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ON THE EXTENSION OF L^2 HOLOMORPHIC FUNCTIONS V

—EFFECTS OF GENERALIZATION

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Abstract. A general extension theorem for L^2 holomorphic bundle-valued top forms is formulated. Although its proof is based on a principle similar to Ohsawa-Takegoshi's extension theorem, it explains previous L^2 extendability results systematically and bridges extension theory and division theory.

Introduction

We would like to continue the study on the extension of holomorphic functions with L^2 growth conditions. In order to clarify the motivation, let us recall some of the earlier results. The study started with the following discovery.

THEOREM 1. (cf. [O-T]) *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, and let Ω' be the intersection of Ω with the complex hyperplane $\{z \in \mathbb{C}^n \mid z_n = 0\}$. Then there exists a constant C which depends only on the diameter of Ω such that, for any plurisubharmonic function φ on Ω, and for any holomorphic function f on Ω' satisfying the condition*

$$\int_{\Omega'} e^{-\varphi(z',0)} |f(z')|^2 d\lambda_{z'} < \infty$$

where $(z', z_n) = (z_1, \ldots, z_n)$ and $d\lambda_{z'}$ denotes the Lebesgue measure, there exists a holomorphic function F on Ω satisfying $F|\Omega' = f$ and

$$\int_{\Omega} e^{-\varphi(z)} |F(z)|^2 d\lambda_z \leq C \int_{\Omega'} e^{-\varphi(z',0)} |f(z')|^2 d\lambda_{z'}.$$

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At first, formulation of similar extension theorems in more general situations has been done in several papers, aiming at a generalization of Theorem 1 for complex manifolds. This was partially done in [O-2, 3], and was more completely done by L. Manivel [M] as an extension theorem for the sections of holomorphic vector bundles on weakly 1-complete manifolds from complex submanifolds of arbitrary codimension. On the other hand, there remained still a demand for refining the extension theorem for domains in $\mathbb{C}^n$. Such a need arose from a question of estimating the growth exponent of the Bergman kernel function. For bounded domains in $\mathbb{C}^n$, we could deduce from Theorem 1 that the Bergman kernel explodes near the boundary, whenever the domain admits a bounded plurisubharmonic exhaustion function (cf. [O-4]). However, for pseudoconvex domains with smooth boundary, Theorem 1 did not even slightly improve an estimate previously given in [D-H-O], whose proof depended on a much more primitive and weaker extension argument in an earlier work [O-1]. The puzzle was settled in [O-5] where an improvement of the L^2 estimate for the $\bar\partial$ operator was found and applied to prove the following.

THEOREM 2. *Let Ω, Ω' and φ be as in Theorem 1. Then, for any plurisubharmonic function ψ on Ω such that $\psi(z)+2\log|z_n|$ is bounded from above, there exists a constant C depending only on $\sup(\psi(z)+2\log|z_n|)$ such that, for any holomorphic function f on Ω' satisfying*

$$\int_{\Omega'} e^{-\varphi(z',0)-\psi(z',0)}|f(z')|^2 d\lambda_{z'} < \infty,$$

there exists a holomorphic function F on Ω satisfying $F|\Omega' = f$ and

$$\int_{\Omega} e^{-\varphi(z)}|F(z)|^2 d\lambda_z \le C \int_{\Omega'} e^{-\varphi(z',0)-\psi(z',0)}|f(z')|^2 d\lambda_{z'}.$$

As for the above mentioned question on the Bergman kernel, from Theorem 2 one can deduce that, if z_0 is a boundary point of a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ whose boundary $\partial\Omega$ is C^∞ smooth and extendable at z_0 in a pseudoconvex way with order ν (cf. [D-H-O]), then the Bergman kernel function K_Ω of Ω satisfies

$$\text{const.} K_\Omega(z) > |z-z_0|^{-2(1+1/\nu)}$$

if $n \ge 2$ and z lies on the inner unit normal to $\partial\Omega$ at z_0. Theorem 2 improves also the regularity of Bonneau-Diederich's integral solution operator for the $\bar\partial$ equation, too (cf. [B-D]).

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Furthermore, the L^2 estimate in the proof of Theorem 2 provided a new insight into a few other classical problems in complex analysis, such as Suita's conjecture on the comparison between the Bergman kernel and logarithmic capacity on Riemann surfaces (cf. [St]), and questions in the theory of interpolation and sampling as developed recently by K. Seip [Sp2]. As for Suita's conjecture, for instance, the following was obtained by the same method as in Theorem 2.

THEOREM 3. (cf. [O-6]) *Let R be a Riemann surface admitting the Green function, and let $K_R(z)|dz|^2$ and $c_R(z)|dz|$ be respectively the Bergman kernel and the logarithmic capacity of R. Then*

$$750\pi K_R(z) > c_R(z)^2.$$

Concerning the interpolation, we gave in [O-7] an alternate proof of the sufficiency part of Seip's theorem in [Sp2], by formulating a generalization of Theorem 2 as an L^2 extension theorem on Stein manifolds.

Unfortunately, this extension theorem was formulated only for the purpose of generalizing Seip's theory to several variables, and did not actually reach the level of a genuinely general theorem. For instance, it is not so easy to see that Theorem 3 follows directly from it, even putting the constant 750π aside. To state the theorem is already laborious.

Therefore it might be desirable to formulate a general extension theorem in such a way that one can immediately deduce all the earlier results on the L^2 extension. The purpose of the present article is to do it as punctually as possible.

Let M be a complex manifold and let (E, h) be a holomorphic Hermitian vector bundle over M. Given a positive measure $d\mu_M$ on M, we shall denote by $A^2(M, E, h, d\mu_M)$ the space of L^2 holomorphic sections of E over M with respect to h and $d\mu_M$. Let S be a closed complex submanifold of M and let $d\mu_S$ be a positive measure on S. The measured submanifold $(S, d\mu_S)$ is said to be a set of interpolation for $(E, h, d\mu_M)$, or for the space $A^2(M, E, h, dV_M)$ if there exists a bounded linear operator I from $A^2(S, ES, h, d\mu_S)$ to $A^2(M, E, h, d\mu_M)$ such that $I(f)|S = f$ for any f. I is called an interpolation operator.

