FOURIER EXPANSIONS

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1. *Introduction.*

The theory of operational solutions of differential equations in applied mathematics suggests a method of developing the theory of Fourier and allied series that is simpler for ordinary applications than the classical development. It may be useful to those whose interests lie in such applications rather than in the deeper analytical processes associated with this subject.

1.1. *Type of function considered.*

The function whose expansion is required will be assumed to be of bounded variation and the period will be taken as 2π. If $g(t)$ denotes the value of the function in the interval $(0, 2\pi)$, the associated periodic function $f(t)$ is defined by the relations :

$$f(t)=g(t), \quad (0<t<2\pi), \quad f(t+2\pi)=f(t).$$

The main part of this paper, however, will be concerned with the practical determination of the Fourier coefficients for a case which is theoretically simple and one most likely to occur in applications. It is defined by the relations :

$$g(t)=g_r(t), \quad (t_{r-1}<t<t_r\,; \quad r=1, 2, \ldots, s\,; \; t_0=0, \, t_s=2\pi),$$

where $g_r(t)$ is a finite linear combination of functions of the type

$$t^p e^{at} \cos bt, \quad t^p e^{at} \sin bt,$$

p being a positive integer or zero.

Thus $g_r(t)$ belongs to the class of functions that are solutions of the ordinary linear differential equation with constant coefficients

$$Q(D)u=0, \quad (D=d/dt).$$

The proof of the Fourier expansion for the general function of bounded variation will be shown to be a natural corollary of the results obtained.

2. *Periodic solutions of $Q(D)u=0$.*

If $u(t)$ is a solution of this equation with the property

$$u(t+2\pi)=u(t)$$

for all values of t, then it must be of the form

$$p_0+\sum_{n=1}^{N}(p_n e^{nit}+q_n e^{-nit})$$

where p_n and q_n are constants. This rather obvious result may be deduced formally from the properties of a finite set of linear independent functions. There will, of course, be no such solution (apart from zero) unless one or more of the numbers

$$0, \; \pm i, \; \pm 2i, \; \pm 3i, \ldots$$

are zeros of $Q(z)$.

When the solution is real, it is more appropriate to take it in the form

$$\tfrac{1}{2}A_0+\sum_{n=1}^{N}(A_n\cos nt+B_n\sin nt)$$

where the coefficients A_n and B_n are real constants.

2.1. *The Laplace transform associated with* $Q(D)u=0$.

Let (i) $u=g(t)$ be a solution of the equation

$$Q(D)u\equiv(D^m+b_1D^{m-1}+b_2D^{m-2}+\ldots+b_m)u=0\ ;$$

(ii) the values of u and its first $m-1$ derivatives at $t=t_0$ be denoted by

$$g_0,\ g_1,\ g_2,\ \ldots,\ g_{m-1}\ \text{respectively}\ ;$$

(iii) C be a simple contour (*e.g.* a polygon or circle) enclosing all the zeros of $Q(z)$.

Then

$$g(t)=\frac{1}{2\pi i}\int_C F(z,t_0)\,e^{z(t-t_0)}\,dz,$$

where $F(z,t_0)=(a_0z^{m-1}+a_1z^{m-2}+\ldots a_{m-1})/Q(z)$ and

$$\begin{aligned}a_0&=g_0,\\ a_1&=g_1+b_1g_0,\\ a_2&=g_2+b_1g_1+b_2g_0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ a_{m-1}&=g_{m-1}+b_1g_{m-2}+b_2g_{m-3}+\ldots+b_{m-1}g_0.\end{aligned}$$

The result is well known, the function $F(z,t_0)$ being the Laplace transform of $g(t)$ at $t=t_0$. That the contour integral is a solution, follows from differentiation under the integral sign. That it is the correct solution is easily deduced by expanding $F(z,t_0)$ near $z=\infty$.

The above solution is appropriate for determining $g(t)$ when $g_0, g_1, \ldots, g_{m-1}$ are given, but in the application that is made here to Fourier analysis it is the function $g(t)$ that is given and the transform that is required. The typical functions occurring in $g(t)$ are $t^pe^{at}\cos bt$ and $t^pe^{at}\sin bt$, and the transforms for these are easily obtained from that of the simple function e^{kt}. Thus

(i) if $g(t)=e^{kt}$, then $F(z,t_0)=\dfrac{e^{kt_0}}{z-k}$;

(ii) if $g(t)=t^pe^{kt}$, differentiation with respect to k of the above result shows that

$$F(z,t_0)=\frac{{t_0}^p}{z-k}+\frac{p{t_0}^{p-1}}{(z-k)^2}+\frac{p(p-1){t_0}^{p-2}}{(z-k)^3}+\ldots+\frac{p!}{(z-k)^{p+1}}e^{kt_0}\ ;$$

(iii) the transforms of $t^pe^{at}\cos bt$ and $t^pe^{at}\sin bt$ are the *apparent* real and imaginary parts of

$$\frac{{t_0}^p}{z-a-ib}+\frac{p{t_0}^{p-1}}{(z-a-ib)^2}+\ldots+\frac{p!}{(z-a-ib)^{p+1}}e^{t_0(a+ib)},$$

(*i.e.*, ignoring for the moment the occurrence of i in z). For example:

(i) $g(t)=P(t)$, a polynomial of degree p,

$$F(z,t_0)=\frac{P(t_0)}{z}+\frac{P'(t_0)}{z^2}+\ldots+\frac{P^{(p)}(t_0)}{z^{p+1}}\ ;$$

(ii) $g(t)=t^2\sin\tfrac{1}{2}t$,

$$F(z,t_0)=\mathrm{I}'\left\{\frac{{t_0}^2}{z-\frac{1}{2}i}+\frac{2t_0}{(z-\frac{1}{2}i)^2}+\frac{2}{(z-\frac{1}{2}i)^3}\right\}e^{\frac{1}{2}it_0},$$

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(the accent denoting that the apparent imaginary part is taken)

$$=\mathrm{I}'\left\{\frac{t_0^2(z+\frac{1}{2}i)}{z^2+\frac{1}{4}}+\frac{2t_0(z^2-\frac{1}{4}+zi)}{(z^2+\frac{1}{4})^2}+\frac{2\{z^3-\frac{3}{4}z+i(\frac{3}{2}z^2-\frac{1}{8})\}}{(z^2+\frac{1}{4})^3}\right\}e^{\frac{1}{2}it_0}.$$

Thus
$$F(z,0)=\frac{16(12z^2-1)}{(4z^2+1)^3},$$

and
$$F(z,\pi)=\frac{4\pi^2 z}{4z^2+1}+\frac{8\pi(4z^2-1)}{(4z^2+1)^2}+\frac{32z(4z^2-3)}{(4z^2+1)^3}.$$

3. *Fourier Expansions.*

In the course of this paper, two main results will be proved. In the first, the Fourier expansion of the function $g(t)$ specified in § 1.1 is obtained as the sum of the residues of an associated function of a complex variable; and in the second, the method of the first is adapted to establish the Fourier expansion of a function of bounded variation.

The first result may be stated as follows:

Let

(i) $g(t)=g_r(t)$, $(t_{r-1}<t<t_r,\ r=1,2,\dots s,\ t_s=0,\ t_s=2\pi)$, and let $f(t)$ be the associated function periodic in 2π;

(ii) the transform of $g_r(t)$ at t' be $F_r(z,t')$;

(iii) $G(z)=F_1(z,0)-F_s(z,2\pi)e^{-2\pi z}+\sum_1^{s-1}\{F_{r+1}(z,t_r)-F_r(z,t_r)\}e^{-zt_r}$ and

$$G_1(z)=\sum_0^{s-1}\{F_{r+1}(z,t_r)-F_r(z,t_r)\}e^{-zt_r};\ F_0(z,0)=F_s(z,2\pi);$$

(iv) $H(z)=G(z)\dfrac{e^{tz}}{e^{2\pi z}-1}$;

(v) the residue of $H(z)$ at a pole z_0 be $\rho(z_0)$.

