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2-GROUPS WITH FEW CONJUGACY CLASSES

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Abstract An old question of Brauer that asks how fast numbers of conjugacy classes grow is investigated by considering the least number c_n of conjugacy classes in a group of order 2^n. The numbers c_n are computed for $n \leqslant 14$ and a lower bound is given for c_{15}. It is observed that c_n grows very slowly except for occasional large jumps corresponding to an increase in coclass of the minimal groups G_n. Restricting to groups that are 2-generated or have coclass at most 3 allows us to extend these computations.

Keywords: p-groups; conjugacy classes; coclass

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1. Introduction

There is a long history to the question of the possible number $k(G)$ of conjugacy classes of a finite group G. It began in 1903 when Landau [**8**] showed that only finitely many groups G have a given $k(G)$. This was made explicit in 1963 by Brauer [**3**] (see also [**4**]), who showed that $k(G) > \log_2 \log_2 |G|$. In general, $k(G)$ will be much larger than this. For example, Bertram [**1**] showed that for a given $\epsilon > 0$ and for almost all integers $n \leqslant x$, as $x \to \infty$, $k(G) > |G|^{1-\epsilon}$ for each group G of order n.

In his paper of 1963, Brauer asked what the 'true' growth of a lower bound for $k(G)$ in terms of $|G|$ might be. One answer to this was provided by Pyber [**14**], who proved the lower bound $k(G) \geqslant \epsilon \log_2 |G|/(\log_2 \log_2 |G|)^8$. Experimentally, López and López [**9, 10**] found that $k(G) > \log_3 |G|$ if $|G| \leqslant 3^{13}$, and in fact no group has been discovered for which this fails. The groups $G = \mathrm{PSL}(3,4)$ and $G = M_{22}$ both satisfy $k(G) = \lceil \log_3 |G| \rceil$.

If we restrict our attention to nilpotent groups (in particular, if we restrict to $|G|$ being a prime power), the ideas of P. Hall immediately give $k(G) > \alpha \log |G|$ for some constant α depending only on p, as described in § 2. This was refined by Sherman [**16**], who showed that if G has nilpotency class c, then $k(G) > c(|G|^{1/c} - 1) + 1$. Kovaćs and Leedham-Green [**7**] produced, for each odd prime p, a group G_p of order p^p with less than $p^3 = (\log_p|G_p|)^3$ classes. A natural question, originally formulated by Pyber [**14**], is whether, for a given p, there exists an absolute constant c and a sequence of p-groups (G_n), where G_n has order p^n and $k(G_n) < cn = c\log_p|G_n|$.

The aim of this paper is to address what Brauer asked in his 1963 paper by attacking the above question with a computer. We focus on 2-groups, where we can make extensive calculations with the help of the computational software package MAGMA [**2**].

Let $c_n = \min\{k : \text{there is a group } G \text{ of order } 2^n \text{ with } k(G) = k\}$. Our approach consists of three searches. In the first search, we find c_n for all $n \leqslant 14$ together with bounds for c_{15}. In the second search, we restrict attention to 2-generated 2-groups and find the smallest number of conjugacy classes for such groups of order 2^{15} and 2^{16}. In the third search, we restrict our attention to 2-groups with coclass at most 3 and with any order.

It appears that c_n grows tightly with n except for large occasional jumps. We provide an explanation for this behaviour. The ultimate answer to Brauer's question will depend on a comparison between the frequency and the size of these jumps.

