.5) if $\delta \leqq 1/(2c_4)$ or $T \geqq T_0$ for some absolute constant T_0. But if these conditions are not satisfied, then, by (4.5)′ and (5.3), (4.7) holds trivially with $z = 0$.

6. Proof of corollary 5. We fix a multiplicative function f of modulus $\leqq 1$ and a function $\alpha(x)$ such that (2.1) holds with $\alpha = \alpha(x)$. By Theorem 4 we have, for any function $\phi(x)$ satisfying (2.3),

$$(6.1) \quad \frac{1}{\phi(x)}(M(f, x) - M(f, x - \phi(x)))$$

$$= \frac{1}{\phi(x)x} M(f_{\alpha(x)}, x) I_{\alpha(x)}(x, \phi(x)) + o(1) \quad (x \to \infty),$$

where

$$(6.2) \quad I_{\alpha(x)}(x, \phi(x)) = \int_{x-\phi(x)}^{x} t^{-i\alpha(x)} dt \ll \phi(x) \min\left(1, \frac{x}{\phi(x)\,|\alpha(x)|}\right).$$

Suppose first that (1.5) and hence (1.3) hold with constants $A \in \mathbf{C}\setminus\{0\}$, $\alpha \in \mathbf{R}$ and a function $L(u)$ satisfying (1.2). Then

$$\lim_{x \to \infty} |M(f, x)|/x = |A| > 0$$

by (1.3), and applying (6.1) and (6.2) with $\phi(x) = x$, we deduce that $\alpha(x)$ is bounded for $x \geqq 3$. But then we have

$$\frac{1}{\phi(x)} I_{\alpha(x)}(x, \phi(x)) = x^{-i\alpha(x)} + o(1) \quad (x \to \infty)$$

for any function $\phi(x)$ satisfying (2.7). Thus the right-hand side of (6.1) is independent of $\phi(x)$, provided $\phi(x)$ satisfies (2.3) and (2.7). Since (1.3) obviously implies the desired relation (2.9) for some function $\phi(x)$ satisfying (2.3) and (2.7), we therefore obtain (2.9) for all such functions $\phi(x)$.

Thus we have proved that, under the hypotheses of Corollary 5, (1.5) (with $A \neq 0$) implies (2.9). To complete the proof of Corollary 5, it suffices to show that (1.4) implies (2.8). The asserted equivalence between (1.4) and (2.8) and between (1.5) and (2.9) (with $A \neq 0$) then follows, since by Theorem 1 either (1.4) or (1.5) holds and the relations (2.8) and (2.9) with $A \neq 0$ cannot hold simultaneously.

To prove the implication (1.4) $\Rightarrow$ (2.8), we assume that (2.8) is not satisfied, so that

$$\liminf_{x\to\infty} \frac{1}{\phi(x)}|M(f, x) - M(f, x - \phi(x))| > 0$$

for some function $\phi(x)$ satisfying (2.3) and (2.7). By (6.1) and (6.2) this implies

$$\liminf_{x\to\infty} |M(f_{\alpha(x)}, x)|/x > 0. \tag{6.3}$$

On the other hand, Theorem 3 yields the upper bound

$$\begin{aligned} &|M(f_{\alpha(x)}, x)| \\ &\ll x\left\{\frac{1}{\log x}\left|F\left(1 + \frac{1}{\log x} - it(x)\right)\right| + \frac{1}{\log x}\right\}^{1/4} \\ &\ll x \exp\left\{-\frac{1}{4} \operatorname{Re} \sum_{p\leqq x} \frac{1 - f(p)p^{it(x)}}{p}\right\} + \frac{x}{(\log x)^{1/4}} \end{aligned} \tag{6.4}$$

for suitable numbers $t(x) \in \mathbf{R}$ satisfying

$$|t(x) - \alpha(x)| \leqq \log x \quad (x \geqq 3)$$

and hence

$$|t(x)| \leqq |\alpha(x)| + \log x \leqq x + \log x \quad (x \geqq 3). \tag{6.5}$$

(6.3) and (6.4) imply

$$\operatorname{Re} \sum_{p\leqq x} \frac{1 - f(p)p^{it(x)}}{p} \ll 1 \quad (x \geqq 3).$$