Then (i) $\rho(0)+\sum_1^\infty\{\rho(ni)+\rho(-ni)\}=\frac{1}{2}\{f(t+0)+f(t-0)\}$ for all values of t;

(ii) the infinite series above takes the form

$$\tfrac{1}{2}A_0+\sum_1^\infty(A_n\cos nt+B_n\sin nt)$$

where $\pi A_n=\mathbf{R}G(in)=\mathbf{R}G_1(in)$: $\pi B_n=-\mathbf{I}G(in)=-\mathbf{I}G_1(in)$ (the residue formulae), a modification being necessary whenever $in\,(n=0,1,2,3,\dots)$ is a pole of $G(z)$;

(iii) $\pi A_n=\int_0^{2\pi}f(t)\cos nt\,dt$: $\pi B_n=\int_0^{2\pi}f(t)\sin nt\,dt$ (the integral formulae);

(iv) when a term t^p occurs in $g_r(t)$, $G(z)$ has a pole at $z=0$. The only coefficient affected is A_0, and the contribution to A_0 for this term is

$$\frac{t_r^{p+1}-t_{r-1}^{p+1}}{p+1}\text{ (by the integral formula);}$$

(v) when terms $t^p\cos mt$, $t^p\sin mt$ $(m=1,2,3,\dots)$ occur in $g_r(t)$, $G(z)$ has a pole at im. The coefficients affected are A_m and B_m. The contributions to these coefficients are respectively

$\frac{1}{2}(A_0'+A'_{2m})$ and $\frac{1}{2}B'_{2m}$ for $t^p\cos mt$

and $\frac{1}{2}B'_{2m}$ and $\frac{1}{2}(A_0+A'_{2m})$ for $t^p\sin mt$,

where $\frac{1}{2}A_0'+\sum_1^\infty\{A_n'\cos nt+B_n'\sin nt\}$ is the Fourier expansion of the function defined to be t^p in the interval (t_{r-1},t_r) and zero elsewhere in $(0,2\pi)$.

Thus $\pi A_0' = (t_r^{p+1} - t_{r-1}^{p+1})/(p+1)$ by the integral formula and A'_{2m} is determined by the residue formula.

3.1. *A particular contour integral.*

The following lemma involving a contour integral is the basis of the subsequent development.

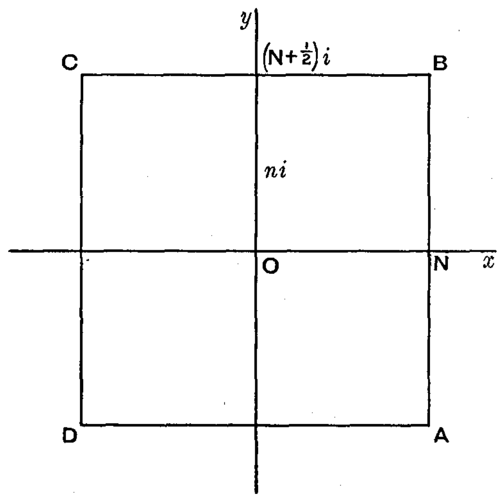

FIG. 1

LEMMA. Let

(i) $F(z) = P(z)/Q(z)$, where
$P(z) = a_0 z^{m-1} + a_1 z^{m-2} + \ldots + a_{m-1}$,
$Q(z) = z^m + b_1 z^{m-1} + \ldots + b_m$.

(ii) Γ_N be the boundary of the rectangle $ABCD$ (Fig. 1) specified by the equations :

$$x = \pm N, \quad y = \pm (N + \tfrac{1}{2}),$$

where N is a positive integer sufficiently large to ensure that all the zeros of $Q(z)$ lie within the rectangle.

(iii) t be a real variable.

Then, when $N \to \infty$,

$$I_N(t) \equiv \frac{1}{2\pi i} \int_{\Gamma_N} F(z) \frac{e^{tz}}{e^{2\pi z} - 1} dz \to \begin{cases} 0, (0 < t < 2\pi), \\ -\tfrac{1}{2} a_0, (t = 0), \\ +\tfrac{1}{2} a_0, (t = 2\pi). \end{cases}$$

On the boundary $zF(z) \to a_0$ as $N \to \infty$.

On AB where $z = N + iy \; (-N - \tfrac{1}{2} \leqslant y \leqslant N + \tfrac{1}{2})$,

$$\left| \frac{e^{tz}}{e^{2\pi z} - 1} \right| \leqslant \frac{e^{tN}}{e^{2\pi N} - 1} \text{ which} \to \begin{matrix} 0, \; (t < 2\pi), \\ 1, \; (t = 2\pi). \end{matrix}$$

Thus $\qquad I_{AB} \to 0, \; (t < 2\pi)$ and $(t = 2\pi, a_0 = 0)$.

On CD where $z = -N + iy \; (-N - \tfrac{1}{2} \leqslant y \leqslant N + \tfrac{1}{2})$,

$$\left| \frac{e^{tz}}{e^{2\pi z} - 1} \right| \leqslant \frac{e^{-tN}}{1 - e^{-2\pi N}} \text{ which} \to \begin{matrix} 0, \; (t > 0), \\ 1, \; (t = 0). \end{matrix}$$

So $\qquad I_{CD} \to 0 \; (t > 0)$ and $(t = 0, a_0 = 0)$.

On BC where $z=x+i(N+\frac{1}{2})$,

$$|I_{BC}|<\frac{1}{2\pi}\int_{-N}^{N}|F(z)|\frac{e^{tx}}{e^{2\pi x}+1}\,dx.$$

But $\displaystyle\int_{-\infty}^{\infty}\frac{e^{tx}}{e^{2\pi x}+1}\,dx$ converges when $0<t<2\pi$.

Therefore $I_{BC}\to 0$ when $0<t<2\pi$.

Also
$$\frac{1}{e^{2\pi x}+1}\text{ and }\frac{e^{2\pi x}}{e^{2\pi x}+1}$$
are bounded as x ranges from $-\infty$ to $+\infty$, and therefore

$$I_{BC}\to 0 \text{ when } t=0 \text{ or } t=2\pi, \text{ if } a_0 \text{ is zero.}$$

Similarly $I_{DA}\to 0$ $(0<t<2\pi)$ and $(0\leqslant t\leqslant 2\pi,\ a_0=0)$.

It has been shown therefore that $I_N(t)$ tends to zero if

$$\text{(i) } 0<t<2\pi \text{ and (ii) } t=0 \text{ or } t=2\pi,\text{ if } a_0=0.$$

There remain the cases $t=0$ and $t=2\pi$ when $a_0\neq 0$.

Let $G(z)=F(z)-a_0/z$.

Then $zG(z)\to 0$ as $N\to\infty$, and the above analysis shows that

$$\int_{\Gamma_N}G(z)\frac{e^{tz}}{e^{2\pi z}-1}\,dz\to 0 \text{ for } 0\leqslant t\leqslant 2\pi.$$

Denoting $\lim I_N(t)$ by $I(t)$, we have

$$I(0)=\lim\frac{1}{2\pi i}\int_{\Gamma_N}\frac{a_0}{z(e^{2\pi z}-1)}\,dz \text{ and } I(2\pi)=\lim\frac{1}{2\pi i}\int_{\Gamma_N}\frac{a_0 e^{2\pi z}}{z(e^{2\pi z}-1)}\,dz.$$

Thus
$$I(2\pi)-I(0)=\lim\frac{1}{2\pi i}\int_{\Gamma_N}\frac{a_0}{z}\,dz=a_0,$$

and by the symmetry of the contour

$$I(2\pi)=\lim\frac{1}{2\pi i}\int_{\Gamma_N}\frac{a_0 e^{-2\pi z}}{z(e^{-2\pi z}-1)}\,dz=\lim\frac{1}{2\pi i}\int_{\Gamma_N}\frac{a_0}{z(1-e^{2\pi z})}\,dz=-I(0),$$

$$\text{i.e., } I(2\pi)=\tfrac{1}{2}a_0 \text{ and } I(0)=-\tfrac{1}{2}a_0.$$

3.2. *Application of the lemma.*

The poles of the integrand are

$$0,\ \pm i,\ \pm 2i,\ \pm 3i,\ \dots \pm Ni,$$
$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \dots,\ \alpha_M,$$

where the numbers α_r are those zeros, if any, of $Q(z)$ that are not of the form $\pm ni$ (n being a positive integer or zero).