Let $n = \dim M$ and let dV_M be a continuous volume form on M. Then we consider a class of continuous functions Ψ from M to the interval $[-\infty, 0)$ such that

1) $\Psi^{-1}(-\infty) \supset S$

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and

2) If S is k-dimensional around a point x, there exists a local coordinate $(z_1, \ldots, z_n)$ on a neighbourhood U of x such that $z_{k+1} = \ldots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} \left| \Psi(z) - (n-k) \log \sum_{k+1}^{n} |z_j|^2 \right| < \infty.$$

The set of such functions Ψ will be denoted by $\#(S)$. Clearly, the condition 2) does not depend on the choice of the local coordinate. For each $\Psi \in \#(S)$, one can associate a positive measure $dV_M[\Psi]$ on S as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S_k} f d\mu \geq \overline{\lim_{t \to \infty}} \frac{2(n-k)}{\sigma_{2n-2k-1}} \int_M f e^{-\psi} \chi_{R(\Psi,t)} dv_M$$

for any nonnegative continuous function f with $\operatorname{supp} f \Subset M$. Here S_k denotes the k-dimensional component of S, σ_m denotes the volume of the unit sphere in $\mathbb{R}^{m+1}$, and $\chi_{R(\Psi,t)}$ denotes the characteristic function of the set

$$R(\Psi, t) = \{x \in M | -t-1 < \Psi(x) < -t\}.$$

Clearly $d\lambda_z[\log |z_n|^2] = d\lambda_{z'}$ for $z = (z', z_n)$.

Let Θ_h be the curvature form of the fiber metric h. We write $\Theta_h \geq 0$ if Θ_h induces a semipositive quadratic form on $T_M^{1,0} \otimes E$. (E, h) is then said to be semipositive in the sense of Nakano. Let $\Delta_h(S)$ be the set of functions $\tilde{\Psi}$ in $\#(S)$ such that, for any point $x \in M$, $e^{-\tilde{\Psi}} h$ is equal to $e^{-\hat{\Psi}} \hat{h}$ around x for some plurisubharmonic function $\hat{\Psi}$ and some fiber metric $\hat{h}$ whose curvature form is semipositive in the sense of Nakano.

Our goal is to prove

THEOREM 4. *Let M be a complex manifold with a continuous volume form dV_M, let E be a holomorphic vector bundle over M with a C^∞ fiber metric h, let S be a closed complex submanifold of M, let $\Psi \in \#(S)$ and let K_M be the canonical line bundle of M. Then $(S, dV_M[\Psi])$ is a set of interpolation for $(E \otimes K_M, h \otimes (dV_m)^{-1}, dV_M)$ if the following are satisfied.*

1) *There exists a closed subset $X \subset M$ such that*

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a) *X is locally negligible with respect to L^2 holomorphic functions, i.e., for any local coordinate neighbourhood $U \subset M$ and for any L^2 holomorphic function f on $U \setminus X$, there exists a holomorphic function $\tilde{f}$ on U such that $\tilde{f}|U \setminus X = f$.*

b) *$M \setminus X$ is a Stein manifold which intersects with every component of S.*

2) *$\Theta_h \geq 0$ in the sense of Nakano.*

3) *$(1+\delta)\Psi \in \Delta_h(S) \cap C^\infty(M \setminus S)$ for some $\delta > 0$.*

Under these conditions there exist a constant C and an interpolation operator from $A^2(S, E \otimes K_M|S, h \otimes (dV_M)^{-1}|S, dV_M[\Psi])$ to $A^2(M, E \otimes K_M, h \otimes (dV_M)^{-1}, dV_M)$ whose norm does not exceed $C\delta^{-3/2}$. If Ψ is plurisubharmonic, the interpolation operator can be chosen so that its norm is less than $2^4\pi^{1/2}$.

It follows immediately from this, that for any bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ satisfying $\sup_\Omega |z_n| < 1$, there exists an interpolation operator from $A^2(\Omega', \Omega' \times \mathbb{C}, e^{-\Psi(z)}, d\lambda_z[\log |z_n|^2])$ to $A^2(\Omega, \Omega \times \mathbb{C}, e^{-\Psi(z)}, d\lambda_z)$ whose norm is bounded by $2^4\sqrt{\pi}$, provided that Ψ is plurisubharmonic and C^∞. Since any plurisubharmonic function φ can be approximated by C^∞ ones from above on Ω, Theorem 1 is obtained.

Similarly we obtain Theorem 2 by letting

$$\Psi(z) = \psi(z) + \log |z_n|^2 - \sup_\Omega(\psi(z) + \log |z_n|^2),$$

$E = \Omega \times \mathbb{C}$, $h = e^{-\varphi(z)}$ and $dV_M = d\lambda_z$. As for Theorem 3, it suffices to put $M = R$, $S = \{q\}$ and $\Psi(\,.\,) = G(\,.\,, q)$. Here $G(p,q)$ denotes the (negative) Green function of R. Recalling that

$$2 \log c_R(z(p)) = \lim_{q \to p}(G(p,q) - \log |z(p) - z(q)|^2)$$

an inequality $2^8\pi K_R(z) > c_R(z)^2$ is obtained from the equality

$$dV_R[G(\,.\,, q)] = c_R(0)^{-2} dV_R[\log |z|^2]$$

which is valid for any continuous volume form dV_R on R and for any local coordinate z around q.

In Section one we shall prove Theorem 4. In Section two, we shall apply Theorem 4 to prove an improved version of Skoda's L^2 division theorem. The point is that we can replace a partial positivity assumption on the curvature by a genuine semipositivity condition. (See remarks after Theorem 7.) The method of transforming the division problem to the extension problem seems to be of independent interest.

In Section three, we generalize the notion of pluricomplex Green function. After discussing some of its elementary properties, we shall formulate a multidimensional variant of Suita's conjecture. Finally we shall present a new proof of an equality between the upper uniform density and the capacity with respect to the canonical potential function for the complex plane and the unit disc. To the author's belief, the approach given here is more flexible than those in [O-7] and [B-O] and suited for the purpose of generalizing the theory of interpolation and sampling to several variables.

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§1. Proof of Main Theorem

1.1. The tool

Let (M, g) be a complete Kähler manifold of dimension n and let (E, h) be a holomorphic Hermitian vector bundle over M. We denote by $C_0^{p,q}(E)$ the set of compactly supported $C^\infty(p, q)$-forms on M with values in E. Exterior differentiation of type (0,1) is denoted by $\bar{\partial}$. $\bar{\partial}$ is naturally identified with its maximal closed extension to the L^2-completion $L^{p,q}(E)$ of $C_0^{p,q}(E)$ with respect to (g, h). For the proof of main theorem, we need to solve an equaiton of the form $\bar{\partial}(\sqrt{\eta}u) = v$, where u is the unknown and η is a positive C^∞ function. In order to prove the existence of a solution with a proper L^2 norm estimate, we need a variant of Nakano's equality which was first established in [O-T]. Let us first recall it.

Let ∂ be the complex exterior derivative of type $(1, 0)$ and ∂^* its adjoint with respect to g. Identifying h with an element of $C^\infty(M, \mathrm{Hom}(E, \bar{E}^*))$ we put $\partial_h = h^{-1} \circ \partial \circ h$. Let Λ be the adjoint of exterior multiplication by the fundamental form of g. By an abuse of notation, we shall identify differential forms with exterior multiplication by them from the left hand side. In what follows, commutators will always be the graded commutators, i.e. $[S, T] := ST - (-1)^{\deg S \deg T} TS$.

