POSITIVE MATRICES AND EIGENVECTORS†

by C. R. PUTNAM

(Received 19 February, 1962)

1. For $i, j = 1, 2, \ldots$, let a_{ij} be real. A matrix $A = (a_{ij})$ will be called positive ($A > 0$) or non-negative ($A \geqq 0$) according as, for all i and j, $a_{ij} > 0$ or $a_{ij} \geqq 0$ respectively. Correspondingly, a real vector $x = (x_1, x_2, \ldots)$ will be called positive ($x > 0$) or non-negative ($x \geqq 0$) according as, for all i, $x_i > 0$ or $x_i \geqq 0$. A matrix A is said to be bounded if $\| Ax \| \leqq M \| x \|$ holds for some constant M, $0 \leqq M < \infty$, and all x in the Hilbert space H of real vectors $x = (x_1, x_2, \ldots)$ satisfying $\| x \|^2 = \sum x_i^2 < \infty$. The least such constant M is denoted by $\| A \|$. If x and y belong to H, then (x, y) will denote as usual the scalar product $\sum x_i y_i$. Whether or not x is in H, or A is bounded, $y = Ax$ will be considered as defined by

$$y_i = \sum_j a_{ij} x_j \tag{1}$$

whenever each of the series of (1) is convergent.

When A is bounded, A^* will denote its adjoint, $A^* = (a_{ji})$, and sp A will denote its spectrum, that is, the set of complex numbers λ for which $A - \lambda I$ fails to have a bounded (right and left, necessarily unique) inverse. The point spectrum consists of those λ in sp A for which $Ax = \lambda x$ holds for some $x \neq 0$ in the Hilbert space H.

It is known, as a generalization of the Perron-Frobenius theory ([**4**], [**5**], [**13**]) for finite matrices, that if $A \geqq 0$, then $\mu = \sup \{ | \lambda | : \lambda \in \text{sp } A \}$ also belongs to sp A; see, e.g., [**2**] (cf. pp. 148 ff.), [**8**], [**14**], [**15**]. In addition, it is known that under certain additional restrictions on A, e.g., that of complete continuity, μ is in the point spectrum of A and there exists a characteristic vector x in H, satisfying $x > 0$ or $x \geqq 0$ according as $A > 0$ or $A \geqq 0$; see [**8**], [**11**], [**14**]. In case $A > 0$, then also μ is a simple eigenvalue.

If it is assumed only that A is bounded and that $A > 0$, then μ need not be an eigenvalue. (The Hilbert matrix cited below is such an example. Also, any Toeplitz matrix $A = (a_{ij})$ given by $a_{ij} = b_{i-j}$, where $\{b_k\}$ ($k = 0, \pm 1, \pm 2, \ldots$) is a sequence of positive numbers for which A is bounded, will do; cf. [**7**], p. 868.) Thus, in general, μ need not have an associated eigenvector *in Hilbert space*. On the other hand, it may happen that there exists a vector x not in Hilbert space for which $Ax = \mu x$. This problem has been considered in particular by Kato [**9**], [**10**] and Rosenblum [**17**]. The Hilbert matrix $A = ((i+j)^{-1})$ satisfies $A > 0$ and is bounded; in fact $\mu = \pi$ (see [**6**], Chapter IX) and, moreover, μ is not in the point spectrum of A ([**12**], [**18**]). It was shown by Kato [**9**] in connection with a problem posed by Taussky [**19**], that μ does however have a positive eigenvector x not belonging to H.

The present paper will consider the problem of the existence of vectors $x > 0$, not necessarily in H, associated with certain bounded $A > 0$, for which

$$Ax \leqq \mu x; \quad \mu = \sup \{ | \lambda | : \lambda \in \text{sp } A \}. \tag{2}$$

In Theorem 1 it will be shown that the inequality of (2), when x is in H, implies equality under

† This research was supported by the U.S. Air Force Office of Scientific Research.

certain circumstances, while in Theorem 2 it will be shown that (2) must always hold for some vector x, not necessarily in Hilbert space. This last result will be combined with a theorem of Kato to yield Theorem 3, giving a necessary and sufficient condition for the boundedness of a symmetric positive matrix.

2. There will be proved the following

THEOREM 1. *Let A be bounded and satisfy $A>0$. Suppose that* (2) *holds for some $x>0$ of the Hilbert space H. Suppose in addition that either μ belongs to the point spectrum of A^*, or only that there exists some v satisfying $|v|=\mu$ and belonging to the point spectrum of A^*. Then necessarily equality must hold in* (2), *so that*

$$Ax=\mu x. \tag{3}$$

Furthermore, μ must then be a simple eigenvalue of both A and A^*, and both have positive eigenvectors (each unique except for a positive multiple).

