A NOTE ON EVALUATION OF THE INTEGRAL

$$\int_0^\infty e^{-kt} I_0^n(t)dt$$

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1. The object of this note is to show that the integral

$$\int_0^\infty e^{-kt} I_0^n(t)dt \quad \text{.................................(1.1)}$$

for certain particular values of k and n can be evaluated in terms of complete elliptic integrals. The integral (1.1) for $n = 1$ can be easily expressed as a binomial and for $n = 2$ in terms of a complete elliptic integral (**2**). However, the corresponding value for other cases does not appear to be given in the literature.

In this note, I use an indirect method to evaluate (1.1) for $k = 3 = n$, in terms of a square of a complete elliptic integral. In the sequel, an interesting case of reducibility of a particular F_c-function (one of the Lauricella's hypergeometric functions of three variables (**1**)) is obtained.

2. Evaluation of (1.1) for $k = 3 = n$

It is very easy to see that (1.1) can be evaluated for positive integral values of n by term by term integration of the n-ple series representing $I_0^n(t)$. In fact, simple algebra shows that

$$\int_0^\infty e^{-kt} I_0^n(t)dt = \frac{1}{k} F_c\left(\tfrac{1}{2}, 1;\ 1, \ldots, 1;\ \frac{1}{k^2}, \ldots, \frac{1}{k^2}\right), \quad \text{...........(2.1)}$$

where F_c is one of the four Lauricella's hypergeometric functions of n-variables.

For $k = 3 = n$, (2.1) gives

$$\int_0^\infty e^{-3t} I_0^3(t)dt = \tfrac{1}{3} F_c(\tfrac{1}{2}, 1;\ 1, 1, 1;\ \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{9}). \quad \text{..............(2.2)}$$

Now, since

$$I_0(t) = \frac{1}{\pi} \int_0^\pi e^{t \cos u} du,$$

the left hand side of (2.2) can be written as

$$\frac{1}{\pi^3}\int_0^\infty e^{-3t}\left[\int_0^\pi\int_0^\pi\int_0^\pi e^{t(\cos u+\cos v+\cos w)}dudvdw\right]dt.$$

Interchanging the order of integration, which is obviously justified, this becomes

$$\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{dudvdw}{3-\cos u-\cos v-\cos w}. \quad\text{..................(2.3)}$$

Evaluating (2.3) by the help of a known integral due to Watson (3), we have

$$\int_0^\infty e^{-3t}I_0^3(t)dt=(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})\left(\frac{2K_2}{\pi}\right)^2, \quad\text{.........(2.4)}$$

where K_2 is the complete elliptic integral with modulus $(2-\sqrt{3})(\sqrt{3}-\sqrt{2})$.

(2.4) gives the desired result.

Incidentally, we have shown that

$$\tfrac{1}{3}F_c(\tfrac{1}{2},\,1;\;1,\,1,\,1;\;\tfrac{1}{9},\,\tfrac{1}{9},\,\tfrac{1}{9})=(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})\left(\frac{2K_2}{\pi}\right)^2. \quad\text{......(2.5)}$$

It does not appear to be easy to establish the reduction in (2.5) by direct transformation of its left hand series.

REFERENCES

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