LEMMA 1. *For any $C^\infty(0,1)$ form θ on M,*

$$[\bar\partial, \theta^*] + [\partial^*, \bar\theta] = [-\sqrt{-1}\bar\partial\bar\theta, \Lambda].$$

Proof. Since $\theta^* = -\sqrt{-1}[\bar\theta, \Lambda]$ and $\partial^* = -\sqrt{-1}[\bar\partial, \Lambda]$, we have

$$\begin{aligned}[\bar\partial, \theta^*] + [\partial^*, \bar\theta] &= [\bar\partial, -\sqrt{-1}[\bar\theta, \Lambda]] + [-\sqrt{-1}[\bar\partial, \Lambda], \bar\theta] \\ &= [-\sqrt{-1}[\bar\partial, \bar\theta], \Lambda] = [-\sqrt{-1}\bar\partial\bar\theta, \Lambda].\end{aligned}$$

PROPOSITION 2. *For any positive C^∞ function η on M,*

$$\begin{aligned}&\bar\partial \circ \eta \circ \bar\partial_h^* + \bar\partial_h^* \circ \eta \circ \bar\partial - \partial_h \circ \eta \circ \partial^* - \partial^* \circ \eta \circ \partial_h \\ (\dagger) \qquad &= [-\sqrt{-1}(\eta\Theta_h - \partial\bar\partial\eta), \Lambda] + (\bar\partial\eta) \circ \bar\partial_h^* \\ &\qquad + \bar\partial \circ (\bar\partial\eta)^* + \partial^* \circ (\partial\eta) + (\partial\eta)^* \circ \partial_h.\end{aligned}$$

Here $\bar\partial_h^*$ denotes the adjoint of $\bar\partial$ with respect to (g, h).

Proof. By the Kähler identities we obtain

$$\begin{aligned}&\bar\partial \circ \eta \circ \bar\partial_h^* + \bar\partial_h^* \circ \eta \circ \bar\partial - \partial_h \circ \eta \circ \partial^* - \partial^* \circ \eta \circ \partial_h \\ &= \eta([\bar\partial, \bar\partial_h^*] - [\partial_h, \partial^*]) + (\bar\partial\eta) \circ \bar\partial_h^* - (\bar\partial\eta)^* \circ \bar\partial - (\partial\eta) \circ \partial^* + (\partial\eta)^* \circ \partial_h.\end{aligned}$$

By Lemma 1 we have

$$-(\partial\eta) \circ \partial^* = \partial^* \circ (\partial\eta) - \bar\partial \circ (\bar\partial\eta)^* - (\bar\partial\eta)^* \circ \bar\partial + [\sqrt{-1}\bar\partial\partial\eta, \Lambda].$$

Combining these equalities we obtain the formula.

PROPOSITION 3. *For any $u \in C_0^{n,q}(E)$*

$$\begin{aligned}&\|\sqrt{\eta}\bar\partial_h^* u\|^2 + \|\sqrt{\eta}\bar\partial u\|^2 - \|\sqrt{\eta}\partial^* u\|^2 \\ &= (\sqrt{-1}(\eta\Theta_h - \partial\bar\partial\eta)\Lambda u, u) + 2\mathrm{Re}(\bar\partial\eta \wedge \bar\partial_h^* u, u).\end{aligned}$$

Proof. Applying both sides of ($\dagger$) to u, we take the inner product with u. Since $\partial\eta \wedge u = \partial_h u = 0$ by the type condition for u, we obtain the required formula by Stokes theorem.

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Corollary 4. *For any positive continuous function c on the interval $(0,\infty)$,*

$$\|\sqrt{\eta+c(\eta)}\bar{\partial}_h^* u\|^2 + \|\sqrt{\eta}\bar{\partial} u\|^2$$

$$\geq (\sqrt{-1}(\eta\Theta_h - \partial\bar{\partial}\eta - c(\eta)^{-1}\partial\eta\wedge\bar{\partial}\eta)\Lambda u, u)$$

for any $u \in C_0^{n,q}(E)$.

Combining this inequality with Hahn-Banach's theorem, we obtain the following variant of Kodaira-Nakano's vanishing theorem.

Theorem 5. *Let (M,g) be a complete Kähler manifold of dimension n and let (E,h) be a holomorphic Hermitian vector bundle over M. If $\kappa := \eta\Theta_h - \mathrm{id}_E \otimes (\partial\bar{\partial}\eta + c(\eta)^{-1}\partial\eta\wedge\bar{\partial}\eta)$ is semipositive in the sense of Nakano as a quadratic form along the fibres of $E\otimes T_M^{1,0}$, for a positive C^∞ function η and a positive continuous function c on $(0,\infty)$, then for any $q \geq 1$ and for any $\bar{\partial}$-closed locally square integrable E-valued (n,q)-form v with $((\sqrt{-1}\kappa\Lambda)^{-1}v, v) < \infty$, there exists a $u \in L^{n,q-1}(E)$ satisfying*

$$\bar{\partial}(\sqrt{\eta+c(\eta)}u) = v$$

and

$$\|u\|^2 \leq ((\sqrt{-1}\kappa\Lambda))^{-1}v, v).$$

1.2. The proof

For simplicity we put $A^2(D) = A^2(D, E\otimes K_M, h\otimes(dV_M)^{-1}, dV_M)$ for any open subset $D \subset M$. By the assumption 1, a) the restriction map $A^2(M) \to A^2(M\setminus X)$ is an isometric isomorphism. Therefore, by 1, b), to show the existence of an interpolation operator with the required properties it suffices to prove that, for any relatively compact Stein open subset $D \subset M\setminus X$ there exists a bounded linear operator I_D from $A^2(S,\Psi) := A^2(S, E\otimes K_M|S, h\otimes(dV_M)^{-1}, dV_M[\Psi])$ to $A^2(D)$ such that $I_D(f)|S\cap D = f|S\cap D$ and $\|I_D(f)\| \leq 2^4\sqrt{\pi}\|f\|$ for any $f \in A^2(S,\Psi)$. Because we shall then obtain an interpolation operator

$$I : A^2(S,\Psi) \longrightarrow A^2(M)$$

as a limit. To be more explicit, we fix a complete orthonormal system $\{f_j\}_{j=1}^\infty$ of $A^2(S,\Psi)$ and any increasing family $\{D_k\}_{k=1}^\infty$ of relatively compact Stein open subsets of $M\setminus X$, and consider the sequence $\{I_{D_k}(f_j)\}_{j,k=1}^\infty$.

Choosing a sequence $\{k_\mu\}_{\mu=1}^\infty \subset \mathbb{N}$ by the diagonal argument in such a way that $\{I_{D_{k_\mu}}(f_j)\}_{\mu=1}^\infty$ converges weakly to some $\tilde{f}_j \in A^2(M)$ for each j. By Cauchy's estimate, an interpolation operator I as above is obtained by putting $I(f) := \lim_{\mu\to\infty} I_{D_{k_\mu}}(f)$ for any $f \in A^2(S,\Psi)$.