2. Basic results

If $|G| = p^{2m+e}$ with $e = 0$ or 1, then a formula of Hall [**13**] states

$$k(G) = m(p^2-1) + p^e + r(G)(p-1)(p^2-1),$$

where $r(G)$ is a non-negative integer. This formula has several implications. First, by noticing that the right-hand side of the equality is at least $m(p^2-1)+p^e$, we get that $k(G) > \alpha\log|G|$, where α is a constant depending only on p. For $p = 2$, we obtain $k(G) = 3(m + r(G)) + 2^e$, so that $k(G) \equiv |G| \pmod 3$. In fact, Poland showed that if G is a p-group such that $r(G) = 0$, then $|G| \leqslant p^{p+2}$ and it has coclass 1. Fernández-Alcober and Shepherd [**5**] recently proved that if $p \geqslant 11$ and $r(G) = 0$, then $|G| \leqslant p^{p+1}$. Computational evidence (such as that provided by this paper) suggests that there are bounds on the order and coclass of p-groups with a given $r(G)$, which, if true, explains phenomena later in this paper. Also, Poland showed that $k(G) > k(Q)$ and $r(G) \geqslant r(Q)$ if Q is a proper quotient of p-group G. We summarize the consequences for c_n. Note that from this point on we consider exclusively 2-groups.

Lemma 2.1. *$c_n \equiv 1 \pmod 3$ if n is even, and $c_n \equiv 2 \pmod 3$ if n is odd. Moreover, $c_n > c_{n-1}$.*

Call a group G of order 2^n with $k(G) = c_n$ a best group. We can establish some properties of sequences of best groups.

Theorem 2.2. *Let G_n $(n = 1, 2, \dots)$ be a sequence of best groups. The coclass of G_n grows without bound as $n \to \infty$.*

This follows by combining the following two lemmas.

Lemma 2.3. *For each positive integer c, there exists a positive real number α_c, such that if G is a 2-group of coclass c, then $k(G) \geqslant \alpha_c|G|$.*

Proof. Shalev's proof [**12**] of the conjectures of Leedham-Green and Newman shows that if G is a 2-group of coclass c, then it contains an abelian normal subgroup A of index bounded by a function $f(c)$. This implies that

$$k(G) \geqslant k(A)/[G:A] = |A|/[G:A] = |G|/[G:A]^2 \geqslant |G|/f(c)^2.$$

Taking $\alpha_c = 1/f(c)^2$ gives the result. □

Lemma 2.4 (see [6]). *Let H_n be the Sylow 2-subgroup of $GL(n, \boldsymbol{F}_2)$. The order of H_n is $2^{T(n)}$, where $T(n) = (n-1)(n-2)/2$ and*

$$2^{(1/12-\epsilon_n)n^2} < k(H_n) < 2^{(1/4+\epsilon_n)n^2},$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Proof of Theorem 2.2. Suppose G_n is a sequence of best groups whose coclasses form a finite set C. Let $\alpha = \min\{\alpha_c \mid c \in C\}$. Then $c_n/2^n = k(G_n)/|G_n| \geqslant \alpha > 0$ for all n. In particular, we have $k(H_n) \geqslant c_{T(n)} \geqslant 2^{T(n)}\alpha$, contradicting Lemma 2.4 for sufficiently large n. □

3. The first search; exhaustive for small n

Theorem 3.1. *The values of c_n for $n \leqslant 14$ are as follows (G_n a best group):*

n	c_n	$r(G_n)$	*coclass*(G_n)
1	2	0	1
2	4	0	1
3	5	0	1
4	7	0	1
5	11	1	1 *or* 2
6	13	1	2
7	14	1	2
8	19	2	2
9	26	4	3 *or* 4
10	28	4	3 *or* 4
11	29	4	3 *or* 4
12	34	5	3 *or* 4
13	35	5	3
14	37	5	3

The aim of this search is to compute c_n for as many n as possible. We originally used CAYLEY but later checked our results with the quicker system MAGMA. The databases of these systems contain all 2-groups of order $\leqslant 256$, and this allows us immediately to find c_n for $n \leqslant 8$. To extend these results, we use the following refinement of Lemma 2.1. We write $n(G) = \log_2(|G|)$.