If $\rho(z_0)$ denotes the residue at z_0, the lemma shows that

$$\sum_{r=1}^{M}\rho(\alpha_r)+\rho(0)+\sum_{n=1}^{N}\{\rho(in)+\rho(-in)\}\to\begin{cases}-\tfrac{1}{2}a_0,\ (t=0),\\ 0,\ (0<t<2\pi),\\ \tfrac{1}{2}a_0,\ (t=2\pi).\end{cases}$$

Some well-known results arise immediately from this formula. For example :

(i) Let $F(z)=1/z$.

$$\frac{1}{2\pi i}\int_{\Gamma_N}\frac{e^{tz}\,dz}{z(e^{2\pi z}-1)}=\frac{t-\pi}{2\pi}+\sum_{n=1}^{N}\frac{\sin nt}{n\pi}\to\begin{cases}-\tfrac{1}{2}, & (t=0),\\ 0, & (0<t<2\pi),\\ +\tfrac{1}{2}, & (t=2\pi),\end{cases}$$

$$i.e.,\ \sum_{n=1}^{\infty}\frac{\sin nt}{n}=\begin{cases}\dfrac{\pi-t}{2}, & (0<t<2\pi),\\ 0, & (t=0 \text{ or } 2\pi).\end{cases}$$

(ii) Let $F(z)=1/z^p$ (p integral>1),

$$I_N\to 0,\ (0\leqslant t\leqslant 2\pi).$$

Now
$$\frac{e^{tz}}{e^{2\pi z}-1}=\frac{1}{2\pi z}\sum_{0}^{\infty}\frac{(2\pi)^r}{r!}B_r(t/2\pi)\,z^r,$$

where B_r is the Bernouillian function of the first order and the rth degree.

It follows immediately that for $0\leqslant t\leqslant 2\pi$

$$\sum_{n=1}^{\infty}\frac{\cos nt}{n^{2m}}=(-1)^{m-1}\frac{(2\pi)^{2m}}{2\,(2m)!}B_{2m}(t/2\pi),$$

$$\sum_{n=1}^{\infty}\frac{\sin nt}{n^{2m+1}}=(-1)^{m-1}\frac{(2\pi)^{2m+1}}{2\,(2m+1)!}B_{2m+1}(t/2\pi),$$

$$\sum_{n=1}^{\infty}\frac{1}{n^{2m}}=\frac{(2\pi)^{2m}}{2\,(2m)!}B_m,$$

where $B_m=(-1)^{m-1}B_{2m}(0)$ is the mth Bernouilli number.

(iii) $F(z)=1/(z^2+a^2)$ (where $a\neq 0$ or integral $\pm$),

$$I_N\to 0 \text{ for } 0\leqslant t\leqslant 2\pi.$$

Then
$$\frac{1}{2\pi a^2}+\frac{e^{iat}}{2ia(e^{2\pi ia}-1)}-\frac{e^{-iat}}{2ia(e^{-2\pi ia}-1)}+\sum_{n=1}^{\infty}\frac{\cos nt}{(a^2-n^2)\pi}=0,$$

giving
$$\frac{\cos a(\pi-t)}{\sin a\pi}=\frac{1}{a\pi}+\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{\cos nt}{a^2-n^2},\ (0\leqslant t\leqslant 2\pi).$$

In particular,

$$(t=0):\qquad \cot a\pi=\frac{1}{a\pi}+\frac{2}{\pi}\sum_{1}^{\infty}\frac{a}{a^2-n^2},$$

$$(t=\pi):\qquad \operatorname{cosec} a\pi=\frac{1}{a\pi}+\frac{2}{\pi}\sum_{1}^{\infty}\frac{(-1)^n a}{a^2-n^2}.$$

4. *Application to Fourier Expansions. Case* $s=1$.

A special simplicity is attached to the case when $g(t)$ is given as a single function (of appropriate type) in the interval $0<t<2\pi$, *i.e.*, the case when $s=1$ (§ 1.1).

Let the transform of $g(t)$ be $F(z, t_0)$ at $t=t_0$, and let

$$T_N(t)=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{F(z,0)\,e^{tz}-F(z,2\pi)\,e^{z(t-2\pi)}}{e^{2\pi z}-1}\,dz.$$

Then

$$T_N(t+2\pi)-T_N(t)=\frac{1}{2\pi i}\int_{\Gamma_N}F(z,0)e^{tz}\,dz-\frac{1}{2\pi i}\int_{\Gamma_N}F(z,2\pi)e^{z(t-2\pi)}\,dz$$
$$=0 \text{ (both integrals being equal to } g(t)\text{)}.$$

Now $T_N(t)$ obviously satisfies the differential equation

$$D(D^2+1)(D^2+4)\ldots(D^2+N^2)\,Q(D)u=0,$$

where $F(z, t_0)=P(z, t_0)/Q(z)$, and

$$P(z, t_0)=a_0z^{m-1}+a_1z^{m-2}+\ldots+a_{m-1},\ (a_0=g(t_0)).$$

Therefore by § 2

$$T_N(t)=\tfrac{1}{2}A_0+\sum_{n=1}^{N}\{A_n\cos nt+B_n\sin nt\},$$

and is the sum of the residues within Γ_N.

$$\text{Also}\quad T_N(t)=\frac{1}{2\pi i}\int_{\Gamma_N}F(z,2\pi)\,e^{z(t-2\pi)}\,dz+\frac{1}{2\pi i}\int_{\Gamma_N}\frac{\{F(z,0)-F(z,2\pi)\}\,e^{tz}}{e^{2\pi z}-1}\,dz$$

$$=g(t)+\frac{1}{2\pi i}\int_{\Gamma_N}\frac{\{F(z,0)-F(z,2\pi)\}\,e^{tz}}{e^{2\pi z}-1}\,dz.$$

Let $N\to\infty$. Then $$T_N(t)\to\begin{cases} g(t),\ (0<t<2\pi),\\ \tfrac{1}{2}\{g(0)+g(2\pi)\},\ (t=0\ \text{and}\ t=2\pi).\end{cases}$$

Defining the periodic function $f(t)$ by the equations:

$$f(t)=g(t),\ (0<t<2\pi);\quad f(t+2\pi)=f(t),$$

so that $$f(+0)=f(2\pi+0)=g(0)$$

and $$f(-0)=f(2\pi-0)=g(2\pi),$$

we have shown that

$$\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos nt+B_n\sin nt)=\tfrac{1}{2}\{f(t+0)+f(t-0)\}$$

for all values of t.

4.1. *Determination of A_n and B_n in the non-resonant case.*

The case when $Q(z)$ has no zeros of the form $\pm ni$ may appropriately be called the non-resonant case. Then the coefficients A_n and B_n may be obtained immediately from the transforms of $g(t)$ at 0 and 2π.