Once for all we fix a relatively compact Stein open subset D in $M \setminus X$. It suffices to show that, for any $f \in A^2(S,\Psi)$ one can find an $F \in A^2(D)$ satisfying $F|D \cap S = f$ and $\|F\| \le 2^4\sqrt{\pi}\|f\|$.

For that we fix a pair (W, ρ), where W is a Stein neighbourhood of $S \setminus X$ in M and ρ is a holomorphic retraction from W onto $S \setminus X$ which is isotopic to the identity map of W. Such a pair exists in virtue of Siu's theorem (cf. [S]). Clearly the retraction naturally induces a linear map, say I_ρ, from $A^2(S,\Psi)$ to the space of holomorphic sections of $E \otimes K_M$ on a neighbourhood U of $S \cap D$ in $M \setminus X$. (Actually, in virtue of the Oka-Grauert principle, one may take W as U. For the Oka-Grauert principle, see [H-L] for instance.)

By the definition of the measure $dV_M[\Psi]$ one can find for any $f \in A^2(S,\Psi)$ and any $\varepsilon > 0$, a positive number t_0 such that

$$\|e^{-\Psi/2} I_\rho(f)\chi_{R(\Psi,t)}\|^2 \le (\pi + \varepsilon)\|f\|^2$$

holds for all $t > t_0$. Here the L^2 norm on the left hand side is measured on U. Let $\lambda : \mathbb{R} \to \mathbb{R}$ be any C^∞ function satisfying $\lambda(t) = 1$ on $(-\infty, 1), \lambda(t) = 0$ on $(2,\infty)$ and $0 \le \lambda(t) \le 1$ on [1,2]. Then we put

$$v_t = \begin{cases} \bar{\partial}(\lambda(\Psi + t + 2)I_\rho(f)) & \text{on } D \cap U \\ 0 & \text{on } D \setminus U \end{cases}$$

by choosing t_0 in advance so that $\{x \in D | -\infty \le \Psi(x) < -t_0\} \Subset U$.

Since $D \setminus S$ admits a complete Kähler metric (cf. [Gr] or [D-F]), we are allowed to apply Theorem 5 for $D \setminus S$.

For simplicity we shall first prove the result when Ψ is plurisubharmonic.

We put then

$$G_t = \log(e^\Psi + e^{-t}) - 1 - \varepsilon$$

and

$$\eta_t = -G_t + \log(-G_t) + 1$$

for $t > t_0$. Since Ψ is negative η_t is well defined, and replacing t_0 by a larger number if necessary, we may assume that $G_t < -1$.

We have

$$\partial\bar\partial G_t = \frac{e^{\Psi}\partial\bar\partial\Psi}{e^{\Psi}+e^{-t}} + \frac{e^{\Psi-t}\partial\Psi\wedge\bar\partial\Psi}{(e^{\Psi}+e^{-t})^2}$$

$$-\partial\bar\partial\eta_t = (1-G_t^{-1})^2\partial\bar\partial G_t + G_2^{-2}\partial G_t\wedge\bar\partial G_t.$$

Combining these equalities with assumptions 2) and 3) we obtain

$$\begin{aligned}
&\eta_t\Theta_h - \mathrm{Id}_E\otimes\partial\bar\partial\eta_t - \mathrm{Id}_E\otimes(\eta_t^{-3}\partial\eta_t\wedge\bar\partial\eta_t)\\
&\geq (e^{\Psi}+e^{-t})^{-2}e^{\Psi-t}\partial\Psi\wedge\bar\partial\Psi\\
&\qquad +G_t^{-2}\partial G_t\wedge\bar\partial G_t - G_t^{-2}(1-G_t)^{-1}\partial G_t\wedge\bar\partial G_t\\
&\geq (e^{\Psi}+e^{-t})^{-2}e^{\Psi-t}\partial\Psi\wedge\bar\partial\Psi \quad (\mathrm{Id}_E \text{ is omitted for simplicity.})
\end{aligned}$$

Therefore for any complete Kähler metric g on $D\setminus S$,

$$\begin{aligned}
&((\sqrt{-1}(\eta_t\Theta_h - \mathrm{Id}_E\otimes\partial\bar\partial\eta_t - \mathrm{Id}_E\otimes\eta_t^{-3}\partial\eta_t\wedge\bar\partial\eta_t)\Lambda)^{-1}v_t, v_t)_{e^{-\Psi}h}\\
&\leq (((\sqrt{-1}\,\mathrm{Id}_E\otimes(e^{\Psi}+e^{-t})^{-2}e^{\Psi-t}\partial\Psi\wedge\bar\partial\Psi)\Lambda)^{-1}v_t, v_t)_{e^{-\Psi}h}\\
&= \|(e^{\Psi}+e^{-t})e^{t/2}\lambda'(\Psi+t+2)I_\rho(f)\|^2_{e^{-2\Psi}h}\\
&= \int_{D\cap U\setminus S}(e^{\Psi}+e^{-t})^2e^{t-2\Psi}\lambda'(\Psi+t+2)^2h(I_\rho(f))\wedge\overline{I_\rho(f)}\\
&\leq (3+e)\left|\int_{D\setminus S}e^{-\Psi}\lambda'(\Psi+t+2)^2h(I_\rho(f))\wedge\overline{I_\rho(f)}\right|\\
&\leq (3+e)\sup|\lambda'|^2\int_{D\setminus S}|I_\rho(f)|^2e^{-\Psi}\chi_{R(\Psi,t)}dV_M\\
&\leq (3+e)(\pi+\varepsilon)\sup|\lambda'|^2\|f\|^2
\end{aligned}$$

if $t>t_0$. (Note that $2m/\sigma_{2m-1}\geq 1/\pi$ for all $m\in\mathbb{N}$.)

Hence by Theorem 5 there exists a $w_t\in L^{n,0}(E)_{e^{-\Psi}h}$ satisfying

$$\bar\partial(\sqrt{\eta_t+\eta_t^3}w_t) = v_t$$

and

$$\|w_t\|^2_{e^{-\Psi}h}\leq(3+e)(\pi+\varepsilon)\sup|\lambda'|^2\|f\|^2.$$

Note that

$$\|\sqrt{\eta_t + \eta_t^3} w_t\|_h^2 \leq (C_0 + \varepsilon)\|w_t\|_{e^{-\Psi}h}^2$$

holds for sufficiently large t, if we put

$$C_0 = \lim_{t\to\infty} \sup_D (\eta_t + \eta_t^3)e^{\Psi}.$$

Since

$$\begin{aligned} C_0 &\leq \sup_{s>0}\{(2 + s + \log(1+s)) + (2 + s + \log(1+s))^3\}e^{-s} \\ &< \sup_{s>0}\{2(1+s) + 8(1+s)^3\}e^{-s} < 2 + 6^3e^{-2} < 2^5, \end{aligned}$$

choosing $\varepsilon \ll 1$ and $t \gg 1$, we obtain an extension

$$F = \lambda(\Psi + t + 2)I_\rho(f) - \sqrt{\eta_t + \eta_t^3} w_t$$

with the desired estimate.

