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GENERALIZED CESÀRO MATRICES

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ABSTRACT. For $\alpha \in [0, 1]$ the operator A_α^* is the operator formally defined on the Hardy space H^2 by

$$(A_\alpha^* f)(z) = (z-\alpha)^{-1} \int_\alpha^z f(s)\, ds, \qquad |z|<1.$$

If $\alpha = 1$, then the usual identification of H^2 with l^2 takes A_1 onto the discrete Cesàro operator. Here we see that $\{A_\alpha : \alpha \in [0, 1]\}$ is not arcwise connected, that $\operatorname{Re} A_\alpha \geq 0$, that A_α is a Hilbert–Schmidt operator if $\alpha \in [0, 1)$, and that A_α is neither normaloid nor spectraloid if $\alpha \in (0, 1)$.

The generalized Cesàro matrices

$$A_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{\alpha}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{\alpha^2}{3} & \frac{\alpha}{3} & \frac{1}{3} & 0 & \cdots \\ \frac{\alpha^3}{4} & \frac{\alpha^2}{4} & \frac{\alpha}{4} & \frac{1}{4} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix},$$

with $\alpha \in [0, 1]$, were introduced in [6], where it was shown that, as operators on l^2, they are bounded; it was shown, furthermore, that if $0 \leq \alpha < 1$, then A_α is compact and has spectrum $\sigma(A_\alpha) = \{1/n\}_{n=1}^{\infty} \cup \{0\}$. The computation

$$\sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \left(\frac{\alpha^{j-1}}{j+1}\right)^2 = \sum_{m=0}^{\infty} \alpha^{2m} \sum_{n=m+1}^{\infty} \frac{1}{n^2} \leq \frac{1}{6} \pi^2 (1-\alpha^2)^{-1} < \infty$$

proves the following slightly stronger result.

THEOREM 1. *A_α, $\alpha \in [0, 1)$, is a Hilbert–Schmidt operator on l^2 with*

$$\|A_\alpha\|_2^2 = \sum_{m=0}^{\infty} \alpha^{2m} \left(\sum_{n=m+1}^{\infty} \frac{1}{n^2} \right),$$

where $\| \ \|_2$ denotes the Hilbert–Schmidt norm [4, pp. 17–20].

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The next theorem implies that the collection $\{A_\alpha: 0 \le \alpha \le 1\}$ is not arcwise connected.

THEOREM 2. *The assignment $\alpha \to A_\alpha$ is continuous (with respect to the topology induced by the operator norm) on $[0, 1)$ but fails to be continuous at 1.*

Proof. If this assignment were continuous from the left at 1, then for any sequence $\{\beta_n\}_{n=1}^\infty \subseteq [0, 1)$ increasing to 1 it would be true that $\|A_{\beta_n} - A_1\| \to 0$ as $n \to \infty$. This would say that A_1, the norm limit of a sequence of compact operators, is compact; A_1 cannot be compact, however, since $\sigma(A_1) = \{\lambda : |1-\lambda| \le 1\}$ [1, p. 130]. It remains to show that if $\alpha \in [0, 1)$, then the assignment $\alpha \to A_\alpha$ is continuous at α. Fix $\lambda \in (\alpha, 1)$. We see that

$$A_\beta - A_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{\beta-\alpha}{2} & 0 & 0 & 0 & \cdots \\ \frac{\beta^2-\alpha^2}{3} & \frac{\beta-\alpha}{3} & 0 & 0 & \cdots \\ \frac{\beta^3-\alpha^3}{4} & \frac{\beta^2-\alpha^2}{4} & \frac{\beta-\alpha}{4} & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix},$$

and hence for $\beta < \lambda$ we have $\|A_\beta - A_\alpha\| \le |\beta - \alpha| \, \|T_\lambda\|$, where T_λ is the following Toeplitz matrix:

$$T_\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ \lambda & 1 & 0 & 0 & \cdots \\ \lambda^2 & \lambda & 1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

If $b_n = \lambda^{n-1}$ for $n = 1, 2, 3, \ldots$, and $b_n = 0$ for $n = 0, -1, -2, \ldots$, then $\sum_n |b_n|^2 = (1-\lambda^2)^{-1} < \infty$, so the b_n's are Fourier coefficients of a function ϕ in $L^2(0, 1)$; the function ϕ is given by

$$\phi(x) = \sum_{n=1}^{\infty} \lambda^{n-1} e^{2\pi i n x} = e^{2\pi i x}(1 - \lambda e^{2\pi i x})^{-1}.$$

Since ϕ is bounded (with $|\phi(x)| \le (1-\lambda)^{-1}$ for all x), the matrix T_λ is bounded [4, p. 24]. Suppose $\varepsilon > 0$; choose $\delta = \min\{\lambda - \alpha, \ \varepsilon \, \|T_\lambda\|^{-1}\}$. If $\beta \in (\alpha - \delta, \alpha + \delta) \cap (0, 1)$, then $\|A_\beta - A_\alpha\| < \varepsilon$, and the proof is complete.