Let $G_1(z)=F(z,0)-F(z,2\pi)$, a function with real coefficients when $g(t)$ is real. Then

$$\rho(0)=\frac{G_1(0)}{2\pi},\quad \rho(in)=\frac{G_1(in)\,e^{int}}{2\pi},\quad \rho(-in)=\frac{G_1(-in)\,e^{-int}}{2\pi}.$$

Thus

$$g(t)=\frac{G_1(0)}{2\pi}+\frac{1}{\pi}\sum_{1}^{\infty}[\mathbf{R}\{G_1(in)\}\cos nt-\mathbf{I}\{G_1(in)\}\sin nt].$$

Examples:

(i) $g(t)=\cos\tfrac{1}{2}t,\quad F(z,t_0)=\dfrac{4z\cos\tfrac{1}{2}t_0-2\sin\tfrac{1}{2}t_0}{4z^2+1},\quad G_1(z)=\dfrac{8z}{4z^2+1},$

$$\textit{i.e.},\quad \cos\tfrac{1}{2}t=\frac{1}{\pi}\sum_{1}^{\infty}\frac{8n}{4n^2-1}\sin nt.$$

(ii) $g(t)=t^2e^{-t}$.

$$F(z,t_0)=\left\{\frac{t_0^2}{z+1}+\frac{2t_0}{(z+1)^2}+\frac{2}{(z+1)^3}\right\}e^{-t_0},$$

$$G_1(z)=\frac{2(1-e^{-2\pi})}{(z+1)^3}-\frac{4\pi e^{-2\pi}}{(z+1)^2}-\frac{4\pi^2e^{-2\pi}}{(z+1)},$$

$$G_1(ni)=\frac{2(1-in)^3(1-e^{-2\pi})}{(n^2+1)^3}-\frac{4\pi(1-in)^2e^{-2\pi}}{(n^2+1)^2}-\frac{4\pi^2(1-in)\,e^{-2\pi}}{(n^2+1)},$$

$$\pi A_n=\frac{2(1-3n^2)(1-e^{-2\pi})}{(n^2+1)^3}-\frac{4(1-n^2)\,e^{-2\pi}}{(n^2+1)^2}-\frac{4\pi^2e^{-2\pi}}{n^2+1},$$

$$\pi B_n=\frac{2n(3-n^2)(1-e^{-2\pi})}{(n^2+1)^3}-\frac{8ne^{-2\pi}}{(n^2+1)^2}-\frac{4\pi^2ne^{-2\pi}}{(n^2+1)},$$

$$\pi A_0=2-e^{-2\pi}(4\pi^2+4\pi+2).$$

4.2. *The resonant case.*

When $g(t)$ contains terms of the type t^p, the integrand has a multiple pole at $z=0$, and when it contains terms of the type $t^p \cos mt$, $t^p \sin mt$ (m integral$\geqslant 1$), the integrand has multiple poles at $z=\pm im$.

(i) If $g(t)=t^p$,

$$F(z, t_0)=\frac{t_0{}^p}{z}+\frac{pt_0{}^{p-1}}{z}+\ldots+\frac{p!}{z^{p+1}},$$

and the integrand is

$$\frac{G(z)\,e^{tz}}{e^{2\pi z}-1},$$

where
$$G(z)=\frac{p!}{z^{p+1}}-\left\{\frac{(2\pi)^p}{z}+\frac{p\,(2\pi)^{p-1}}{z^2}+\ldots+\frac{p!}{z^{p+1}}\right\}e^{-2\pi z}$$

and
$$G_1(z)=-\frac{(2\pi)^p}{z}-\frac{p\,(2\pi)^{p-1}}{z^2}-\ldots-\frac{p\,(p-1)\ldots 3\,.\,2}{z^{p+1}},$$

$$\left.\begin{aligned}\pi A_n&=\mathbf{R}G_1(in)=\frac{p\,(2\pi)^{p-1}}{n^2}-\frac{p\,(p-1)\,(p-2)\,(2\pi)^{p-3}}{n^4}-\ldots,\\ \pi B_n&=-\mathbf{I}G_1(in)=-\frac{(2\pi)^p}{n}+\frac{p\,(p-1)\,(2\pi)^{p-2}}{n^3}-\ldots\end{aligned}\right\}\ (n\geqslant 1).$$

The value of A_0 may be determined by calculating the residue at $z=0$, a pole of order $p+1$; and it is a simple exercise to show in this way that πA_0 is equal to $(2\pi)^{p+1}/(p+1)$. However, it is obviously easier to use the integral formula proved in the next paragraph to obtain this result.

(ii) If $g(t)=t^p \cos mt$ or $t^p \sin mt$ (m integral$\geqslant 1$) there are multiple poles at $z=\pm im$, the residues at which determine the coefficients A_m and B_m.

By (i) above let the expansion of t^p be

$$\tfrac{1}{2}A_0{}'+\sum_1^{\infty}(A_n{}'\cos nt+B_n{}'\sin nt).$$

It then follows immediately by the integral formulae that for $t^p \cos mt$

$$A_m=\tfrac{1}{2}(A_0{}'+A'_{2m})\ ;\quad B_m=\tfrac{1}{2}B'_{2m},$$

and for $t^p \sin mt$

$$A_m=\tfrac{1}{2}B'_{2m}\ ;\qquad B_m=\tfrac{1}{2}(A_0{}'-A'_{2m}).$$

Example. $g(t)=t^2\cos 3t$.

$$F(z, t_0)=\mathbf{R}'\left\{\frac{t_0{}^2}{z-3i}+\frac{2t_0}{(z-3i)^2}+\frac{2}{(z-3i)^3}\right\}e^{3it_0},$$

$$\begin{aligned}G_1(z)&=F(z,0)-F(z,2\pi)\\ &=\mathbf{R}'\left\{-\frac{(2\pi)^2}{(z-3i)}-\frac{4\pi}{(z-3i)^2}\right\}\\ &=-\frac{4\pi^2 z}{(z^2+9)}-\frac{4\pi\,(z^2-9)}{(z^2+9)^2}.\end{aligned}$$

Also
$$\pi A_3=\tfrac{1}{2}\left\{\frac{(2\pi)^3}{3}+\frac{2\,(2\pi)}{36}\right\},\quad \pi B_3=-\tfrac{1}{2}\,\frac{(2\pi)^2}{6}.$$

Thus $A_n=\dfrac{4\,(n^2+9)}{(n^2-9)^2}$, $B_n=-\dfrac{4\pi n}{(n^2-9)}$, $(n\neq 3)$; and $A_3=\dfrac{4\pi^2}{3}+\dfrac{1}{18}$, $B_3=-\dfrac{\pi}{3}$.

4.3. *The integral formulae.*

It has already been shown that

$$\tfrac{1}{2}A_0+\sum_{n=1}^{N}(A_n\cos nt+B_n\sin nt)=g(t)+\frac{1}{2\pi i}\int_{\Gamma_N}\frac{\{F(z,0)-F(z,2\pi)\}\,e^{zt}}{e^{2\pi z}-1}\,dz$$

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for all finite values of t, N being finite but sufficiently large to ensure that all the zeros of $Q(z)$ lie within Γ_N.

The range of integration is finite and we can integrate with respect to the real variable t from 0 to 2π, the integrand being in fact an analytic function of both variables within their ranges.

Thus
$$\pi A_0 = \int_0^{2\pi} g(t)\,dt + \frac{1}{2\pi i}\int_{\Gamma_N} \frac{F(z,0)-F(z,2\pi)}{z}\,dz$$
$$= \int_0^{2\pi} g(t)\,dt,$$
since the residue at ∞ of the integrand of the contour integral is zero.

Similarly by multiplying through by $\cos nt$ and $\sin nt$, integrating from 0 to 2π and using the results
$$\int_0^{2\pi} e^{zt}\cos nt\,dt = \frac{(e^{2\pi z}-1)z}{z^2+n^2}; \quad \int_0^{2\pi} e^{zt}\sin nt\,dt = -\frac{(e^{2\pi z}-1)n}{z^2+n^2},$$
we find that
$$\pi A_n = \int_0^{2\pi} g(t)\cos nt\,dt, \quad \pi B_n = \int_0^{2\pi} g(t)\sin nt\,dt.$$

It will be found, however, that except for the coefficient A_0 in the case of a polynomial, the residue method is more rapid than the integral method.

For example, let $g(t) = \sin^5(t/2)$.

By approximation near $t=0$ and $t=2\pi$, we have
$$g(0)=g'(0)=g''(0)=g'''(0)=g^{\text{iv}}(0)=0; \quad g^{\text{v}}(0)=15/4,$$
$$g(2\pi)=g'(2\pi)=g''(2\pi)=g'''(2\pi)=g^{\text{iv}}(2\pi)=0; \quad g^{\text{v}}(2\pi)=-15/4,$$
$$G(z) = \frac{15/2}{(z^2+\frac{1}{4})(z^2+\frac{9}{4})(z^2+\frac{25}{4})},$$
$$\pi\sin^5(t/2) = 16/15 + \sum_1^\infty 480\cos nt/\{(1-4n^2)(9-4n^2)(25-4n^2)\}.$$

4.4. *Odd and even functions.*

(i) If $g(t)=g(2\pi-t)$, the corresponding periodic function $f(t)$ is even.