If Ψ is not plurisubharmonic, we replace η_t by $\eta_t + \delta^{-1}$ in the above argument and obtain an interpolation operator similarly, but with a worse estimate for the norm.

Remark 1. Condition 1) of the main theorem is satisfied if M is pseudoconvex (i.e. M carries a continuous plurisubharmonic exhaustion function) and holomorphically embeddable into a complex projective space. In fact one may take as X a generic hyperplane section. Concerning a criterion for a pseudoconvex manifold to be embeddable into a projective space, see [T].

Remark 2. For many purposes it is desirable to remove the regularity assumption for Ψ in condition 3). This is possible if rank $E = 1$. Because, for any singular fiber metric h of a holomorphic line bundle over a Stein manifold D, and for any strictly plurisubharmonic function Ψ on D, such that h is locally of the form $e^{-\varphi}$ for some plurisubharmonic function φ, there exists an increasing family of C^∞ fiber metrics h_ν converging to h such that

$$\Theta_{h_\nu} \geq -\partial\bar{\partial}\psi$$

for all ν. Thus, if rank $E = 1$, conditions 2) and 3) can be replaced respectively by

2′) $\quad i\Theta_h$ is a positive current and

3′) $\quad \Psi \in \Delta_h(S)$.

Here the definition of $\#_h(S)$ for the singular fiber metric h is done in an obvious way.

§2. Extension and division

Let $g : E \to Q$ be a surjective morphism between holomorphic vector bundles, of rank p and q, respectively, over a complex manifold N of dimension n, and let L be a holomorphic line bundle over N. Let h and b be respectively the C^∞ fiber metrics of E and L, and let a be the fiber metric of Q induced from h. In [Sk-3], H. Skoda proved the following.

Theorem 6. *Assume that N admits a Kähler metric and a plurisubharmonic exhaution function of class C^2, (E, h) is semi-positive in the sense of Griffiths, and $\Theta_b - \Theta_{\det h} - k\Theta_{\det a} \geq 0$ for some $k > \inf(n, p-q)$. Then the morphism*

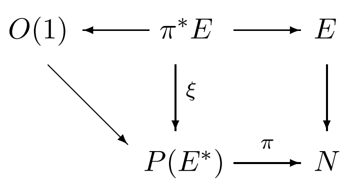
$$g_* : H^0(N, E \otimes K_N \otimes L) \longrightarrow H^0(N, Q \otimes K_N \otimes L)$$

is surjective.

We shall show that the following is a straightforward consequence of

Theorem 7. *Under the hypothesis of Theorem 6, assume moreover that N is holomorphically embeddable into a complex projective space. Then the morphism $g_* : H^0(N, E \otimes K_N \otimes L) \to H^0(N, Q \otimes K_N \otimes L)$ is surjective.*

Proof. Let $P(E^*) \to N$ be the projectification of the dual bundle $E^* \to N$. Then there exists a diagram



Here the morphism $\pi^* E \to O(1)$ associates $v \in \pi^* E$ the class $v + \operatorname{Ker} \xi(v)$ One has then $H^0(P(E^*), O(1)) \simeq H^0(N, E)$. To show that the morphism g_* is surjective, it suffices to show that the restriction morphism

$$H^0(P(E^*), O(1) \otimes \pi^*(K_N \otimes L)) \longrightarrow H^0(P(Q^*), O(1) \otimes \pi^*(K_N \otimes L)|P(Q^*))$$

induced by the injective morphism $G^* : Q^* \to E^*$ is surjective.

For $M = P(E^*)$ and $S = P(Q^*)$, condition 1) of is satisfied since we assume that N is embeddable into a complex projective space. As for 2), one can see from

$$\begin{aligned} &O(1) \otimes \pi^*(K_N \otimes L) \\ \simeq\ & K_{P(E^*)} \otimes K^*_{P(E^*)} \otimes O(1) \otimes \pi^*(K_N \otimes L) \\ \simeq\ & K_{P(E^*)} \otimes O(p) \otimes \pi^*(\det E^* \otimes K_N^*) \otimes O(1) \otimes \pi^*(K_N \otimes L) \\ \simeq\ & K_{P(E^*)} \otimes O(p+1) \otimes \pi^*(\det E^* \otimes L) \end{aligned}$$

that 2) is satisfied by the metrized line bundle $(O(p+1)\otimes\pi^*(\det E^*\otimes L), h_0)$ with respect to the induced metric h_0, since

$$\Theta_b - \Theta_{\det h} \geq k\Theta_{\det a} \geq 0$$

by assumption.

Therefore, in view of 3′), the conclusion of Theorem 7 will follow if $\Delta_{h_0 \exp(-\varphi)}(P(Q^*)) \neq \emptyset$ for some continuous plurisubharmonic exhaustion function φ on $P(E^*)$.

To proceed, let us assume first that $p - q = 1$ for the sake of simplicity. Then there is a canonical isomorphism between $\pi^*(\operatorname{Ker} g)^* \otimes O(1)$ and the line bundle $[P(Q^*)]$ associated to the divisor $P(Q^*) \subset P(E^*)$. Hence $\Delta_{e^{-\varphi}h_0}(P(Q^*))$ is nonempty for some φ if and only if $O(p) \otimes \pi^*(\det E^* \otimes L \otimes \operatorname{Ker} g)$ is semipositive. Since the curvature form of $\operatorname{Ker} g$ is given by

$$\Theta_h | \operatorname{Ker} g + A^* \wedge A$$

in terms of Θ_h and a $\operatorname{Hom}(\operatorname{Ker} g, Q)$-valued $(1,0)$ form A, where $A^* := \sum_j A_j^* d\bar{z}_j$, if $A = \sum_j A_j dz_j$, $\det Q \otimes \operatorname{Ker} g$ is semipositive if

$$(*) \qquad \operatorname{Tr} A \wedge A^* + A^* \wedge A \geq 0.$$

But this is an obvious inequality. Hence we are done if $p - q = 1$.

If $p - q > 1$, by appealing to Skoda's lemma ("Lemma fondamental" in [Sk-2]) instead of $(*)$, we obtain the Nakano semipositivity of $(\det Q)^k \otimes \operatorname{Ker} g$ for any integer $k \geq \inf(n, p-q)$. Therefore, by a similar argument as above, after reducing the codimension of $P(Q^*)$ to one by blowing up, we apply Theorem 4. The detail in routine and may well be left to the reader.

Remarks. We note that the above proof shows that the assumption

$$\Theta_b - \Theta_{\det h} - k\Theta_{\det a} \geq 0$$

for some $k > \inf(n, p-q)$, can be weakened to

$$\Theta_b - \Theta_{\det h} - \inf(n, p-q)\Theta_{\det a} \geq 0$$

if $p-q=1$ and $[P(Q^*)]\otimes O(1)$ is semipositive. Moreover, the extendability of sections needed in the above proof is still valid even if we drop the projective embeddability of N. In fact, we could have formulated an extension theorem which has literally Theorem 6 as a corollary. This minor modification may well be left to the reader as an exercise.