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REMARK. The following proposition provides an alternate way of showing that the assignment $\alpha \to A_\alpha$ is not continuous at 1.

PROPOSITION 1. $\|A_1 - A_\alpha\| = 2$ *for all* $\alpha \in [0, 1)$.

Proof. It is clear that $\|A_1 - A_\alpha\| \le \|A_1\| = 2$ [1, p. 130]. Since $A_1 - (A_1 - A_\alpha) = A_\alpha$ is compact and A_1 has no eigenvalues, it follows from [3, Problem 143] that if $\lambda \in \sigma(A_1)$, then $\lambda \in \sigma(A_1 - A_\alpha)$; therefore $\|A_1 - A_\alpha\| \ge r(A_1 - A_\alpha) \ge r(A_1) = 2$, where $r(\cdot)$ denotes spectral radius; this completes the proof.

PROPOSITION 2. $\|\text{Im } A_\alpha\| \le 2\alpha$.

Proof. Take $B_\alpha \equiv A_\alpha - A_0$. It was shown in [6] that $\|B_\alpha\| \le 2\alpha$. Since $\text{Im } A_\alpha = (2i)^{-1}(A_\alpha - A_\alpha^*) = (2i)^{-1}(B_\alpha - B_\alpha^*)$, it is easy to see that $\|\text{Im } A_\alpha\| \leqq \frac{1}{2}(\|B_\alpha\| + \|B_\alpha^*\|) \le 2\alpha$.

The *numerical range* $W(A)$ of the operator A is defined to be the set $\{\langle Af, f\rangle : \|f\| = 1\}$; the *numerical radius* $\omega(A)$ of A is the number sup $\{|\lambda| : \lambda \in W(A)\}$. It is a consequence of the preceding proposition that $W(A_\alpha) \subseteq \{\lambda : |\text{Im } \lambda| \le 2\alpha\}$; the next theorem implies that $W(A_\alpha)$ is a subset of the right half-plane $\{\lambda : \text{Re } \lambda \ge 0\}$.

THEOREM 3. $\text{Re } A_\alpha \ge 0$ *for* $\alpha \in [0, 1]$; *that is*, $\langle (\text{Re } A_\alpha) f, f\rangle \ge 0$ *for all* $f \in l^2$.

Proof. It suffices to show that $A_\alpha + A_\alpha^* \ge 0$. The matrix $A_\alpha + A_\alpha^*$ is positive if and only if all of its finite sections

$$S_n = \begin{bmatrix} 2 & \frac{\alpha}{2} & \frac{\alpha^2}{3} & \frac{\alpha^3}{4} & \cdots & \frac{\alpha^{n-2}}{n-1} & \frac{\alpha^{n-1}}{n} \\ \frac{\alpha}{2} & 1 & \frac{\alpha}{3} & \frac{\alpha^2}{4} & \cdots & \frac{\alpha^{n-3}}{n-1} & \frac{\alpha^{n-2}}{n} \\ \frac{\alpha^2}{3} & \frac{\alpha}{3} & \frac{2}{3} & \frac{\alpha}{4} & \cdots & \frac{\alpha^{n-4}}{n-1} & \frac{\alpha^{n-3}}{n} \\ \frac{\alpha^3}{4} & \frac{\alpha^2}{4} & \frac{\alpha}{4} & \frac{1}{2} & \cdots & \frac{\alpha^{n-5}}{n-1} & \frac{\alpha^{n-4}}{n} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{\alpha^{n-2}}{n-1} & \frac{\alpha^{n-3}}{n-1} & \frac{\alpha^{n-4}}{n-1} & \frac{\alpha^{n-5}}{n-1} & \cdots & \frac{2}{n-1} & \frac{\alpha}{n} \\ \frac{\alpha^{n-1}}{n} & \frac{\alpha^{n-2}}{n} & \frac{\alpha^{n-3}}{n} & \frac{\alpha^{n-4}}{n} & \cdots & \frac{\alpha}{n} & \frac{2}{n} \end{bmatrix}$$

have positive determinants. Multiply the second column of S_n by α and

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subtract from the first column, then multiply the third column by α and subtract from the second, and continue in this way through the columns. The resulting matrix

$$T_n=\begin{bmatrix} 2-\frac{\alpha^2}{2} & \frac{\alpha}{2}-\frac{\alpha^3}{3} & \frac{\alpha^2}{3}-\frac{\alpha^4}{4} & \cdots & \frac{\alpha^{n-2}}{n-1}-\frac{\alpha^n}{n} & \frac{\alpha^{n-1}}{n} \\ -\frac{\alpha}{2} & 1-\frac{\alpha^2}{3} & \frac{\alpha}{3}-\frac{\alpha^3}{4} & \cdots & \frac{\alpha^{n-3}}{n-1}-\frac{\alpha^{n-1}}{n} & \frac{\alpha^{n-2}}{n} \\ 0 & -\frac{\alpha}{3} & \frac{2}{3}-\frac{\alpha^2}{4} & \cdots & \frac{\alpha^{n-4}}{n-1}-\frac{\alpha^{n-2}}{n} & \frac{\alpha^{n-3}}{n} \\ 0 & 0 & -\frac{\alpha}{4} & \cdots & \frac{\alpha^{n-5}}{n-1}-\frac{\alpha^{n-3}}{n} & \frac{\alpha^{n-4}}{n} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \frac{2}{n-1}-\frac{\alpha^2}{n} & \frac{\alpha}{n} \\ 0 & 0 & 0 & \cdots & -\frac{\alpha}{n} & \frac{2}{n} \end{bmatrix}$$