By means of the elementary inequality

$$\begin{aligned} \operatorname{Re}\left(1 - \frac{z_1}{z_2}\right) &= \frac{1}{2}\left|1 - \frac{z_1}{z_2}\right|^2 \leqq |1 - z_1|^2 + |1 - z_2|^2 \\ &\leqq 2\{\operatorname{Re}(1 - z_1) + \operatorname{Re}(1 - z_2)\} \end{aligned}$$

$$(|z_1| = |z_2| \leqq 1)$$

we deduce

$$\operatorname{Re} \sum_{p \leqq x} \frac{1 - p^{i(t(x)-t(x'))}}{p} \ll 1 \quad (x' \geqq x \geqq 3).$$

The left-hand side in this estimate equals

$$\log \frac{\left|\zeta\left(1 + \dfrac{1}{\log x}\right)\right|}{\left|\zeta\left(1 + \dfrac{1}{\log x} - i(t(x) - t(x'))\right)\right|} + O(1)$$

$$\geqq \operatorname{loglog} x - \frac{2}{3} \operatorname{loglog}(\,|t(x) - t(x')\,| + 3)$$

$$+ \log\left|\frac{1}{\log x} - i(t(x) - t(x'))\right| + O(1)$$

by (3.19). In view of (6.5) it follows that

$$t(x) = t(x') + O\left(\frac{1}{\log x}\right) \quad (3 \leqq x \leqq x' \leqq x^2).$$

Hence the numbers $t(x)$ are bounded, $|t(x)\,| \leqq K$ $(x \geqq 3)$, say, and from (6.3) and (6.4) we conclude that

$$F^*(\sigma, K) = \sup_{|t| \leqq K} |F(\sigma + it)\,| \gg \frac{1}{\sigma - 1} \quad (1 < \sigma < 1.5),$$

which contradicts (1.4). This proves the implication (1.4) $\Rightarrow$ (2.8).

7. A negative result. By Theorem 1, the limit

$$\lim_{x \to \infty} |M(f, x)\,|/x$$

exists for any multiplicative function f of modulus $\leqq 1$, and Corollary 5 shows that if this limit is non-zero, then, for any function $\phi(x)$ satisfying (2.3) and (2.7), the limit

$$\lim_{x \to \infty} \frac{1}{\phi(x)} |M(f, x) - M(f, x - \phi(x))\,|$$

exists and is non-zero. We show here that in the case f has mean value zero, the above limit does not exist in general.

THEOREM 5. *Let* $\phi(x)$ *satisfy* (2.3) *and* (2.7). *Then there exists a multiplicative function* f *of modulus* $\leqq 1$, *for which the mean value* (1.1) *exists and equals zero, but*

(7.1) $$\liminf_{x\to\infty} \frac{1}{\phi(x)} |M(f, x) - M(f, x - \phi(x))| = 0$$

and

(7.2) $$\limsup_{x\to\infty} \frac{1}{\phi(x)} |M(f, x) - M(f, x - \phi(x))| = 1.$$

Proof. Let $\phi(x)$ be given as in the theorem. Put $x_0 = \alpha_0 = 1$, and for $k \geqq 1$ choose x_k and α_k inductively such that

(7.3) $$x_k/(\phi(x_k)\alpha_{k-1}) \geqq k,$$

(7.4) $$\alpha_k \geqq k$$

and

(7.5) $$|p^{i\alpha_k} - p^{i\alpha_j}| \leqq \frac{1}{kx_k} \quad (0 \leqq j \leqq k - 1, x_j \leqq p < x_{j+1}).$$

The condition (7.3) can be met in view of the assumption (2.7) on $\phi(x)$. The existence of a number $\alpha_k \geqq k$ satisfying (7.5) follows from Kronecker's theorem.

Define a completely multiplicative function f by

$$f(p) = p^{i\alpha_k} \quad (x_k < p \leqq x_{k+1}).$$

By (7.5) we have, for $n \leqq x_k$,

$$f(n) = n^{i\alpha_{k-1}} + O\left(\frac{\Omega_{k-1}(n)}{kx_{k-1}}\right),$$

where

$$\Omega_k(n) = \sum_{\substack{p^m | n \\ p \leqq x_k}} 1.$$

It follows that for $y \leqq x_k$

(7.6) $$M(f, x_k) - M(f, x_k - y)$$
$$= \sum_{x_k - y < n \leqq x_k} n^{i\alpha_{k-1}} + O\left(\frac{1}{kx_{k-1}} \sum_{x_k - y < n \leqq x_k} \Omega_{k-1}(n)\right).$$

The error term can be estimated by

$$\ll \frac{1}{kx_{k-1}} \sum_{p \leqq x_{k-1}} \sum_{\substack{m \geqq 1 \\ p^m \leqq x_k}} \sum_{\substack{x_k - y < n \leqq x_k \\ p^m | n}} 1 \ll \frac{y + \log x_k}{k \log x_{k-1}}$$

and hence is of order $o(y)$ as $k \to \infty$, uniformly for $\sqrt{x_k} \leqq y \leqq x_k$.