There exist other variants of division theorem in [Sk-1, 2, 3] and [D]. It seems that not all of them are corollaries of the corresponding extension theorems.

Lastly, to show another advantage of our approach, we formulate below an analogue of the solution of mixed boundary value problems in real analysis.

THEOREM 8. *Let $g : E \to Q$ be a surjective morphism between holomorphic vector bundles over a pseudoconvex manifold M, let L be a holomorphic line bundle over M, and let S be a closed complex submanifold of M. Assume that M is holomorphically embeddable into a projective space,*[1] *that E and L admit fiber metrics satisfying the conditions of Theorem 6, and that, with respect to the fiber metric h of E, there exists a continuous plurisubharmonic exhaustion function φ of M such that $\Delta_{e^{-\varphi}h \det h}(S) \neq \emptyset$. Then the morphism*

$$\begin{aligned} &H^0(M, E\otimes K_M \otimes L) \\ &\qquad \longrightarrow H^0(S, E\otimes K_M\otimes L) \times_{H^0(S, Q\otimes K_M \otimes L)} H^0(M, Q\otimes K_M\otimes L), \end{aligned}$$

induced by the restriction and g, is surjective.

Sketch of Proof. Main theorem is to be applied to $P(E^*) \setminus (P(Q^*) \cap \pi^{-1}(S))$ and its submanifold $P(Q^*) \cup \pi^{-1}(S) \setminus (P(Q^*) \cap \pi^{-1}(S))$. The rest is similar as in Theorem 7.

[1]This can be replaced by the existence of a Kähelr metric on M.

§3. Generalized pluricomplex Green function

Let Ω be a domain in $\mathbb{C}^n$ and let $w \in \Omega$. If $\Delta_1(\{w\})$ $(= \Delta_1(\{w\}, \Omega))$ is nonempty, we put

$$G(z, w) = \sup\{u(z) : u \in \Delta_1(\{w\})\}.$$

The function $g = G/2n$ is called the pluricomplex Green function with pole at w. Based on this well known notion we introduce the following new terminology.

Definition. Let M be a complex manifold, let (E, h) be a holomorphic Hermitian vector bundle over M, and let S be a closed complex submanifold of M. The generalized pluricomplex Green function of (E, h) with poles along S is

$$G_h(z, S) = \sup\{u(z) : u \in \Delta_h(S)\}$$

if $\Delta_h(S) \neq \emptyset$, and $G_h(z, S) = -\infty$ if $\Delta_h(S) = \emptyset$.

When $E = \mathbb{C} \times M$ and $h = 1$, $G_h(z, S)$ will be denoted by $G(z, S)$.

Proposition 9. *If $\psi \in \Delta_h(S)$, then $G_h(z, S) - \psi(z)$ is locally bounded on M.*

Proof. That $G_h - \psi \geq 0$ follows from the definition. To see that $G_h - \psi$ is bounded from above, we take a local coordinate z around x such that

$$S \cap U = \{z | z' := (z_1, \dots, z_k) = 0\}$$

and $u(z) - k \log \|z'\|^2$ is bounded on $U = \{z \mid \max |z_j| < 1\}$ for any $u \in \Delta_h(S)$, where $k = \operatorname{codim}_x S$.

Let $U' = \{z \in U \mid \|z'\| > 1/2\}$, and let A be a positive number such that $-\Theta_h + A \operatorname{Id}_E \otimes \partial\bar{\partial}\|z\|^2 \geq 0$ on U. Then, for each $z = (z', z'') \in U'$, the function $A\|z\|^2 + \psi(z) - k \log \|z'\|^2$ is subharmonic on the disc $\Delta^z := \{(\lambda z', z'') \mid |\lambda| < 1\}$. Since $u < 0$ we have

$$\sup_U (u(z) - k \log \|z'\|^2) \leq -k \log 4 + nA$$

by the maximum principle for subharmonic functions. Hence

$$\sup_U (G_h(z, S) - k \log \|z'\|^2) \leq -k \log 4 + nA,$$

so that $G_h - \psi$ is also bounded from above for any $\psi \in \Delta_h(S)$.

PROPOSITION 10. *If* rank $E = 1$, *then* $G_h(z, S)$ *is continuous as a function with values in* $[-\infty, 0)$.

Proof. If rank $E = 1$ and ψ_1, $\psi_2 \in \Delta_h(S)$, then $\max\{\psi_1, \psi_2\}$ also belongs to $\Delta_h(S)$ by a basic property of plurisubharmonic functions. Therefore $G_h(\,.\,, S)$ is, in this case, the limit of an increasing sequence of continuous functions. Hence $G_h(\,.\,, S)$ is lower semicontinuous. Since the upper envelope of $G_h(\,.\,, S)$ is then continuous, it will belong to $\Delta_h(S)$. By the definition of G_h, G_h is thus equal to its upper envelope. Therefore G_h itself must be continuous.

Let dV_M be any continuous volume form on M and let $\{\sigma_j\}_{j=1}^{\infty}$ (resp. $\{\tau_j\}_{j=1}^{\infty}$) be a complete orthonormal system of $A^2(M, K_M, dV_M^{-1}, dV_M)$ (resp. $A^2(S, K_M \otimes S, dV_M^{-1}, dV_M[G(\,.\,, S)])$) and put

$$\kappa_M = \sum_{j=1}^{\infty} \sigma_j \otimes \bar{\sigma}_j \in C^{\omega}(M, K_M \otimes \bar{K}_M)$$
$$(\text{resp. } \kappa_{M/S} = \sum_{j=1}^{\infty} \tau_j \otimes \bar{\tau}_j \in C^{\omega}(S, K_M \otimes \bar{K}_M)).$$

Clearly, κ_M and $\kappa_{M/S}$ do not depend on the choice of dV_M.

CONJECTURE. $(\pi^k/k!)\kappa_M(x) \geq \kappa_{M/S}(x)$ *for any* $x \in S$, *if* M *is pseudoconvex. Here* $k = \operatorname{codim}_x S$.

If $\dim M = 1$ and $S = \{x\}$, we have

$$\kappa_{M/S}(x) = e^{\gamma} dz \otimes d\bar{z}|_x$$

where $\gamma = \lim_{y \to x}(G(y, x) - \log |z(y) - z(x)|^2)$. Hence the above conjecture is an extension of (still unsolved) Suita's conjecture for the Bergman kernel of Riemann surfaces.

By the main theorem we have $2^8 \pi \kappa_M(x) \geq \kappa_{M/S}(x)$ if M satisfies the condition 1). It is known that the conjecture is true if M is the unit open ball in $\mathbb{C}^n$ and S is a point of M, or if M is an annulus and S is a point. There is a variational approach to this problem (see [St] and [G-K]).