has the same determinant as S_n. We now multiply the second row of T_n by α and subtract from the first row, then multiply the third row by α and subtract from the second, and continue in this way through the rows. The resulting matrix

$$Z_n=\begin{bmatrix} 2 & -\frac{\alpha}{2} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\alpha}{2} & 1 & -\frac{\alpha}{3} & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\alpha}{3} & \frac{2}{3} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{\alpha}{4} & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \frac{2}{n-2} & -\frac{\alpha}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{\alpha}{n-1} & \frac{2}{n-1} & -\frac{\alpha}{n} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{\alpha}{n} & \frac{2}{n} \end{bmatrix}$$

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has the same determinant as T_n. It is easy to check that $2/(k-1) > \alpha/(k-1)+\alpha/k$ for $k>2$ and $\alpha \in [0, 1]$, and hence z_{n+1} is diagonally dominant [5, Exercise 3, p. 227]; since Z_{n+1} is hermitian, diagonally dominant, and all of its diagonal elements are positive, it follows that Z_{n+1} is positive definite [5, Exercise 5, p. 228]. Its principal minor det Z_n must then be positive [5, p. 96], so $\det S_n = \det Z_n > 0$, as needed.

Since $W(A_\alpha)$ is convex [3, p. 110] and contains the set of eigenvalues $\pi_0(A_\alpha) = \{1/n\}_{n=1}^{\infty}$ [6, p. 407], it is easy to see that $(0, 1] \subseteq W(A_\alpha)$; in fact, it is routine to check that $W(A_0) = (0, 1]$. We also find that the numerical range of the Cesàro operator A_1 is not hard to compute; for this computation we need the following lemma.

Lemma. *If T is an operator and λ is a complex number such that $|\lambda| = \|T\|$ and $\lambda \in W(T)$, then λ is an eigenvalue of T.*

The proof of this lemma appears in [3, p. 319].

Theorem 4. *The numerical range of the Cesàro operator A_1 is the open disk $\{\lambda : |1-\lambda| < 1\}$.*

Proof. If λ is an eigenvalue of A_1^*, then it is clear that $\bar{\lambda} \in W(A_1)$; hence $\{\lambda : |1-\lambda| < 1\} \subseteq W(A_1)$ by [1, Theorem 2, p. 130]. Assume $|\lambda - 1| = 1$; then $|\lambda - 1| = \|A_1 - I\|$, but $\lambda - 1$ is not an eigenvalue of $A_1 - I$ by [1, p. 130]. It follows from the lemma that $\lambda - 1 \notin W(A_1 - I)$ and hence $\lambda \notin W(A_1)$ when $|\lambda - 1| = 1$. If $|\lambda - 1| > 1$, then $\lambda \notin W(A_1)$ since $W(A_1)$ is convex.

We now turn to the case $0 < \alpha < 1$. It is easy to see that if B is the matrix $\langle b_{ij} \rangle$ $(i, j = 0, 1, 2, \ldots)$ with $b_{00} = 1$, $b_{10} = \alpha/2$, $b_{11} = \frac{1}{2}$, and $b_{ij} = 0$ for all other values of i, j, then $W(B) \subseteq W(A_\alpha)$. It follows from [2] that $W(B)$ is the closed elliptical disk bounded by the curve

$$\frac{(x-\frac{3}{4})^2}{1+\alpha^2} + \frac{y^2}{\alpha^2} = \frac{1}{16};$$

since the major axis is $\frac{1}{2}(1+\alpha^2)^{1/2}$ we find that $\omega(B) = \frac{3}{4} + \frac{1}{4}(1+\alpha^2)^{1/2}$, and hence $\omega(A_\alpha) \geq \omega(B) > 1$ if $\alpha > 0$. Since $r(A_\alpha) = 1 < \|A_\alpha\|$ [6], we must have $\|A_\alpha\| > \omega(A_\alpha)$ [3, Problem 173]. In summary, we have found that $r(A_\alpha) < \omega(A_\alpha) < \|A_\alpha\|$; we state this result in Halmos' terminology [3, pp. 114–115].

Theorem 5. *A_α is not* normaloid *(since $\omega(A_\alpha) \neq \|A_\alpha\|$) and not* spectraloid *(since $r(A_\alpha) \neq \omega(A_\alpha)$) for $0 < \alpha < 1$.*

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