(ii) If $g(t)=-g(2\pi-t)$, the function $f(t)$ is odd.

In the former case we expect that B_n is zero and in the latter that A_n is zero.

Now
$$g(t) = \frac{1}{2\pi i}\int_{\Gamma_N} F(z,0)\,e^{tz}\,dz = -\frac{1}{2\pi i}\int_{\Gamma_N} F(-z,0)\,e^{-tz}\,dz.$$
Therefore
$$g(2\pi - t) = -\frac{1}{2\pi i}\int_{\Gamma_N} F(-z,0)\,e^{z(t-2\pi)}\,dz.$$
Also
$$g(t) = \frac{1}{2\pi i}\int_{\Gamma_N} F(z,2\pi)\,e^{z(t-2\pi)}\,dz.$$
Since the transform $F(z,t_0)$ is unique, it follows that

(i) for an even periodic function, $F(z,2\pi) = -F(-z,0)$;

(ii) for an odd periodic function, $F(z,2\pi) = F(-z,0)$.

In (i) $G_1(z) = F(z,0)+F(-z,0)$
and, except when ni is a pole of $F(z,0)$,
$$\pi g(t) = F(0) + 2\sum_1^\infty \mathrm{R}\{F(ni)\}\cos nt.$$

In (ii) $G_1(z)=F(z, 0)-F(-z, 0)$

and $$\pi g(t)=-2\sum_1^\infty \mathrm{I}\{F(ni)\}\sin nt.$$

Examples :

(i) $g(t)=\sin\frac{1}{2}t$: $F(z, 0)=2/(4z^2+1)$;

$$\pi\sin\tfrac{1}{2}t=2+4\sum_1^\infty \cos nt/(1-4n^2).$$

(ii) $g(t)=t(t-\pi)(t-2\pi)$. Odd. $F(z, 0)=2\pi^2/z^2-6\pi/z^3+6/z^4$.

$$t(t-\pi)(t-2\pi)=12\sum_1^\infty \sin nt/n^3.$$

5. *Discontinuities within the interval. Case* $s\geqslant 1$.

We now take the more general case given in § 1.1 when $g(t)$ is given by the relations

$$g(t)=g_r(t), \quad (t_{r-1}<t<t_r, \quad r=1, 2, \ldots, s; \; t_0=0, t_s=2\pi)$$

and $s\geqslant 1$.

Consider the integral

$$T_N(t)=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{F(z, t_1)e^{z(t-t_1)}-F(z, t_2)e^{z(t-t_2)}}{e^{2\pi z}-1}\,dz$$

where $0\leqslant t_1<t_2\leqslant 2\pi$; and $F(z, t_0)$ is the transform, say, of a function $h(t)$.

Then $$T_N(t+2\pi)-T_N(t)=\frac{1}{2\pi i}\int_{\Gamma_N}F(z, t_1)e^{z(t-t_1)}\,dz-\frac{1}{2\pi i}\int_{\Gamma_N}F(z, t_2)e^{z(t-t_2)}\,dz$$
$$=0,$$

since both integrals are equal to $h(t)$.

Therefore, as before,

$$T_N(t)=\tfrac{1}{2}A_0+\sum_{n=1}^N(A_n\cos nt+B_n\sin nt).$$

By § 3, $T_N(t)\to 0$ in the interval common to

$$t_1<t<2\pi+t_1 \quad\text{and}\quad t_2<t<2\pi+t_2,$$

i.e., in the interval $t_2<t<2\pi+t_1$.

Using the periodic property of $T_N(t)$ we deduce that $T_N(t)$ tends to zero also in the interval $t_2-2\pi<t<t_1$. Now

$$T_N(t)=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{F(z, t_1)e^{z(t-t_1)}-F(z, t_2)e^{z(t-t_2+2\pi)}}{e^{2\pi z}-1}\,dz+\frac{1}{2\pi i}\int_{\Gamma_N}F(z, t_2)e^{z(t-t_2)}\,dz,$$

the latter integral being equal to $h(t)$.

Therefore $T_N(t)\to h(t)$ in the interval common to

$$0<t-t_1<2\pi \text{ and } 0<t-t_2+2\pi<2\pi,$$

i.e., in the interval $t_1<t<t_2$. Also

$$T_N(t_1)\to-\tfrac{1}{2}h(t_1)+h(t_1),\ \textit{i.e.},\ \tfrac{1}{2}h(t_1)\text{ ; and}$$
$$T_N(t_2)\to-\tfrac{1}{2}h(t_2)+h(t_2),\ \textit{i.e.},\ \tfrac{1}{2}h(t_2).$$

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Confining our attention to the interval $(0, 2\pi)$, we have shown that

$$T(t) \equiv \lim T_N(t) = \left.\begin{array}{ll} 0, & 0 \leqslant t < t_1, \\ \tfrac{1}{2}h(t_1), & t = t_1, \\ h(t), & t_1 < t < t_2, \\ \tfrac{1}{2}h(t_2), & t = t_2, \\ 0, & t_2 < t \leqslant 2\pi, \end{array}\right\}$$

the first interval being non-existent when $t_1 = 0$ and the last when $t_2 = 2\pi$.

The result for the more general case in which

$$g(t) = g_r(t) \quad (t_{r-1} < t < t_r, \; r = 1, 2, \ldots, s, \; t_0 = 0, \; t_s = 2\pi)$$

follows by addition.

Thus if $F_r(z, \theta)$ is the transform of $g_r(t)$ at $t = \theta$, then the Fourier expansion of $g(t)$ is the limit $T(t)$ of the sum of the residues of

$$\sum_{r=1}^{s} \frac{F_r(z, t_{r-1})\, e^{z(t-t_{r-1})} - F_r(z, t_r)\, e^{z(t-t_r)}}{e^{2\pi z} - 1},$$

and if $f(t)$ is the periodic function associated with $g(t)$, then

$$T(t) = \tfrac{1}{2}\{f(t+0) + f(t-0)\}$$

for all values of t.

Using the more appropriate notation of

$$F(z, t_r - 0) \text{ for } F_r(z, t_r) \text{ and } F(z, t_r + 0) \text{ for } F_{r+1}(z, t_r),$$

we can write the integrand as

$$\frac{G(z)\, e^{zt}}{e^{2\pi z} - 1},$$

where $G(z) = F(z, +0) + \sum_{1}^{s-1} \{F(z, t_r + 0) - F(z, t_r - 0)\}\, e^{-zt_r} - F(z, 2\pi - 0)\, e^{-2\pi z}$.

The coefficients are then given by the relation

$$\pi(A_n - iB_n) = G(in),$$

with the appropriate modification when in is a pole of $G(z)$.

5.1. *The integral formulae.*

Integrate the relation

$$T_N(t) = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{F(z, t_1)\, e^{z(t-t_1)} - F(z, t_2)\, e^{z(t-t_2)}}{e^{2\pi z} - 1}\, dz$$

from $t = 0$ to $t = 2\pi$.

Then $\pi A_0 = \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma} \{F(z, t_1)\, e^{-zt_1} - F(z, t_2)\, e^{-zt_2}\} dz/z.$

Now $h(t) = \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_N} F(z, t_1)\, e^{z(t-t_1)}\, dz$, and therefore

$$\int_0^{t_1} h(t)\, dt = \frac{1}{2\pi i} \int_{\Gamma_N} F(z, t_1)\, dz/z - \frac{1}{2\pi i} \int_{\Gamma_N} F(z, t_1)\, e^{-zt_1}\, dz/z$$
$$= -\frac{1}{2\pi i} \int_{\Gamma_N} F(z, t_1)\, e^{-zt_1}\, dz/z,$$

since the first integrand has no pole at ∞.