It can be remarked that in case A is completely continuous, then μ belongs to the point spectrum of both A and A^*, so that, in particular, the hypothesis of the theorem concerning A^* is fulfilled.

Proof of Theorem 1. Let $A^*y=vy$, where $|v|=\mu$ for some $y=(y_1, y_2, \ldots)\neq 0$ of the Hilbert space H. If $|y|$ is defined by $|y|=(|y_1|, |y_2|, \ldots)$, it is clear that $|y|$ is also in H and that

$$A^*|y|\geqq\mu|y|. \tag{4}$$

Hence, by (2),

$$(\mu x, |y|)\geqq(Ax, |y|)=(x, A^*|y|)\geqq(x, \mu|y|). \tag{5}$$

Thus the inequalities of (5) become equalities. In particular the last yields

$$(x, A^*|y|-\mu|y|)=0$$

and hence, by (4) and the fact that $x>0$, $A^*|y|=\mu|y|$. But $A^*>0$, and this implies that $|y|>0$. The first relation of (5) now becomes $(Ax-\mu x, |y|)=0$, which yields (3), as a consequence of (2) and $|y|>0$. The last assertion of the theorem can be proved as in **[14]** (cf. p. 590).

The argument used above is similar to that used in [**16**, pp. 78–80, 82] for integral equations.

3. In this section it will be shown that, for *every* bounded $A>0$, there exists *some* positive vector x, not necessarily in Hilbert space, for which (2) holds. Whether there exists a relation corresponding to (3) under conditions similar to those of Theorem 1 will remain undecided however. There will be proved the following

THEOREM 2. *Let A be bounded and satisfy $A>0$. Then there exists a vector $x>0$, not necessarily in Hilbert space, for which* (2) *holds.*

Proof of Theorem 2. Let λ be real and satisfy $\lambda>\mu$. Since the resolvent $R(\lambda)=(A-\lambda I)^{-1}$ satisfies $(A-\lambda I)R(\lambda)=I$, it follows that

$$AR(\lambda)y=\lambda R(\lambda)y+y, \tag{6}$$

where $y=(y_k)$ is any vector of Hilbert space. If $R(\lambda)=(r_{ij}(\lambda))$ and if $z=R(\lambda)y$, so that

$$z_k = \sum_m r_{km}(\lambda) y_m \quad (k = 1, 2, \ldots), \tag{7}$$

then, by (6),

$$(Az)_k = \lambda z_k + y_k. \tag{8}$$

Since $\lambda > \mu$ (μ being the spectral radius of A), $R(\lambda)$ is given by $R(\lambda) = -\sum_{n=0}^{\infty} A^n \lambda^{-n-1} < 0$. Choose $\lambda_1 > \lambda_2 > \ldots \to \mu + 0$ and let, for each $n = 1, 2, \ldots, y^{(n)} = (y_k^{(n)})$ be defined by

$$y_k^{(n)} = (r_{1k}(\lambda_n))^{-1} \text{ or } 0 \quad \text{according as} \quad k = n \text{ or } k \neq n. \tag{9}$$

Then, if $z^{(n)} = (z_k^{(n)})$, one obtains from (7) the relation

$$z_k^{(n)} = \sum_m r_{km}(\lambda_n) y_m^{(n)} = r_{kn}(\lambda_n)/r_{1n}(\lambda_n) > 0, \tag{10}$$

and, in particular,

$$z_1^{(n)} = 1 \quad \text{for} \quad n = 1, 2, \ldots . \tag{11}$$

Since $y_k^{(n)} \to 0$ as $n \to \infty$ for each fixed $k = 1, 2, \ldots$, it follows that $\{\lambda z_1^{(n)} + y_1^{(n)}\}$ is a bounded sequence of numbers. According to (8), for $k = 1$, this last expression is equal to $\{\sum a_{1m} z_m^{(n)}\}$; hence, since $A > 0$ and $z_k^{(n)} > 0$, $\{z_k^{(n)}\}$ is a bounded sequence of numbers for each fixed $k = 1, 2, \ldots$. By the diagonal selection process there exists a sequence $\mu_1 > \mu_2 > \ldots \to \mu + 0$ for which

$$x_k = \lim_{n \to \infty} Z_k^{(n)} \quad \text{exists for each } k = 1, 2, \ldots , \tag{12}$$

where $Z_k^{(n)}$ is defined by (10) with $Z_k^{(n)} = z_k^{(n)}$ and λ_n replaced by μ_n. Clearly $x = (x_1, x_2, \ldots) \geqq 0$. In addition, it follows from (8) that

$$\mu x_k = \lim_{n \to \infty} \left(\sum_m a_{km} Z_m^{(n)} \right) \geqq (Ax)_k, \quad \text{for } k = 1, 2, \ldots . \tag{13}$$