§4. Density and capacity

4.1. Notions of density and capacity

Definition. A subset $\Gamma \subset \mathbb{C}$ is said to be uniformly discrete if $\inf\{|z-w| \mid z, w \in \Gamma, z \neq w\} > 0$.

For any uniformly discrete subset Γ of $\mathbb{C}$ we put

$$D^+(\Gamma) = \limsup_{r\to\infty} \sup_w \frac{\#\{z \in \Gamma \mid |z-w| < r\}}{\pi r^2}$$

$D^+(\Gamma)$ is called the upper uniform density of Γ.

For the simplicity of notation we put

$$A^2_\alpha = A^2(\mathbb{C}, \mathbb{C}\times\mathbb{C}, e^{-\alpha|z|^2}, d\lambda_z).$$

Theorem 11. *Let Γ be a uniformly discrete subset of $\mathbb{C}$ and let δ_Γ be the Dirac mass supported on Γ. Then, (Γ, δ_Γ) is a set of interpolation for A^2_α if and only if $\alpha > \pi D^+(\Gamma)$.*

For the proof the reader is referred to [Sp-W] and [Sp-1].

For the unit disc $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ we put

$$A^2_{\alpha,\Delta} = A^2(\Delta, \mathbb{C}\times\Delta, (1-|z|^2)^\alpha, d\lambda_z).$$

Definition. A subset $\Gamma \subset \Delta$ is said to be uniformly discrete if

$$\inf\left\{\left|\frac{z-w}{z\bar{w}-1}\right| \,\middle|\, z, w \in \Gamma, z \neq w\right\} > 0.$$

we put $\rho(z,w) = |z - w/z\bar{w} - 1|$ and

$$D^+_\Delta(\Gamma) = \limsup_{r\to 1} \sup_z \frac{\sum_{\xi\in\Gamma} \frac{1}{2} < \rho(z,\xi) < r^{\log \rho(z,\xi)}}{\log(1-r)}.$$

Theorem 12. *Let Γ be a uniformly discrete subset of Δ. Then $(\Gamma, (1-|z|^2)\delta_\Gamma)$ is a set of interpolation for $A^2_{\alpha,\Delta}$ if and only if $\alpha > 2D^+(\Gamma)$.*

For the proof, see [Sp-2].

Definition. For any discrete subset $\Gamma \subset \mathbb{C}$ we put

$$C^+(\Gamma) = \inf\{\alpha | \Delta_{e^{-\alpha|z|^2}}(\Gamma) \neq \emptyset\}.$$

$C^+(\Gamma)$ will be called the canonical capacity of Γ. The canonical capacity of a discrete subset of Δ is defined by

$$C^+_\Delta(\Gamma) = \inf\{\alpha | \Delta_{(1-|z|^2)^\alpha}(\Gamma) \neq \emptyset\}.$$

4.2. Equivalence of the notions

We shall deduce from the main theorem, Theorem 11 and Theorem 12, that the above mentioned notions of density and capacity coincide.

THEOREM 13. *For any uniformly discrete subset* $\Gamma \subset \mathbb{C}$ (*resp.* $\Gamma \subset \Delta$), *one has* $\pi D^+(\Gamma) = C^+(\Gamma)$ (*resp.* $2D^+(\Gamma) = C^+_\Delta(\Gamma)$).

Proof. We first prove the inequality $C^+(\Gamma) = \pi D^+(\Gamma)$ by showing that $\Delta_{e^{-\alpha|z|^2}}(\Gamma) \neq \emptyset$ for any $\alpha > \pi D^+(\Gamma)$.

Let χ_ε be a C^∞ function on $\mathbb{R}$ such that

$$\begin{aligned} \chi_\varepsilon(t) &= 1 \qquad \text{for} \quad t \leq \varepsilon \\ \chi_\varepsilon(t) &= 0 \qquad \text{for} \quad t > 1+\varepsilon \\ \chi'_\varepsilon(t) &= -1 \qquad \text{for} \quad 2\varepsilon < t < 1 \\ \chi''_\varepsilon(t) &\leq 0 \qquad \text{for} \quad t \leq 1/2 \end{aligned}$$

and

$$\chi''_\varepsilon(t) \leq 2/\varepsilon \qquad \text{for} \quad t > 1.$$

Here $\varepsilon \in (0, 1/100)$. Then we put

$$\psi^\varepsilon_R(z) = \sum_{\xi \in \Gamma_\alpha} \chi_\varepsilon\left(\frac{|z-\xi|^2}{R^2}\right) \log\left|\frac{z-\xi}{(1+\varepsilon)R}\right|^2 - 1.$$

Here $\Gamma_\alpha = \Gamma \cup \frac{1}{\alpha}(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$. It is clear that $\psi^\varepsilon_R \in \#(\Gamma_\alpha)$. Furthermore

$$\psi^\varepsilon_R(z) = \sum_{\xi \in \Gamma_\alpha} \left(A_\xi \log\left|\frac{z-\xi}{(1+\varepsilon)R}\right|^2 + B_\xi\right) dz \wedge d\bar{z},$$

where

$$A_\xi = \frac{|z-\xi|^2}{R^4}\chi''_\varepsilon\left(\frac{|z-\xi|^2}{R^2}\right) + \frac{1}{R^2}\chi'_\varepsilon\left(\frac{|z-\xi|^2}{R^2}\right)$$

and

$$B_\xi = \frac{2}{R^2}\chi'_\varepsilon\left(\frac{|z-\xi|^2}{R^2}\right).$$

We have

$$\sum_{\xi \in \Gamma_\alpha} \frac{|z-\xi|^2}{R^4}\chi''_\varepsilon\left(\frac{|z-\xi|^2}{R^2}\right) \log\left|\frac{z-\xi}{(1+\varepsilon)R}\right|^2$$

$$\geq - \sum_{\substack{\xi\in\Gamma_\alpha \\ R\leq|z-\xi|\leq(1+\varepsilon)R}} \frac{|z-\xi|^2}{R^4}\frac{2}{\varepsilon}\log(1+\varepsilon)^2$$

$$\geq -8\varepsilon\alpha$$

for $R \gg 1$. On the other hand, for any $\delta > 0$,

$$\sum_{\xi\in\Gamma_\alpha} \frac{1}{R^2}\chi'\left(\frac{|z-\xi|^2}{R^2}\right)\log\left|\frac{z-\xi}{(1+\varepsilon)R}\right|^2 \geq \alpha - \frac{\delta}{2}$$

if we choose $\varepsilon \ll 1$ and $R \gg 1/\varepsilon$, because $\int_0^1 |\log t|dt = 1$. Clearly $\sum_{\xi\in\Gamma_\alpha} B_\xi \geq -2\alpha - \delta/2$ if $\varepsilon \ll 1$ and $R \gg 1/\varepsilon$. Hence we have

$$\psi_R^\varepsilon(z) \in \Delta_{e^{-(\alpha+\delta)|z|^2}}(\Gamma_\alpha)$$

if $R \gg 1/\varepsilon \gg 1$. Hence $\Delta_{e^{-\alpha'|z|^2}}(\Gamma_\alpha) \neq \emptyset$ for any $\alpha' > \alpha > \pi D^+(\Gamma_\alpha)$. Since

$$\Delta_{e^{-\alpha'|z|^2}}(\Gamma) \supset \Delta_{e^{-\alpha'|z|^2}}(\Gamma_\alpha)$$

we have $\Delta_{e^{-\alpha'|z|^2}}(\Gamma) \neq \emptyset$ for any $\alpha > \pi D^+(\Gamma)$. Hence $\pi D^+(\Gamma) \geq C^+(\Gamma)$.