Similarly

$$\int_0^{t_2} h(t)\,dt = -\frac{1}{2\pi i}\int_{\Gamma_N} F(z, t_2)\,e^{-zt_2}\,dz/z,$$

and therefore

$$\pi A_0 = \int_{t_1}^{t_2} h(t)\,dt.$$

Again, integration of $T_N(t)\cos nt$ from 0 to 2π gives

$$\pi A_n = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{z\{F(z, t_1)\,e^{-zt_1} - F(z, t_2)\,e^{-zt_2}\}}{z^2+n^2}\,dz.$$

Also

$$\int_0^{t_1} h(t)\cos nt\,dt = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{F(z, t_1)(z\cos nt_1 + n\sin nt_1)}{z^2+n^2}\,dz - \frac{1}{2\pi i}\int_{\Gamma_N} \frac{F(z, t_1)\,ze^{-zt_1}}{z^2+n^2}\,dz$$

$$= -\frac{1}{2\pi i}\int_{\Gamma_N} \frac{F(z, t_1)\,ze^{-zt_1}}{z^2+n^2}\,dz,$$

since the first integrand has no pole at ∞.

Similarly

$$\int_0^{t_2} h(t)\cos nt\,dt = -\frac{1}{2\pi i}\int_{\Gamma_N} \frac{F(z, t_2)\,ze^{-zt_2}}{z^2+n^2}\,dz,$$

i.e.,

$$\pi A_n = \int_{t_1}^{t_2} h(t)\cos nt\,dt.$$

Similarly

$$\pi B_n = \int_{t_1}^{t_2} h(t)\sin nt\,dt.$$

The corresponding result in the more general case when $g(t)$ is given by a set of functions $g_r(t)$ in the various sub-intervals follows by addition.

5.2. *Illustrations.*

Since $e^{2\pi ni} = 1$, we may, when ni is not a pole of $G(z)$, replace $G(z)$ by the more compact expression $G_1(z)$ where

$$G_1(z) = \sum_0^{s-1}\{F(z, t_r+0) - F(z, t_r-0)\}\,e^{-zt_r}.$$

(*a*) Polynomials.

Suppose $g_r(t)$ is a polynomial, $r = 1$ to s:

$$\pi A_0 = \sum_1^{s}\int_{t_{r-1}}^{t_r} g_r(t)\,dt,$$

$$G_1(z) = \sum_0^{s-1}\left\{\frac{g_{r+1}(t_r) - g_r(t_r)}{z} + \frac{g'_{r+1}(t_r) - g_r'(t_r)}{z^2}\cdots\right\}e^{-zt_r},\ g_0(t_0) = g_s(2\pi),$$

$$\pi(A_n - iB_n) = G_1(in),\quad (n > 0).$$

Example 1.

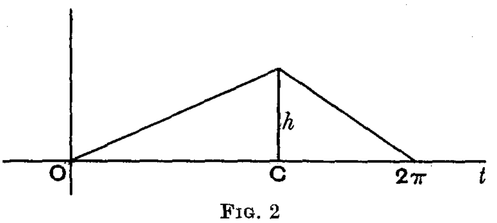

FIG. 2

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$$g(t)=\frac{ht}{c},\quad (0\leqslant t\leqslant c)\,;\quad \frac{h(2\pi-t)}{2\pi-c},\quad (c\leqslant t\leqslant 2\pi)\,;$$

$$A_0=h,$$

$$G_1(z)=\frac{(m_1-m_2)(1-e^{-zc})}{z^2},$$

where m_1 and m_2 are the gradients of the lines.

Thus $$A_n=-\frac{2h(1-\cos nc)}{c(2\pi-c)n^2}\,;\quad B_n=\frac{2h\sin nc}{c(2\pi-c)n^2}.$$

Example 2.

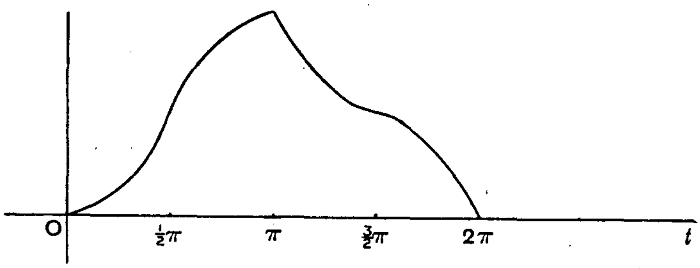

FIG. 3

$$g(t)=\begin{cases} t^2, & (0\leqslant t\leqslant \tfrac{1}{2}\pi)\\ \tfrac{1}{2}\pi^2-(t-\pi)^2, & (\tfrac{1}{2}\pi\leqslant t\leqslant \pi)\\ \tfrac{1}{4}\pi^2+(t-\tfrac{3}{2}\pi)^2, & (\pi\leqslant t\leqslant \tfrac{3}{2}\pi)\\ \tfrac{1}{4}\pi^2-(t-\tfrac{3}{2}\pi)^2, & (\tfrac{3}{2}\pi\leqslant t\leqslant 2\pi). \end{cases}$$

The graph consists of four similar parabolic arcs, and it is obvious from the figure that $\pi A_0=\frac{1}{2}\pi^3$.

The calculations of A_n and B_n, $(n>0)$, by the integral formulae are tedious, but by the residue method we need only the measure of discontinuity of $g(t)$ and its derivatives. Thus

$$G_1(z)=\frac{0-0}{z}+\frac{0+\pi}{z^2}+\frac{2+2}{z^3}+\left\{\frac{0}{z}+\frac{0}{z^2}+\frac{-2-2}{z^3}\right\}e^{-\frac{1}{2}z\pi}$$
$$+\left\{\frac{0}{z}+\frac{-\pi-0}{z^2}+\frac{2+2}{z^3}\right\}e^{-z\pi}+\left\{\frac{0}{z}+\frac{0}{z^2}+\frac{-2-2}{z}\right\}e^{-3z\pi/2},$$

$$\pi A_n=-\frac{\pi}{n^2}-\frac{4\sin\frac{1}{2}n\pi}{n^3}+\frac{\pi\cos n\pi}{n^2}-\frac{4\sin\frac{3}{2}n\pi}{n^3},$$

$$A_n=-(1-\cos n\pi)/n^2,$$

$$-\pi B_n=4(1-\cos\tfrac{1}{2}n\pi+\cos n\pi-\cos\tfrac{3}{2}n\pi)/n^3,$$

$=16/n^3$ if n is of the form $4m-2$ and is otherwise zero.

$$f(t)=\tfrac{1}{4}\pi^2-2\left(\cos t+\frac{\cos 3t}{3^2}+\frac{\cos 5t}{5^2}\ldots\right)$$
$$-\frac{16}{\pi}\left(\frac{\sin 2t}{2^3}+\frac{\sin 6t}{6^3}+\frac{\sin 10t}{10^3}\ldots\right).$$

Example 3. $g(t)=\sin t$, $(0\leqslant t\leqslant\pi)$; 0, $(\pi\leqslant t\leqslant 2\pi)$.
This is a resonant case, where the integrand is

$$\frac{(1-e^{-z\pi})}{z^2+1}\,\frac{e^{tz}}{e^{2\pi z}-1}$$
$$=\frac{e^{z(t-\pi)}}{(z^2+1)(e^{\pi z}-1)}.$$

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The poles are 0, $\pm i$, $\pm 2i$, $\pm 4i$, $\pm 6i$, ... , and

$$f(t)=\frac{1}{\pi}+\frac{\sin t}{2}+\frac{2}{\pi}\sum_{1}^{\infty}\frac{\cos 2mt}{1-4m^2}.$$

5.3. *Sine and Cosine Series.*

If $g(t)$ is given for the interval $0<t<\pi$, and its specification for the interval $\pi<t<2\pi$ is determined by the relation $g(2\pi-t)=g(t)$ (so that $f(t)=f(-t)$), then the Fourier expansion is a cosine series. Similarly if $g(2\pi-t)=-g(t)$ (so that $f(-t)=-f(t)$), the expansion is a sine series.