Hence (2) holds and so $x > 0$ by virtue of (11) and $A > 0$. This completes the proof of Theorem 2.

4. As a consequence of a result of Kato [**10**, p. 576] and Theorem 2 of the present paper there will be proved

THEOREM 3. *Let $A = (a_{ij})$ be any symmetric positive matrix ($0 < a_{ij} = a_{ji}$), not assumed to be bounded. Then a necessary and sufficient condition that A be bounded is that there exist some real constant ν and a vector $x > 0$, not necessarily in Hilbert space, for which*

$$Ax \leqq \nu x. \tag{14}$$

Proof of Theorem 3. The sufficiency follows from the result of Kato mentioned above, even if the hypothesis $A > 0$ is weakened to $A \geqq 0$. In fact, it is shown there that

$$\| A \| \leqq \nu. \tag{15}$$

The necessity follows from Theorem 2 above. In fact, if A is bounded and positive, even if A is not symmetric, then (2) holds for some $x>0$. This completes the proof of Theorem 3.

Incidentally, it is clear that relation (2) for some $x>0$ implies (14) for the same x and all real $v>\mu$. Since $\|A\|=\mu$, it follows from (15) that (14) then holds for some fixed $x>0$ and for all $v\geqq\mu$, but that (14) does not hold for any $x>0$ if $v<\mu$.

REFERENCES

1. F. F. Bonsall, Endomorphisms of partially ordered vector spaces, *J. London Math. Soc.* **30** (1955), 133–144.

2. F. F. Bonsall, Endomorphisms of a partially ordered vector space without order unit, *ibid.* **30** (1955), 144–153.

3. F. F. Bonsall, Linear operators in complete positive cones, *Proc. London Math. Soc.*, Ser. 3, **8** (1958), 53–75.

4. G. Frobenius, Ueber Matrizen aus positiven Elementen, *Sitzungsberichte Preuss. Akad. Wiss.* (Berlin, 1908), 471–476; (1909), 514–518.

5. G. Frobenius, Ueber Matrizen aus nicht negativen Elementen, *ibid.* (1912), 456–477.

6. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities* (Cambridge, 1952).

7. P. Hartman and A. Wintner, The spectra of Toeplitz matrices, *Amer. J. Math.* **76** (1954), 867–882.

8. S. Karlin, Positive operators, *J. Math. and Mech.* **8** (1959), 907–937.

9. T. Kato, On the Hilbert matrix, *Proc. Amer. Math. Soc.* **8** (1957), 73–81.

10. T. Kato, On positive eigenvectors of positive infinite matrices, *Comm. Pure and Applied Math.* **11** (1958), 573–586.

11. M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space *Uspehi Mat. Nauk* (N. S.) **3,** No. 1 (23) (1948), 3–95, Amer. Math. Soc. Trans. No. 26.

12. W. Magnus, On the spectrum of Hilbert's matrix, *Amer. J. Math.* **72** (1950), 699–704.

13. O. Perron, Zur Theories der Matrices, *Math. Annalen* **64** (1907), 248–263.

14. C. R. Putnam, On bounded matrices with non-negative elements, *Canadian J. Math.* **10** (1958), 587–591.

15. C. R. Putnam, A note on non-negative matrices, *ibid.* **13** (1961), 59–62.

16. M. A. Rutman, Sur les opérateurs totalement continus linéaires laissant invariant un certain cone, *Mat. Sbornik* (N. S.) **8**, no. 1 (50) (1940), 77–96.

17. M. Rosenblum, On the Hilbert matrix, I, *Proc. Amer. Math. Soc.* **9** (1958), 137–140.

18. O. Taussky, A remark concerning the characteristic roots of the finite segments of the Hilbert matrix, *Quart. J. Math.*, Oxford Ser., vol. 20 (1949), 80–83.

19. O. Taussky, Research Problem 60–3–12, *Bull. Amer. Math. Soc.*

PURDUE UNIVERSITY