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Taking in turn $y = x_k$ and $y = \phi(x_k)$ in (7.6), we obtain, by (7.3) and (7.4),

$$\lim_{k\to\infty} \frac{1}{x_k}|M(f, x_k)| = \lim_{k\to\infty} \frac{1}{x_k}\left|\sum_{n\leqq x_k} n^{i\alpha_{k-1}}\right| = 0$$

and

$$\lim_{k\to\infty} \frac{1}{\phi(x_k)}|M(f, x_k) - M(f, x_k - \phi(x_k))|$$
$$= \lim_{k\to\infty} \frac{1}{\phi(x_k)}\left|\sum_{x_k-\phi(x_k)<n\leqq x_k} n^{i\alpha_{k-1}}\right| = 1.$$

The second relation proves (7.2), while the first implies, by Theorem 1 and Corollary 5, that f has mean value zero and satisfies (7.1).

Appendix. Proof of theorem 1. We confine ourselves to the proof that (1.4) implies, under the assumptions of Theorem 1, that f has mean value zero. This is the hard part of Theorem 1, for which Halász' analytical method had been so far the only available method of proof. The proof of the remaining assertions of Theorem 1 is comparatively simple. It is straightforward to verify that for any multiplicative function of modulus $\leqq 1$ either (1.4) or (1.5) (with $A \neq 0$) holds (cf. [**5**, Vol. I, p. 244]). The implication (1.5) $\Rightarrow$ (1.3) (in the case $A \neq 0$) can be proved by elementary convolution arguments.

We fix a multiplicative function f of modulus $\leqq 1$ and assume that (1.4) holds. Put $m(x) = M(f, x)/x$, and for $0 < \epsilon \leqq 1/2$ define

$$R(\epsilon) = \{\min(\log(1/\epsilon), \log\log x)\}^{-1/2}.$$

Our proof is based on the relation

$$m(x) = m\left(\frac{x}{t}\right)t^{i\alpha} + O(R(\epsilon)) \quad (1 \leqq t \leqq x^{\epsilon}),$$

where $\alpha = \alpha(\epsilon, x)$ is a suitable real number of modulus $\leqq x$. This is a variant of (3.8) and can be established in the same way, the only difference being a different choice of the parameters Q, κ and ϵ in the application of Lemma 2. Multiplying both sides of this relation by t^{-1} and integrating over $1 \leqq t \leqq x^{\epsilon}$, we get

$$m(x) \log x^{\epsilon}$$
$$= \frac{x^{i\alpha}}{1 + i\alpha}\left(\sum_{x^{1-\epsilon}<n\leqq x} \frac{f(n)n^{-i\alpha}}{n}\right) + O(1 + R(\epsilon) \log x^{\epsilon}).$$

To obtain the desired result $m(x) = o(x)$ $(x \to \infty)$, it now suffices to show that (1.4) implies

$$\max_{|\alpha| \leqq K} \left| \sum_{n \leqq x} \frac{f(n) n^{i\alpha}}{n} \right| = o(\log x) \quad (x \to \infty)$$

for every $K > 0$. But this is a consequence of the following lemma.