This readily follows from the integral formulae or from the fact that

(i) if $g(2\pi-t)=g(t)$, then $F(z, t_0)=-F(-z, 2\pi-t_0)$,

and

(ii) if $g(2\pi-t)=-g(t)$, then $F(z, t_0)=F(-z, 2\pi-t_0)$.

If therefore

$$g(t)=\begin{cases} g_r(t), & (t_{r-1}<t<t_r\,;\quad r=1, 2, \ldots, m\,;\quad t_0=0,\ t_m=\pi),\\ g(2\pi-t), & (\pi<t<2\pi),\end{cases}$$

the integrand is

$$\frac{G(z)\,e^{tz}}{e^{2\pi z}-1}$$

where $G(z)=F_1(z, 0)+F_1(-z, 0)\,e^{-2\pi z}$

$$+\sum_{r=1}^{m-1}\{F_{r+1}(z, t_r)-F_r(z, t_r)\}e^{-zt_r}+\{F_{r+1}(-z, t_r)-F_r(-z, t_r)\}\,e^{-z(2\pi-t_r)}\}$$
$$-\{F_m(z, \pi)\,e^{-\pi z}+F_m(-z, \pi)\,e^{\pi z}\},$$

and the expansion is obviously a cosine series.

Also if

$$K(z)=F(z, 0)+\sum_{1}^{m-1}\{F_{r+1}(z, t_r)-F_r(z, t_r)\}\,e^{-zt_r}-F_m(z, \pi)\,e^{-z\pi},$$

then $A_n=\dfrac{2}{\pi}\mathbf{R}K(in)$, when in is not a pole of $K(z)$.

The corresponding value of $G(z)$ for an odd function $f(t)$ is obtained by replacing $F(-z, t_r)$ by $-F(-z, t_r)$ and $B_n=-\dfrac{2}{\pi}\mathbf{I}K(in)$.

Example. 1.

$$g(t)=\begin{cases} t & (0\leqslant t\leqslant \frac{1}{3}\pi)\\ \pi-2t & (\frac{1}{3}\pi\leqslant t\leqslant \frac{2}{3}\pi)\\ t-\pi & (\frac{2}{3}\pi\leqslant t\leqslant \pi),\end{cases}$$

$$A_0=0,$$

$$K(z)=\{1-3e^{-\pi z/3}+3e^{-2\pi z/3}-e^{-\pi z}\}/z^2,$$

$$A_n=\frac{2}{\pi n^2}\{-1+3\cos n\pi/3-3\cos 2n\pi/3+\cos n\pi\},$$

i.e., $$f(t)=\frac{2}{\pi}\left\{\cos t-\frac{8\cos 3t}{3^2}+\frac{\cos 5t}{5^2}+\frac{\cos 7t}{7^2}-\frac{8\cos 9t}{9^2}+\frac{\cos 11t}{11^2}+\ldots\right\}$$

and $$B_n=\frac{6}{\pi n^2}\{\sin n\pi/3-\sin 2n\pi/3\},\ \text{giving}$$

$$f(t)=\frac{6\sqrt{3}}{\pi}\left\{\frac{\sin 2t}{2^2}-\frac{\sin 4t}{4^2}+\frac{\sin 8t}{8^2}-\frac{\sin 10t}{10^2}\ldots\right\}.$$

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Example 2. Express $\sin t$ as a cosine series.

The integrand is

$$\frac{1+2e^{-\pi z}+e^{-2\pi z}}{z^2+1}\cdot\frac{e^{tz}}{e^{2\pi z}-1}$$
$$=\frac{e^{\pi z}+1}{z^2+1}\cdot\frac{e^{z(t-2\pi)}}{e^{z\pi}-1}.$$

Poles are $0,\ \pm 2i,\ \pm 4i,\ \pm 6i,\ \ldots$, giving

$$\frac{\pi}{2}\left|\sin t\right| = 1-2\sum_{1}^{\infty}\frac{\cos 2mt}{4m^2-1}.$$

5.4. *Step Functions.*

Let $g(t)=c_r\,(t_{r-1}<t<t_r,\ r=1,2,\ldots,m:\ \ t_0=0,\ t_m=2\pi)$.

It is important to note that the variation of t takes place in the open intervals specified by the discontinuities. Although the value at t_r may be prescribed in an application, this has no necessary relationship with the determinate value $\frac{1}{2}(c_r+c_{r+1})$ given by the Fourier expansion.

The integrand is

$$\sum_{r=1}^{r=m}\frac{c_r\{e^{z(t-t_{r-1})}-e^{z(t-t_r)}\}}{z(e^{2\pi z}-1)},$$

and the Fourier expansion is

$$\sum_{r=1}^{m}\frac{c_r(t_r-t_{r-1})}{2\pi}+\sum_{n=1}^{\infty}\sum_{r=1}^{m}\frac{c_r\{\sin n(t-t_{r-1})-\sin n(t-t_r)\}}{n\pi}.$$

The use of the above integrand suggests the method of establishing the Fourier expansion of a function of bounded variation.

6. *Functions of bounded variation.*

We recall some essential properties of these functions. Let $g(t)$ be a function of bounded variation in the interval $a\leqslant t\leqslant b$. Then

(i) The limits $g(t+0)$, $g(t-0)$ exist at all points of the interval. The points of discontinuity are of the first kind and form a set of measure zero.

(ii) If the interval is subdivided at the points

$$t_1,\ t_2,\ \ldots,\ t_{m-1},\ \ (t_0=a,\ t_m=b),$$

$\sum_{1}^{m}|\,g(t_{r-1})-g(t_r)\,|$ has a finite upper bound K independent of m and for all choices of t_1, $t_2, \ldots, t_{m-1}$.

(iii) $g(t)$ is integrable (R) between a and b; *i.e.*, if t_r' be any point in the interval $t_{r-1}\leqslant t\leqslant t_r$ the integral is the limit of

$$\sum_{1}^{m} g(t_r')(t_r-t_{r-1}),$$

when m tends to infinity in such a way that $\max(t_r-t_{r-1})$ tends to zero.

6.1. *The Fourier expansion for* $g(t)$.

Let $g(t)$ be a function of bounded variation in the interval $0\leqslant t\leqslant 2\pi$ and let θ be a point interior to the interval.

Divide the interval $(0,\theta)$ at the points $t_1, t_2, \ldots, t_{m-1}$ $(t_0=0,\ t_m=\theta)$.

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Let t_r' be a point interior to the interval (t_{r-1}, t_r), and consider the step-function defined by the equations

$$h(t)=\begin{cases} g(t_r'), & t_{r-1}<t<t_r, \quad r=1, 2 \dots m, \\ 0, & \theta<t<2\pi. \end{cases}$$

The selection of an interior point does not, of course, alter the value of the limit

$$\sum_1^m g(t_r')(t_r-t_{r-1})$$

that defines the integral of $g(t)$ from 0 to θ, but it will be recalled that in the elementary cases considered earlier the given functions $g(t)$ were specified only in the open intervals, and that the values of the Fourier expansions at the ends of the intervals were then determinate.