Lemma 3.2. *If Q is a quotient of the 2-group G, then $2k(G)-3n(G) \geqslant 2k(Q)-3n(Q)$ if $n(G)$ is even, whereas $2k(G)-3n(G) \geqslant 2k(Q)-3n(Q)-1$ if $n(G)$ is odd.*

Proof. If $|G| = 2^{2m+e}$ and $|Q| = 2^{2n+f}$ with $e, f \in \{0, 1\}$, then

$$\begin{aligned}(2k(G)-3n(G))-(2k(Q)-3n(Q)) &= 6(r(G)-r(Q))+(2^{e+1}-3e)-(2^{f+1}-3f)\\ &= 6(r(G)-r(Q))+f-e.\end{aligned}$$

Since $r(G) \geqslant r(Q)$, this is non-negative except, possibly, if $f = 0$ and $e = 1$, in which case it is at least -1. □

The p-group generation process of O'Brien [**12**] creates, for each positive integer d, a tree whose vertices are the d-generated 2-groups (counted once up to isomorphism). An edge exists from P to Q if P is isomorphic to $Q/\gamma_c(Q)$, where $\gamma_c(Q)$ is the last non-trivial term of the lower exponent-p central series of Q. In that case, we call Q an immediate descendant of P. If there is a path from P to Q, then we say Q is a descendant of P. O'Brien's process allows us to compute immediate descendants (and so descendants) of any given 2-group.

We use Lemma 3.2 and O'Brien's trees to compute c_n for increasing n. To test, for instance, if there is a group of order 2^{12} with $\leqslant 31$ conjugacy classes, we use O'Brien's routine to compute all 2-groups Q with smaller order and $2k(Q)-3n(Q) \leqslant 26$. If our group existed, then it would be an immediate descendant of such a Q. A computational check shows that no such Q has an immediate descendant of order 2^{12} with $\leqslant 31$ conjugacy classes. So $c_{12} \geqslant 34$ and we find all best groups of order 2^{12} by using O'Brien's routine to find all groups with $2k(Q)-3n(Q) \leqslant 32$ and $|Q| \leqslant 2^{12}$.

This works well until we try to find c_{15}. The bound on $2k(Q)-3n(Q)$ becomes so large that we have to consider too many groups in O'Brien's trees for this computation to be feasible. The case of 2-generated groups alone (see § 4) took a few months to complete. The best we have is that $53 \leqslant c_{15} \leqslant 68$.

We have data on the best groups of order 2^n ($n \leqslant 14$) that may be obtained by request from the authors. A few observations are in order. For each n there are 2-generated best groups. For $n = 9, 10, 11, 12$, there are also 3-generated best groups (these being the ones of coclass 4 of those orders). Note that jumps in c_n are apparently accompanied by jumps in coclass. The best groups of order 2^{14} are extensions of the same point group of order 2^6 by a normal subgroup isomorphic to the direct product C_4^4.

4. The second search; 2-generated groups

Since the search for the best groups of order 2^{15} ultimately involved too many groups to be feasible, we decided to restrict our attention to 2-generated groups. This permits a lengthy but successful search.

Theorem 4.1. *There are 142 2-generated groups of order 2^{15} with 68 conjugacy classes. No 2-generated group of order 2^{15} has fewer conjugacy classes. There are 92*

2-generated groups of order 2^{16} with 70 conjugacy classes. No 2-generated group of order 2^{16} has fewer conjugacy classes. Every 2-generated group of order 2^{17} has $\geqslant 74$ conjugacy classes.

This arises by use of the method of the previous section. We inductively construct all 2-generated 2-groups Q with $2k(Q) - 3n(Q) \leqslant 92$. The largest of these have order 2^{16}. The groups of order 2^{15} are of coclass 3 or 4. The ones of order 2^{16} are of coclass 4. Note that if G_n is one of these groups, then $r(G_n) = 15$. We know of no d-generated groups, for $d \geqslant 3$, that have the same order as, but fewer conjugacy classes than, the above 2-generated groups.

5. The third search; groups of coclass $\leqslant 3$

The paper of Newman and O'Brien [**11**] presents a method for obtaining all 2-groups of coclass 3. Since many of our best 2-groups have coclass $\leqslant 3$, we decided to do an exhaustive study of the number of conjugacy classes of these groups using [**11**]. All but 1782 sporadic examples naturally fall into 82 families as follows. There are 82 pro-2 groups, which correspond to the infinite ends of the subtrees of coclass $\leqslant 3$ groups of O'Brien's trees for $1 \leqslant d \leqslant 3$. Mainline groups in family $\#i$ are obtained by taking the exponent-p central quotients of pro-2 group $\#i$. The rest are obtained by taking descendants of these mainline groups. Above a certain vertex (the periodic root), the pattern of descendants is conjecturally periodic. This regularity allows us to find the 2-groups of coclass 3 with fewest conjugacy classes for all orders n. Since the largest of the 1782 sporadic groups has order 2^{14}, we need not consider them when working with $n > 14$. The case of $n \leqslant 14$ was covered in § 2.