LEMMA. *Uniformly for all multiplicative functions f of modulus $\leqq 1$ and all $x \geqq 3$ we have*

$$\left| \sum_{n \leqq x} \frac{f(n)}{n} \right| \ll 1 + \left(\frac{1}{\log x} \left| F\left(1 + \frac{1}{\log x}\right) \right| \right)^{1/2} \log x.$$

Proof. The proof is based on a simple convolution argument. Putting $g = f * 1$ (where $*$ denotes the Dirichlet convolution), we have

$$\left| \sum_{n \leqq x} \frac{f(n)}{n} \right| = \left| \frac{1}{x} \sum_{n \leqq x} f(n) \sum_{m \leqq x/n} 1 \right| + O\left(\frac{1}{x} \sum_{n \leqq x} |f(n)| \right)$$

$$= \left| \frac{1}{x} \sum_{n \leqq x} g(n) \right| + O(1) \leqq \frac{1}{x} \sum_{n \leqq x} |g(n)| + O(1).$$

By an elementary estimate of Halberstam and Richert [**9**], the last expression is

$$\ll \exp\left(\sum_{p \leqq x} \frac{|g(p)| - 1}{p} \right) + 1.$$

Now

$$\sum \frac{|g(p)|}{p} = \sum \frac{|1 + f(p)|}{p}$$

$$\leqq \left(\sum \frac{|1 + f(p)|^2}{p} \right)^{1/2} \left(\sum \frac{1}{p} \right)^{1/2}$$

$$\leqq \left(\sum \frac{4 - 2(1 - \operatorname{Re} f(p))}{p} \right)^{1/2} \left(\sum \frac{1}{p} \right)^{1/2}$$

$$= 2\left(\sum \frac{1}{p} \right) \left(1 - \frac{1}{2} S \left(\sum \frac{1}{p} \right)^{-1} \right)^{1/2} \leqq 2 \sum \frac{1}{p} - \frac{1}{2} S,$$

where the summations run over all primes $\leqq x$ and

$$S = \sum_{p \leqq x} \frac{1 - \operatorname{Re} f(p)}{p}.$$

Hence we get

$$\left| \sum_{n \leqq x} \frac{f(n)}{n} \right| \ll \exp\left\{ \sum_{p \leqq x} \frac{1}{p} - \frac{1}{2} S \right\} + 1 \ll e^{-S/2} \log x.$$

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If now $f(2^m) = 0$ for all $m \geqq 1$, then a standard argument yields

$$e^{-S} \asymp \frac{1}{\log x}\left|F\left(1 + \frac{1}{\log x}\right)\right|,$$

and the asserted estimate follows. The general case can be readily derived from this, using the identity

$$\sum_{n \leqq x} \frac{f(n)}{n} = \sum_{m \geqq 0} \frac{f(2^m)}{2^m} \sum_{\substack{n \leqq x2^{-m} \\ (n,2)=1}} \frac{f(n)}{n}$$

$$= \sum_{m \geqq 0} \frac{f(2^m)}{2^m} \sum_{\substack{n \leqq x \\ (n,2)=1}} \frac{f(n)}{n} + O(1).$$

REFERENCES

1. G. J. Babu, *Probabilistic methods in the theory of arithmetical functions*, Macmillan Lectures in Mathematics, New Delhi (1978).
2. ——— *On the distribution of arithmetic functions*, Acta Arith. *29* (1976), 97-104.
3. ——— *On the mean values and distributions of arithmetic functions*, Acta Arith. *40* (1981), 63-77.
4. H. Delange, *Sur les fonctions arithmétiques multiplicatives*, Ann. Scient. Ec. Norm. Sup. *78* (1961), 273-304.
5. P. D. T. A. Elliott, *Probabilistic number theory I, II* (Springer, New York, 1979, 1980).
6. ——— *Arithmetic functions and integer products* (Springer, New York, 1985).
7. G. Halász, *Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen*, Acta Math. Acad. Sci. Hung. *19* (1968), 365-403.
8. ——— *On the distribution of additive arithmetic functions*, Acta Arith. *27* (1975), 143-152.
9. H. Halberstam and H.-E. Richert, *On a result of R. R. Hall*, J. Number Theory *11* (1979), 76-89.
10. A. Hildebrand, *On Wirsing's mean value theorem for multiplicative functions*, Bull. London Math Soc. *18* (1986), 147-152.
11. M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. *15* (1972), 164-170.
12. H. L. Montgomery, *A note on mean values of multiplicative functions*, Preprint. Mittag-Leffler Institute (1978).
13. Y. Motohashi, *On the sum of the Möbius function in a short segment*, Proc. Japan Acad. *52* (1976), 477-479.
14. K. Ramachandra, *Some problems of analytic number theory*, Acta Arith. *31* (1976), 313-323.
15. P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. Reine Ang. Math. *313* (1980), 161-170.
16. E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen II*, Acta Math. Acad. Sci. Hung. *18* (1967), 411-467.

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