The Fourier expansion of the step-function suggests the consideration of the finite contour integral

$$\frac{1}{2\pi i}\int_{\Gamma_N} \frac{\sum_1^m \{g(t_r')(e^{z(t-t_{r-1})}-e^{z(t-t_r)}\}}{z(e^{2\pi z}-1)}\,dz.$$

Calculation of the residues as in § 5.4 shows that

$$\sum_1^m \frac{g(t_r')(t_r-t_{r-1})}{2\pi}+\sum_{n=1}^N\sum_{r=1}^m \frac{g(t_r')\{\sin n(t-t_{r-1})-\sin n(t-t_r)\}}{n\pi}$$
$$=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{g(t_1')\,e^{zt}\,dz}{z(e^{2\pi z}-1)}-\frac{1}{2\pi i}\int_{\Gamma_N}\frac{g(t_m')\,e^{z(t-\theta)}\,dz}{z(e^{2\pi z}-1)}$$
$$+\sum_1^{m-1}\frac{1}{2\pi i}\int_{\Gamma_N}\frac{\{g(t'_{r+1})-g(t_r)\}\,e^{z(t-t_r)}}{z(e^{2\pi z}-1)}\,dz.$$

This equation may be written

$$\tfrac{1}{2}\alpha_0+\sum_1^N(\alpha_n\cos nt+\beta_n\sin nt)=I_0-I_m+\sum_1^{m-1}I_r,$$

where
$$\pi\alpha_0=\sum_1^m g(t_r')\;(t_r-t_{r-1}),$$

$$\pi\alpha_n=\sum_1^m\frac{g(t_r')\;(\sin nt_r-\sin nt_{r-1})}{n},\quad \pi\beta_n=\sum_1^m\frac{g(t_r')\;(\cos nt_{r-1}-\cos nt_r)}{n},$$

$$I_0=\frac{1}{2\pi i}\int_{\Gamma_N} g(t_1')\,\frac{e^{zt}\,dz}{z(e^{2\pi z}-1)},\quad I_m=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{g(t_m')\,e^{z(t-\theta)}\,dz}{z(e^{2\pi z}-1)},$$

$$I_r=\int_{\Gamma_N}\frac{g(t'_{r+1})-g(t_r')}{2\pi i}\cdot\frac{e^{z(t-t_r)}}{z(e^{2\pi z}-1)}\,dz,\quad (r=1, 2, \dots, m-1).$$

Now in § 3 it was proved that

$$\frac{1}{2\pi i}\int_{\Gamma_N}\frac{e^{zt}\,dz}{z(e^{2\pi z}-1)}\to\begin{cases}-\tfrac{1}{2}, & t=0,\\ 0, & 0<t<2\pi,\\ +\tfrac{1}{2}, & t=2\pi,\end{cases}$$

as $N\to\infty$.

Recapitulation of that proof will show that the integral tends uniformly to zero in the interval $0<\delta\leqslant t\leqslant 2\pi-\delta<2\pi$, *i.e.*, for any given $\epsilon\,(>0)$ a value N_0 can be found such that

$$\left|\frac{1}{2\pi i}\int_{\Gamma_N}\frac{e^{zt}\,dz}{z(e^{2\pi z}-1)}\right|<\epsilon$$

for all $N\geqslant N_0$ and for all values of t in the interval $\delta\leqslant t\leqslant 2\pi-\delta$.

It may be noted, however, that

$$\frac{1}{2\pi i}\int_{\Gamma_N}\frac{dz}{z(e^{2\pi z}-1)}=-\tfrac{1}{2} \quad\text{and}\quad \frac{1}{2\pi i}\int_{\Gamma_N}\frac{e^{2\pi z}\,dz}{z(e^{2\pi z}-1)}=+\tfrac{1}{2}$$

for all $N>0$.

Putting $t=\theta$ in the residue equation we have

$$\tfrac{1}{2}\alpha_0+\sum_1^N(\alpha_n\cos nt+\beta_n\sin nt)=\sum_0^{m-1}I_r-I_m,$$

where

$$I_0=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{g(t_1')\,e^{\theta z}\,dz}{z(e^{2\pi z}-1)},\quad I_r=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{\{g(t'_{r+1})-g(t_r')\}\,e^{z(\theta-t_r)}\,dz}{z(e^{2\pi z}-1)},$$

$$I_m=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{g(t_m')\,dz}{z(e^{2\pi z}-1)}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_p \ldots$ and $\mu_1, \mu_2, \ldots, \mu_q, \ldots$ be two sequences of positive numbers tending to zero.

Choose t_{m-1} so that $\theta-t_{m-1}=\lambda_N$, and the other points $t_1, t_2, \ldots, t_{m-2}$, so that

$$\max\,(t_r-t_{r-1})<\mu_m,\quad r=1, 2, \ldots, (m-2).$$

Since $\quad 0<\lambda_p<\theta-t_r\leqslant\theta<2\pi,\quad (r=0, 1, \ldots, m-2),$

$$\left|\sum_0^{m-1}I_r\right|<\{|g(t_1')|+\sum_1^{m-1}|g(t'_{r+1})-g(t_r')|\}\lambda_N$$

$$<K\lambda_N,\ (N\geqslant N_0),$$

where K is independent of m and for all methods of subdivision ; *i.e.*,

$$\lim_{N\to\infty}\lim_{m\to\infty}\sum_0^{m-1}I_r=0.$$

Again $g(t_m')=g(\theta-k\lambda_N)\quad(0<k<1)$,

$$\textit{i.e.,}\quad \lim_{N\to\infty}I_m=-\tfrac{1}{2}g(\theta-0).$$

Also $\lim\alpha_0$ when m tends to infinity

$$=\lim\frac{1}{\pi}\sum_1^m g(t_r')(t_r-t_{r-1})=\frac{1}{\pi}\int_0^\theta g(\tau)\,d\tau,$$

$$\lim\alpha_n=\lim\frac{1}{\pi}\sum_1^m\frac{g(t_r')}{n}(\sin nt_r-\sin nt_{r-1}),$$

$$=\lim\frac{1}{\pi}\sum_1^m\frac{g(t_r')}{n}\cos nt_r''\,(t_r-t_{r-1}),$$ where t_r'' is some point interior to the interval (t_{r-1}, t_r),

$$=\frac{1}{\pi}\int_0^\theta g(\tau)\cos n\tau\,d\tau,$$

since $g(t)\cos nt$ is of bounded variation and $\cos nt$ is uniformly continuous.

Similarly $$\lim\beta_n=\frac{1}{\pi}\int_0^\theta g(\tau)\,\sin n\tau\,d\tau.$$

Thus when we let N tend to infinity,

$$T_0^\theta(t)\equiv\int_0^\theta g(\tau)\,d\tau+\sum_{n=1}^\infty\int_0^\theta g(\tau)\cos n(t-\tau)\,d\tau=\tfrac{1}{2}\pi g(\theta-0),$$

when $t=\theta$, and 0, when $\theta<t<2\pi$.

By taking $t=2\pi$ and using a similar argument for the finite contour integral we deduce that $T_0{}^\theta(2\pi)=\frac{1}{2}\pi g(+0)$, and therefore since T is periodic $T_0{}^\theta(0)$ is also $\frac{1}{2}\pi g(+0)$.

By dividing up the interval from θ to 2π, we may prove similarly that

$$T_\theta^{2\pi} \equiv \int_\theta^{2\pi} g(\tau)\, d\tau + \sum_{n=1}^{\infty} \int_0^{2\pi} g(\tau) \cos n(t-\tau)\, d\tau = \begin{cases} \frac{1}{2}\pi g(\theta+0) & \text{when } t=\theta, \\ \frac{1}{2}\pi g(2\pi-0) & \text{when } t=0 \text{ or } 2\pi, \\ 0 & \text{when } 0<t<\theta. \end{cases}$$

By addition therefore

$$T(t) \equiv \int_0^{2\pi} g(\tau)\, d\tau + \sum_{n=1}^{\infty} \int_0^{2\pi} g(\tau) \cos n(t-\tau)\, d\tau = \begin{cases} \frac{1}{2}\pi\{g(\theta-0)+g(\theta+0)\}, & t=\theta, \\ \frac{1}{2}\pi\{g(+0)+g(2\pi-0)\}, & t=0,\ 2\pi. \end{cases}$$

But θ is any point within the interval. Therefore if $f(t)$ is the periodic function associated with $g(t)$,

$$\int_0^{2\pi} f(\tau)\, d\tau + \sum_{n=1}^{\infty} \int_0^{2\pi} f(\tau) \cos n(t-\tau)\, d\tau = \tfrac{1}{2}\pi\{f(t-0)+f(t+0)\}$$

for all values of t.

NOTE. It has been tacitly assumed above that the infinite series is obtained by letting m and N tend to infinity in this order. This is readily justified, however, if we assume that $\lambda_p = O(1/p)$.

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