For family $\#i$, for each i, we compute the number $f_i(n)$ of conjugacy classes of its mainline quotient of order 2^n for sufficiently large n. For instance, family $\#2$ yields dihedral groups and so $f_2(n) = 2^{n-2} + 3$.

There is a formula for $f_i(n)$ of the form $a2^n +$ lower terms (a independent of n). For instance, for family $\#34$, setting $x = 2^{[n/4]}$, $f_{34}(n) = 2^{n-12} + cx^2 + dx + 9$, where the values of c and d depend on $n \pmod 4$ as follows: if $n \equiv 0 \pmod 4$, then $c = 27/128$ and $d = 51/16$; if $n \equiv 1 \pmod 4$, then $c = 3/8$ and $d = 27/8$; if $n \equiv 2 \pmod 4$, then $c = 27/64$ and $d = 33/8$; if $n \equiv 3 \pmod 4$, then $c = 3/4$ and $d = 39/8$.

It is easy to see then that no group in family $\#2$ beats even the mainline groups in family $\#34$. We carry this method through with all 82 families. It is interesting to observe that $f_{29}(n) = f_{34}(n) + 6$, and lengthy computations show that these are the only two families that compete for best coclass 3 groups, in the following sense.

Let $c_n^{(3)} = \min\{k :$ there is a group G of order 2^n and coclass $\leqslant 3$ with $k(G) = k\}$. If the group G has order 2^n, coclass $\leqslant 3$, and $k(G) = c_n^{(3)}$, then G will be called a best coclass 3 group.

Theorem 5.1. *For $n \leqslant 14$, $c_n^{(3)}$ is given by Theorem 3.1, and, for $15 \leqslant n \leqslant 26$, the values of $c_n^{(3)}$ are given by*

n	$c_n^{(3)}$
15	68
16	76
17	110
18	148
19	242
20	373
21	617
22	1 123
23	1 493
24	4 993
25	6 341
26	11 911

The best groups of coclass 3 of order 2^{15} are located in families #30, #32, #35 and #42. The best of order 2^{16} are in family #42 and of orders 2^{17}, 2^{18}, 2^{20} and 2^{21} in family #20. As for $n = 19$ and $n \geqslant 22$, the following claim is verified for $n \leqslant 26$ and is expected to hold in general. (A rigorous check of it would be far too lengthy; even the computational evidence for it takes several weeks to obtain.)

Claim 5.2. *Suppose $n = 19$ or $n \geqslant 22$. The best coclass 3 groups of order 2^n depend on $n \pmod 4$ as follows.*

(i) *If $n \equiv 1 \pmod 4$, then they are descendants of the mainline group of order 2^{n-2} of family #34.*

(ii) *If $n \equiv 2 \pmod 4$, then they are descendants of the mainline group of order 2^{n-3} of family #29.*

(iii) *If $n \equiv 3 \pmod 4$, then they are descendants of the mainline group of order 2^{n-4} of family #29.*

(iv) *If $n \equiv 0 \pmod 4$, then they are descendants of the mainline group of order 2^{n-5} of family #34.*

In each of the four cases, there is a formula for $c_n^{(3)}$ of the form $c_n^{(3)} = 2^{n-13} + bx^3 + cx^2 + dx + e$, where $x = 2^{[n/4]}$ and where b, c, d and e are rational numbers depending only on the congruence class of n ($\pmod 4$). For instance, it appears that for $n \equiv 1 \pmod 4$, $b = 107/14336$, $c = -11/256$, $d = 119/16$ and $e = -81/7$.

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