On the other hand, let $\alpha > C^+(\Gamma)$. Then, by Hörmander's method of solving the $\bar\partial$ equation, it is easily seen that there exists a constant C such that for any $\xi \in \Gamma$ there exists a holomorphic function $f_\xi \in A^2_\alpha$ satisfying $f_\xi|\Gamma = 0$, $f'_\xi(\xi) = 1$ and $|f_\xi(z)|^2 e^{-\alpha|z|^2} \leq C$ everywhere on $\mathbb{C}$. This obviously means that the generalized Green function $G_\alpha = G_{e^{-\alpha|z|^2}}(\,.\,,\Gamma)$ satisfies that $d\lambda_z[G_\alpha] \leq \frac{e^{-\alpha|z|^2}}{C}\delta_\Gamma$.

Hence (Γ, δ_Γ) is a set of interpolation for A^2_α by the main theorem. Therefore, by Theorem 11 one has $\alpha > \pi D^+(\Gamma)$. This proves the inequality $C^+(\Gamma) \geq \pi D^+(\Gamma)$.

Proof of the equality $2D^+(\Gamma) = C^+(\Gamma)$ is similar. Namely, to prove that $2D^+(\Gamma) \geq C^+(\Gamma)$ we put

$$\psi_r^\varepsilon(z) = \sum_{\xi\in\Gamma_*} \chi_\varepsilon(\log(1-\rho(z,\xi)^2)/\log(1-r^2))\log\rho(z,\xi)^2.$$

Here Γ_* is the union of Γ and a noneuclidean lattice. We shall leave the detail to the reader. The converse inequality follows from the Hörmander's L^2 theorem and Theorem 12 in a similar way as above.

Remark. It seems to be a fruitful task to generalize the above equalities between densities and capacities to higher dimensions.

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References

[B-D] P. Bonneau and K. Diederich, *Integral solution operators for the Cauchy-Riemann equations on pseudoconvex domains*, Math. Ann., **286** (1990), 77–100.

[B-O] B. Berndtsson and J.M. Ortega, *On interpolation and sampling in Hilbert space of analytic functions*, J. Reine. Angew. Math., **464** (1995), 109–128.

[D] J.-P. Demailly, *Scindage holomorphe d'un morphisme de fibrés vectoriells semi-positifs avec estimations L^2*, Séminaire Pierre Lelong-Henri Skoda (Analysis), 1980/1981, and Colloquium at Wimereux, May 1981, pp.77–107. LNM, **919** Springer, Berlin-New York, 1982.

[D-F] K. Diederich and J.E. Fornaess, *On the nature of thin complements of complete Kähler metrics*, Math. Ann., **264** (1984), 475–495.

[D-H-O] K. Diederich, G. Herbort and T. Ohsawa, *The Bergman kernel on uniformly extendable pseudoconvex domains*, Math. Ann., **273** (1986), 371–384.

[G] H. Grauert, *Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik*, Math. Ann., **131** (1956), 38–75.

[G-K] R.E. Greene and S. Krantz, *Deformation of complex structures, estimates for the equation, and stability of the Bergman kernel*, Adv. in Math., **43** (1982), 1–86.

[H-L] G. Henkin and J. Leiterer, *The Oka-Grauert principle without induction over the base dimension*, Math. Ann., **311** (1998), 71–93.

[M] L. Manivel, *Un théorème de prolongement L^2 des sections holomorphes d'un fibré hermitein*, Math. Z., **212** (1993), 107–122.

[O-1] T. Ohsawa, *On complete Kähler domains with C^1 boundary*, Publ.. RIMS, Kyoto Univ., **15** (1980), 929–940.

[O-2] ———, *On the extenson of L^2 holomorphic functions II*, Publ. RIMS, Kyoto Univ., **24** (1988), 265–275.

[O-3] ———, The existence of right inverses of residue homomorphisms, Complex Analysis and Geometry, Plenum Press, New York, 1993, pp. 285–291.

[O-4] ———, *On the Bergman kernel of hyperconvex domains*, Nagoya Math. J., **129** (1993), 43–52.

[O-5] ———, *On the extension of L^2 holomorphic functions III: negligible weights*, Math. Z., **219** (1995), 215–225.

[O-6] ———, *Addendum to "On the Bergman kernel of hyperconvex domians"*, Nagoya Math. J., **137** (1995), 145–148.

[O-7] ———, *On the extension of L^2 holomorphic functions IV: a new density concept*, Geometry and Analysis on Complex Manifolds, Festschrift for Prof. S. Kobayashi, World Sci. 1994, pp. 157–170.

[O-T] T. Ohsawa and K. Takegoshi, *On the extension of L^2 holomorphic funcitons*, Math. Z., **195** (1987), 197–204.

[Sp-1] K. Seip, *Density theorems for sampling and interpolation in the Bargmann-Fock space I*, J. Reine Angew. Math., **429** (1992), 91–106.

[Sp-2] ———, *Beurling type density theorems in the unit disc*, Invent. Math., **113** (1993), 21–39.

[Sp-W] K. Seip and R. Wallstén, *Density theorems for sampling and interpolation in the Bargmann-Fock space II*, J. Reine Angew. Math., **429** (1992), 107–113.

[S] Y.T. Siu, *Every Stein subvariety admits a Stein neighbourhood*, Invent. Math., **38** (1976/77), 89–100.

[Sk-1] H. Skoda, *Application des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids*, Ann. Sci. Ec. Norm. Sup., **5** (1972), 545–579.

[Sk-2] ———, *Morphismes surjectifs de fibrés vectoriels semi-positifs*, Ann. Sci. Ecole Norm. Sup., **11** (1978), 577–611.

[Sk-3] ———, *Relèvement des sections globales dans les fibrés semi-positifs*, Séminaire Pierre Lelong-Henri Skoda (Analyse), Années 1978/79 LNM 822, Springer, Berlin 1980 pp.259–301.

[St] N. Suita, *Capacities and kernels on Riemann surfaces*, Arch. Rational Mech. Anal., **46** (1972), 212–217.

[T] S. Takayama, *Adjoint linear series on weakly 1-complete Kähler manifolds I: Global projective embedding*, Math. Ann., **311** (1998), 501–531.

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